Gibbs measures with double stochastic integrals on a path space

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We investigate Gibbs measures relative to Brownian motion in the case when the interaction energy is given by a double stochastic integral. In the case when the double stochastic integral is originating from the Pauli-Fierz model in nonrelativistic quantum electrodynamics, we prove the existence of its infinite volume limit.

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1. Preliminaries

1.1. *Gibbs measures relative to Brownian motions*

Gibbs measures relative to Brownian motion appeared in [22], where they have been introduced to study a particle system linearly coupled to a scalar quantum field. A systematic study of such measures has been started from [6], where by making use of this measure the spectrum of the so-called Nelson model is investigated. Since then there has been growing activity and interest in the study of various types of these measures [4,5,12,13,21].

One way to understand Gibbs measures relative to Brownian motion is to view them as the limit of a one-dimensional chain of unbounded interacting spins, with the distance between the spins going to zero. As a simple example, which will be instructive in what follows, let us take $\mathbb{R}^d$ for the spin space, and fix a (finite or infinite) a priori measure ν_0 on $\mathbb{R}^d$ as well as smooth, bounded functions $\mathbb{V} : \mathbb{R}^d \to \mathbb{R}$ and $\mathbb{W} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. On the lattice $\varepsilon\mathbb{Z} \cap [-T, T]$ with spacing ε and $n = 2T/\varepsilon$

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sites, we define the measure $\nu^{\mathbb{W}}$ through

$$\nu^{\mathbb{W}}(\mathrm{d}x_{-n}\dots\mathrm{d}x_n) = \frac{1}{Z_{\nu^{\mathbb{W}}}}\prod_{|i|\le n}\nu_0(\mathrm{d}x_i)\mathrm{e}^{-\varepsilon\sum_i \mathbb{V}(x_i)-\frac{1}{\varepsilon}\sum_i(x_{i+1}-x_i)^2+\varepsilon^2\sum_{i,j}\mathbb{W}(j-i,x_j-x_i)}. \tag{1.1}$$

Here $Z_{\nu^{\mathbb{W}}}$ normalizes $\nu^{\mathbb{W}}$ to a probability measure, and for finite ε, $\nu^{\mathbb{W}}$ is just a chain of interacting spins. However, the scaling becomes very important when $\varepsilon \to 0$. Then, formally each spin configuration $(x_i)_{|i|\le n} \in \varepsilon Z \cap [-T,T]$ becomes a function $x(\cdot)$ on $[-T,T]$, and the particular scaling of the quadratic term above gives rise to the term $\lim_{\varepsilon\to 0}\varepsilon\sum_i(x_{i+1}-x_i)^2/\varepsilon^2 = \int_{-T}^T(\mathrm{d}x(s)/\mathrm{d}s)^2\mathrm{d}s$. It is this term that prevents the measure $\nu^{\mathbb{W}}$ from being concentrated on more and more rough functions when $\varepsilon\to 0$, ensuring continuity of $x(t)$ in the limit. Indeed, when ν_0 is chosen as the Lebesgue measure on $\mathbb{R}^d$, it is not difficult to show that part of the normalization along with the quadratic term give converge to Wiener measure $\mathcal{W}$, so that in the limit, $\varepsilon\to 0$, we obtain

$$\nu_T^{\mathbb{W}}(\mathrm{d}B) = \frac{1}{Z_{\nu_T^{\mathbb{W}}}}\mathrm{e}^{-\int_{-T}^T\mathbb{V}(B_s)\,\mathrm{d}s-\int_{-T}^T\mathrm{d}s\int_{-T}^T\mathrm{d}t\mathbb{W}(B_t-B_s,t-s)}\,\mathrm{d}\mathcal{W}. \tag{1.2}$$

Here $(B_t)_{t\ge 0}$ is now a Brownian motion, hence we call (1.2) Gibbs measures relative to Brownian motion. Indeed, the measure appearing in [22] is of the above type, and most of the subsequent works cited above have been concerned with measures of the form (1.2).

In this paper we study another type of Gibbs measures, arising from a very similar discrete spin system. Namely, let us now define

$$\begin{aligned}
&\nu^{\mathbb{W}_M}(\mathrm{d}x_{-n}\dots\mathrm{d}x_n)\\
&= \frac{1}{Z_{\nu^{\mathbb{W}_M}}}\prod_{|i|\le n}\nu_0(\mathrm{d}x_i)\exp\Bigg(-\varepsilon\sum_i\mathbb{V}(x_i)-\frac{1}{\varepsilon}\sum_i(x_{i+1}-x_i)^2\\
&\qquad\qquad +\sum_{i,j}(x_{i+1}-x_i)\cdot\mathbb{W}_M(j-i,x_j-x_i)(x_{j+1}-x_j)\Bigg).
\end{aligned}\tag{1.3}$$

Now $\mathbb{W}_M$ is a $d\times d$ matrix, but otherwise the expression looks very similar to (1.1). The crucial point is, however, that now the scaling of the term involving $\mathbb{W}_M$ is different. The ε^2 which ensured convergence to a double Riemann integral is gone by sandwiching $\varepsilon^2\mathbb{W}_M$ between $(x_{i+1}-x_i)/\varepsilon$ and $(x_{j+1}-x_j)/\varepsilon$, and replaced by the increments of the spins themselves. Since these increments will eventually converge to Brownian motion increments, as discussed above, they are of order $\sqrt{\varepsilon}$, so the scaling is indeed different. So after taking the limit $\varepsilon\to 0$, we informally obtain

$$\nu_T^{\mathbb{W}_M}(\mathrm{d}B) = \frac{1}{Z_{\nu_T^{\mathbb{W}_M}}}\mathrm{e}^{-\int_{-T}^T\mathbb{V}(B_s)\,\mathrm{d}s-\int_{-T}^T\int_{-T}^T\mathrm{d}B_s\cdot\mathbb{W}_M(B_t-B_s,t-s)\,\mathrm{d}B_t}\mathrm{d}\mathcal{W}. \tag{1.4}$$

As a consequence, taking the limit $\varepsilon\to 0$ yields a double stochastic integral in place of the double Riemann integral (1.2).

1.2. *Definition of double stochastic integrals*

From now on we assume that $d = 3$ and specify the pair potential $\mathbb{W}_M = W = W(X,t) = (W_{\mu\nu}(X,t))_{1\le\mu,\nu\le 3}$ given by

$$W_{\mu\nu}(X,t) := \int \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-\omega(k)|t|} e^{ik\cdot X} \delta^{\perp}_{\mu\nu}(k) \mathrm{d}^3 k, \tag{1.5}$$

where $\delta^{\perp}(k) = (\delta^{\perp}_{\mu\nu}(k))_{1\le\mu,\nu\le 3}$ is given by

$$\delta^{\perp}_{\mu\nu}(k) := \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}. \tag{1.6}$$

Measures of the type (1.4) with pair potential (1.5) appear in the study of the so-called non-relativistic quantum electrodynamics, and have been introduced on a formal level in [10,14,24]. However we notice that there are some difficulties in the expression (1.4): For $t > s$, the integrand is not adapted to the natural filtration $\mathcal{F}_T = \sigma(B_r; r \le T)$, so as a stochastic integral or any of its obvious transformations the double stochastic integral such as (1.4) does not make sense. So the right-hand side of (1.4) is just an informal symbol.

In [16] and [18] , however, the firm mathematical definition of (1.4) has been given through a Gaussian random process associated with an Euclidean quantum field. We outline it below. A Gaussian random process $\mathcal{A}^E(f)$ labeled by $f \in \oplus^3 L^2(\mathbb{R}^{3+1})$ on some probability space (Q_E, Σ_E, μ_E) is introduced, which has mean zero and covariance $\mathbb{E}_{\mu_E}[\mathcal{A}(f)\mathcal{A}(g)] = q(f,g)$ given by

$$q(f,g) := \frac{1}{2}\int \overline{\hat{f}(k,k_0)} \cdot \delta^{\perp}(k)\hat{g}(k,k_0)\mathrm{d}^3 k \mathrm{d}k_0 \tag{1.7}$$

for $f, g \in \oplus^3 L^2(\mathbb{R}^{3+1})$, where $\hat{}$ denotes the Fourier transformation. Let

$$K_t = \oplus^3_{\mu=1} \int_0^t j_s \varphi(\cdot - B_s) \mathrm{d}B^\mu_s \tag{1.8}$$

be the $\oplus^3 L^2(\mathbb{R}^{3+1})$-valued stochastic integral defined in the similar way as standard stochastic integrals, where $j_s : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^{3+1})$ denotes the isometry satisfying

$$(j_s f, j_t g)_{L^2(\mathbb{R}^{3+1})} = (\hat{f}, e^{-|t-s|\omega}\hat{g})_{L^2(\mathbb{R}^3)}. \tag{1.9}$$

See Subsection 3.1 for the details.

Definition 1.1. Let W be the pair potential defined in (1.5). The double stochastic integral is defined by

$$\int_0^t \int_0^t dB_s \cdot W(B_t - B_s, t-s) dB_t := q(K_t, K_t). \tag{1.10}$$

We would like to express (1.10) as an iterated stochastic integral in this paper.

1.3. *Main results*

Let us define the Wiener measure $\mathcal{W}$ on $X := C(\mathbb{R};\mathbb{R}^3)$, cf. also [23] . Let $H_0 = -(1/2)\Delta$. Suppose that $f_1, ..., f_{n-1} \in L^\infty(\mathbb{R}^3)$ with compact support. Then there exists a measure $\mathcal{W}$ on X such that

$$\begin{aligned}&(f_0, e^{-(t_1-t_0)H_0} f_1 \cdots f_{n-1} e^{-(t_n-t_{n-1})H_0} f_n)_{L^2(\mathbb{R}^3)}\\ &= \int_X \overline{f_0(B_{t_0})} f_1(B_{t_1}) \cdots f_n(B_{t_n}) \mathrm{d}\mathcal{W}. \end{aligned}\qquad (1.11)$$

A path with respect to this measure is denoted by $B_t(w) = w(t)$ for $w \in X$. Note that Wiener measure is not a probability measure, indeed it has infinite mass. If P_W^{x,t_0} denotes the measure of standard Brownian motion starting from $x \in \mathbb{R}^3$ at time t_0, then

$$\int_X f_0(B_{t_0}) \cdots f_n(B_{t_n}) \mathrm{d}\mathcal{W} = \int_{\mathbb{R}^3} dx \int_{C([t_0,\infty);\mathbb{R}^3)} f_0(B_{t_0}) \cdots f_n(B_{t_n}) \mathrm{d}P_W^{x,t_0}.$$

Let $\psi \in L^2(\mathbb{R}^3)$ be a nonnegative function and we fix it throughout this paper. In the case of (1.2), the existence of the weak limit of the measure on X,

$$d\nu_T^{\mathbb{W}} := \frac{1}{Z_{\nu_T^{\mathbb{W}}}} \psi(B_{-T})\psi(B_T) e^{-\int_{-T}^T ds \int_{-T}^T dt \mathbb{W}(B_t - B_s, t-s)} e^{-\int_{-T}^T \mathbb{V}(B_s) ds} \mathrm{d}\mathcal{W},$$

as $T \to \infty$ has been investigated for various kinds of $\mathbb{V}$ and $\mathbb{W}$, and the limiting measure, $\nu_\infty^{\mathbb{W}}$, proved to be useful to study the ground state φ_{g} of some particle system linearly coupled to a scalar quantum field. Namely for a suitable operator $\mathcal{O}$, we can express the expectation $(\varphi_{\mathrm{g}}, \mathcal{O}\varphi_{\mathrm{g}})$ as $\int_X f_{\mathcal{O}} d\nu_\infty^{\mathbb{W}}$ with some integrand $f_{\mathcal{O}}$. So, beyond the existence of a measure of the form (1.4), one is interested in the limit as $T \to \infty$, at least along a subsequence. In other words, one would like to prove the tightness of the family of measures (1.4). This is by no means an easy task, given that there are very few good general estimates on single stochastic integrals, let alone double integrals.

The purpose of our present paper is to point out that there is at least one special case where there is a comparatively easy way to construct both the finite volume Gibbs measure and the infinite volume limit, namely the case when $\mathbb{W}_M = W = (W_{\mu\nu})_{1\le\mu,\nu\le 3}$ is given in (1.5). Fortunately, this special case is the one that motivated the whole theory of Gibbs measures with double stochastic integrals. The main results in this paper are

(1) we give an iterated stochastic integral expression of (1.10);

(2) we show the tightness of the family of measures

$$\frac{1}{Z_T} \psi(B_T)\psi(B_{-T}) \mathrm{e}^{-\int_{-T}^T V(B_s)ds - \alpha^2 \int_{-T}^T \int_{-T}^T dB_s \cdot W(B_t - B_s, t-s) \mathrm{d}B_t} \mathrm{d}\mathcal{W}$$

for a general class of V including the Coulomb potential $V(x) = -1/|x|$, and arbitrary values of coupling constant $\alpha \in \mathbb{R}$.

There has been recent progress both of the above topics: M. Gubinelli and J. Lőrinczi [13] employ the concepts of stochastic currents rough paths in order to define

(1.4) rigorously for finite volume, and use a cluster expansion in order to construct the infinite volume limit. While these are impressive results, the techniques used are rather advanced, and the use of cluster expansion comes with strong assumptions on single site potentials V and coupling constants. The advantage of our methods is that we can avoid some restrictions needed in [13]; in particular we need not restrict to single site potentials that grow faster than quadratically at infinity, and we need no small coupling constant in front of the double stochastic integral. In particular our results include the Coulomb potential which is the most reasonable single site potential. On the other hand, of course the range of potentials W that is treated in [13] is much greater than ours.

The paper is organized as follows: In Section 2 we will construct the finite volume Gibbs measure as the marginal of a measure with single stochastic integral on a larger state space. This construction is well known [25], but has not been carried out rigorously so far. In Section 3, we rely on the detailed results available about the Pauli-Fierz model [20,11] in order to show that our family of Gibbs measures is tight, giving the existence of an infinite volume measure. While we expect that the general method of enlarging the state space should allow us to define and prove infinite volume limits for many more models than just Pauli-Fierz, this is not all straightforward. We will comment on this issue at the end of Section 3.

2. Iterated expression of finite volume measures

In this section we will specify the measure μ_T that we are working with, and identify it as the marginal of another measure ν_T on a larger state space. Let us start by introducing an infinite dimensional Ornstein-Uhlenbeck process which will serve as the reference measure for the auxiliary degrees of freedom. Put

$$\omega(k) = \sqrt{|k|^2 + m^2} \tag{2.1}$$

for $m \geq 0$, and let $X_s(f)$ be the Gaussian random process on a probability space $(Q, \Sigma, \mathcal{G})$ labeled by measurable function $f = (f_1, f_2, f_3)$ with mean zero and covariance given by

$$\mathbb{E}_{\mathcal{G}}[X_s(f)X_t(g)] = \int \mathrm{d}^3k \frac{1}{2\omega(k)} \mathrm{e}^{-\omega(k)|t-s|} \overline{\hat{f}(k)} \cdot \delta^{\perp}(k)\hat{g}(k). \tag{2.2}$$

Here $\hat{f}$ denotes the Fourier transform of f and we assume that $\hat{f}_\mu/\sqrt{\omega}, \hat{g}_\nu/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, $\mu, \nu = 1, 2, 3$.

Remark 2.1. Let $Y_s(f)$ be the Gaussian random process on (Q_E, Σ_E, μ_E) defined by

$$Y_s(f) := \mathcal{A}^E(j_s(\hat{f}/\sqrt{\omega})^\vee). \tag{2.3}$$

Then $Y_s(f)$ is mean zero and its covariance is

$$\mathbb{E}_{\mu_E}[Y_s(f)Y_t(f)] = \mathbb{E}_{\mathcal{G}}[X_s(f)X_t(g)]. \tag{2.4}$$

Hence $Y_s(f)$ and $X_s(f)$ are isomorphic as Gaussian random processes.

We will now couple $\mathcal{G}$ to the Wiener measure $\mathcal{W}$. For this we use a coupling function φ with the assumption below:

Assumption (A):

(1) $\hat{\varphi}(k) = \hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$ and $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$.
(2) $\hat{\varphi}$ is rotation invariant, i.e., $\hat{\varphi}(Rk) = \hat{\varphi}(k)$ for all $R \in O(3)$.

Let us now define the quantity

$$J_{[0,T]}(X) := \int_0^T X_s(\varphi(\cdot - B_s)) \cdot \mathrm{d}B_s.$$

The proper definition of $J_{[0,T]}$ reads

$$J_{[0,T]}(X) := \lim_{n\to\infty} \sum_{j=1}^n X_{(j-1)T/n}(\varphi(\cdot - B_{(j-1)T/n})(B_{jT/n} - B_{(j-1)T/n})), \qquad (2.5)$$

where the right hand side strongly converges in $L^2(X \times Q; \mathcal{G} \otimes P_W^{x,0})$. This is proved by showing that the right-hand side of (2.5) is Cauchy by making use of (2.2). In the same way, we can define

$$J_T(X) := \int_{-T}^T X_s(\varphi(\cdot - B_s)) \cdot \mathrm{d}B_s.$$

The coupling between the Gaussian process and Brownian motion is given by the measure ν on $X \times Q$ with

$$\mathrm{d}\nu_T = \frac{1}{Z_T} \exp\left(\mathrm{i}\alpha \int_{-T}^T X_s(\varphi(\cdot - B_s)) \cdot \mathrm{d}B_s \right) \psi(B_{-T})\psi(B_T) \mathrm{d}\mathcal{W} \otimes \mathrm{d}\mathcal{G}, \qquad (2.6)$$

where $\psi \in L^2(\mathbb{R}^3)$ is an arbitrary nonnegative function, Z_T the normalizing constant, and α is a coupling constant. In order to guarantee that the density in (2.6) is integrable with respect to $\mathcal{W}$, we chose the boundary function ψ to be of rapid decrease at infinity.

We are now in the position to define our finite volume Gibbs measure. We will introduce an on-site potential V which we take Kato-decomposable [7], i.e. we require that the negative part V_- is in the Kato class while the positive part V_+ is the locally Kato class [23]. This ensures e.g. that

$$\sup_x \mathbb{E}_{P_W^{x,0}} \left[\exp\left(- \int_0^t V(B_s)\, \mathrm{d}s \right) \right] < \infty. \qquad (2.7)$$

Definition 2.1. Let $V : \mathbb{R}^3 \to \mathbb{R}$ be Kato-decomposable and $\alpha \in \mathbb{R}$ a coupling constant. Then the measure μ_T^V on X is defined through

$$\begin{aligned} \mathrm{d}\mu_T^V &:= \frac{1}{Z_T} \mathrm{e}^{-\int_{-T}^T V(B_s)\,\mathrm{d}s} \mathbb{E}_{\mathcal{G}}[\mathrm{d}\nu_T] \\ &= \frac{1}{Z_T} \psi(B_{-T})\psi(B_T) \mathrm{e}^{-\int_{-T}^T V(B_s)\,\mathrm{d}s} \mathbb{E}_{\mathcal{G}} \left[\exp\left(\mathrm{i}\alpha \int_{-T}^T X_s(\varphi(\cdot - B_s)) \cdot \mathrm{d}B_s \right) \right] \mathrm{d}\mathcal{W}. \end{aligned} \tag{2.8}$$

We want to show that the measure μ_T^V we just defined is a Gibbs measure with double stochastic integral as given in Section 1. The key to doing this is the fact that we will be actually able to calculate the Gaussian integral $\int_Q \exp(\mathrm{i}J_T(X))\mathrm{d}\mathcal{G}(X)$, and thus are left with an expression involving Brownian motion paths only. In doing so, we will set $\alpha = 1$ for a simpler notation.

Let us give the heuristic presentation first. By the standard formula we have

$$\mathbb{E}_{\mathcal{G}}[\mathrm{e}^{\mathrm{i}J_T}] = \exp\left(-\frac{1}{2} \mathbb{E}_{\mathcal{G}}[J_T^2] \right) \tag{2.9}$$

and formally, by Remark 2.1, we have

$$\mathbb{E}_{\mathcal{G}}[J_T^2] = \frac{1}{2} \int_{-T}^T \int_{-T}^T \mathrm{d}B_s \cdot W(B_t - B_s, t-s) \mathrm{d}B_t, \tag{2.10}$$

where W is given in (1.5). As it stands, there are problems with the right-hand side of formal expression (2.10), mainly because the integrand is not adapted. The resolution is to use symmetry of W and break up the integral into two parts, one where $s < t$ and one where $s > t$, which are then proper iterated Itô integrals. This leaves the diagonal part, which gives a non-vanishing contribution by the unbounded variation of B_t.

We define the iterated stochastic integral S_T by

$$\begin{aligned} S_T := \int \mathrm{d}^3k \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} \int_{-T}^T \mathrm{e}^{\mathrm{i}k\cdot B_s} \mathrm{d}B_s \cdot \int_{-T}^s e^{-\omega(k)(s-r)} \mathrm{e}^{-\mathrm{i}k\cdot B_r} \delta^\perp(k) \mathrm{d}B_r \\ + T \int \mathrm{d}^3k \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} \end{aligned} \tag{2.11}$$

S_T is the well-defined expression that will replace (2.8). The above line of reasoning and (2.11) are not new [25], except that (2.11) is usually not written out but instead just referred to as the double stochastic integral with the diagonal removed. Nevertheless, (2.11) can be considered as known. However, the derivation above is mathematically not rigorous, since the ill-defined expression (2.10) appears along the way. To avoid this, one has to derive (2.11) directly from $\mathbb{E}_{\mathcal{G}}[\mathrm{e}^{\mathrm{i}J_T}]$. This is what we do in the next theorem.

Theorem 2.1. *For almost every $w \in X$, we have*

$$\mathbb{E}_{\mathcal{G}}[\mathrm{e}^{\mathrm{i}J_T}] = e^{-S_T}. \tag{2.12}$$

Proof. Let us replace the time interval $[-T,T]$ with $[0,T]$ for notational convenience. We employ (2.5) and use dominated convergence to get

$$\begin{aligned}\mathbb{E}_{\mathcal{G}}[e^{\mathrm{i}J_T}] &= \lim_{n\to\infty}\mathbb{E}_{\mathcal{G}}\left[\exp\left(\mathrm{i}\sum_{j=1}^n X_{\Delta_j}(\varphi(\cdot-B_{\Delta_j}))\cdot\delta B_j\right)\right]\\ &= \lim_{n\to\infty}\exp\left(-\frac{1}{2}\mathbb{E}_{\mathcal{G}}\left[\sum_{j=1}^n X_{\Delta_j}(\varphi(\cdot-B_{\Delta_j}))\cdot\delta B_j\right]^2\right),\end{aligned}$$

where we set $\delta B_j = B_{jT/n} - B_{(j-1)T/n}$ and $\Delta_j = (j-1)T/n$, $j=1,...,N$. Now

$$\begin{aligned}&\mathbb{E}_{\mathcal{G}}\left[\sum_{j=1}^n X_{\Delta_j}(\varphi(\cdot-B_{\Delta_j}))\cdot\delta B_j\right]^2\\ &= \int \mathrm{d}^3k\,\frac{|\hat\varphi(k)|^2}{2\omega(k)}\sum_{j=1}^n\sum_{l=1}^n \mathrm{e}^{-|\Delta_j-\Delta_l|\omega(k)}\mathrm{e}^{\mathrm{i}k(B_{\Delta_j}-B_{\Delta_l})}\delta B_j\cdot\delta^\perp(k)\delta B_l\\ &= 2\sum_{j=1}^n\int \mathrm{d}^3k\,\frac{|\hat\varphi(k)|^2}{2\omega(k)}\mathrm{e}^{-\Delta_j\omega(k)+\mathrm{i}kB_{\Delta_j}}\delta B_j\cdot\delta^\perp(k)\left(\sum_{l=1}^{j-1}\mathrm{e}^{\Delta_l\omega(k)-\mathrm{i}kB_{\Delta_l}}\delta B_l\right) \qquad (2.13)\\ &\qquad + \sum_{j=1}^n \delta B_j\cdot\left(\int \mathrm{d}^3k\,\frac{|\hat\varphi(k)|^2}{2\omega(k)}\delta^\perp(k)\right)\delta B_j. \qquad (2.14)\end{aligned}$$

For the diagonal term in the last line above we note that

$$\int \frac{|\hat\varphi(k)|^2}{2\omega(k)}\delta^\perp_{\mu\nu}(k)\mathrm{d}^3k = \delta_{\mu\nu}\frac{2}{3}\int\frac{|\hat\varphi(k)|^2}{2\omega(k)}\mathrm{d}^3k$$

by the rotation invariance of $\hat\varphi$. Now as $n\to\infty$,

$$\sum_{j=1}^n |\delta B_j|^2 \to 3T, \qquad (2.15)$$

for almost every $w\in X$. Thus for almost every $w\in X$, we find

$$\lim_{n\to\infty}\sum_{j=1}^n\delta B_j\cdot\left(\int \mathrm{d}^3k\,\frac{|\hat\varphi(k)|^2}{2\omega(k)}\delta^\perp(k)\right)\delta B_j = T\int \mathrm{d}^3k\,\frac{|\hat\varphi(k)|^2}{\omega(k)}.$$

For the off-diagonal term, we start by noting that by the definition of the Itô integral for locally bounded functions $f,g:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$, we can see that

$$\mathbb{E}_{P_W^{0,0}}\left[\int_0^t ds\left|f(s,B_s)\int_0^s g(r,B_r)\mathrm{d}B_r\right|^2\right] < \infty.$$

Hence the stochastic integral of $\rho(s) = f(s, B_s) \int_0^s g(r, B_r) \mathrm{d}B_r$ exists and it holds that

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\Delta_j, B_{\Delta_j}) \delta B_j \cdot \delta^{\perp}(k) \int_0^{\Delta_j} g(r, B_r) \mathrm{d}B_r$$
$$= \int_0^T f(s, B_s) \mathrm{d}B_s \cdot \delta^{\perp}(k) \int_0^s g(r, B_r)\, \mathrm{d}B_r \tag{2.16}$$

strongly in $L^2(P_W^{0,0})$. By the independence of Brownian increments and the fact that $\mathbb{E}_{P_W^{0,0}}[\delta B_j]^2 = 1/n$, $\mathbb{E}_{P_W^{0,0}}[\delta B_j] = 0$, we can estimate the $L^2(X; P_W^{0,0})$-difference of (2.16) and the off-diagonal term:

$$\mathbb{E}_{P_W^{0,0}} \left[\sum_{j=1}^{n} f(\Delta_j, B_{\Delta_j}) \delta B_j \cdot \delta^{\perp}(k) \left(\int_0^{\Delta_j} g(r, B_r)\, \mathrm{d}B_r - \sum_{l=1}^{j} g(\Delta_l, B_{\Delta_l}) \delta B_l \right) \right]^2$$
$$= \sum_{j=1}^{n} \sum_{\mu,\nu=1}^{3} (\delta^{\perp}_{\mu\nu}(k))^2 \mathbb{E}_{P_W^{0,0}} \left[f(\Delta_j, B_{\Delta_j})^2 \right] \mathbb{E}_{P_W^{0,0}} \left[(\delta B_j^{\mu})^2 \right]$$
$$\times \mathbb{E}_{P_W^{0,0}} \left[\left(\int_0^{\Delta_j} g(r, B_r)\, \mathrm{d}B_r - \sum_{l=1}^{j} g(\Delta_l, B_{\Delta_l}) \delta B_l \right)^2 \right]$$
$$\leq 9 \left\| f^2 \right\|_\infty \frac{1}{n} \sum_{\nu=1}^{3} \sum_{j=1}^{n} \mathbb{E}_{P_W^{0,0}} \left[\int_0^{\Delta_j} g(r, B_r)\, \mathrm{d}B_r^{\nu} - \sum_{l=1}^{j} g(\Delta_l, B_{\Delta_l}) \delta B_l^{\nu} \right]^2 . \tag{2.17}$$

Then the right-hand side above converges to zero as $n \to \infty$ and (2.13) converges to

$$2 \int_0^t \mathrm{d}B_s \cdot \left(f(s, B_s) \delta^{\perp}(k) \int_0^s g(r, B_r)\, \mathrm{d}B_r \right)$$

strongly in $L^2(X; P_W^{0,0})$. By putting $f(t, x) = \mathrm{e}^{\mathrm{i}k\cdot x} \mathrm{e}^{-\omega(k)t}$ and $g(t, x) = \mathrm{e}^{-\mathrm{i}k\cdot x} \mathrm{e}^{\omega(k)t}$, the proof is finished. □

Remark 2.2. It is interesting that we know that $|e^{-S_T}| = |\mathbb{E}_{\mathcal{G}}[e^{iJ}]| \leq 1$ almost surely. This is not obvious from the iterated integral representation e^{-S_T}.

Let us summarize:

Proposition 2.1. *Let μ_T^V be the measure on X from Definition 2.1. Then*

$$\mathrm{d}\mu_T^V = \frac{1}{Z_T} \psi(B_{-T}) \psi(B_T) \mathrm{e}^{-\alpha^2 \hat{S}_T} \mathrm{e}^{-\int_{-T}^{T} V(B_s)\, \mathrm{d}s}\, \mathrm{d}\mathcal{W},$$

where $\hat{S}_T$ is defined by S_T with the diagonal part removed:

$$\hat{S}_T := \int \mathrm{d}^3k \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} \int_{-T}^{T} \mathrm{e}^{\mathrm{i}k\cdot B_s} \mathrm{d}B_s \cdot \int_{-T}^{s} e^{-\omega(k)(s-r)} \mathrm{e}^{-\mathrm{i}k\cdot B_r} \delta^{\perp}(k) \mathrm{d}B_r.$$

Or

$$\hat{S}_T := \int_{-T}^{T} Z(s,w) \cdot \mathrm{d}B_s,$$

where

$$Z(s,w) = \int_{-T}^{s} \mathrm{d}B_r \left(\int \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} \delta^{\perp}(k) e^{-(s-r)\omega(k)} e^{-ik\cdot(B_r - B_s)} \mathrm{d}^3k \right).$$

Remark 2.3. In Proposition 2.1, the diagonal term $t \int \mathrm{d}^3k \frac{|\hat{\varphi}(k)|^2}{2\omega(k)}$ is absorbed in the normalization constant, since it does not depend on the Brownian path B. Moreover from Remark 2.2 it follows that

$$|\exp\left(-\hat{S}_T\right)| \le \exp\left(T \int \mathrm{d}^3k \frac{|\hat{\varphi}(k)|^2}{2\omega(k)}\right).$$

3. The infinite volume limit

3.1. *Tightness and the Pauli-Fierz model*

The idea of the proof of the infinite volume limit we are about to give is not straightforward. We will show that it follows from showing that the bottom of the spectrum of a self-adjoint operator is eigenvalue. Actually, in the case of pair potential W under consideration, associated self-adjoint operator is realized as the Pauli-Fierz Hamiltonian H in the non-relativistic quantum electrodynamics. Fortunately it is established that H has the unique ground state for not only confining external potential V, e.g., $V(x) = |x|^2$, but also the Coulomb $V(x) = -1/|x|$, which is the most important case.

Let us begin with defining the Pauli-Fierz Hamiltonian with form factor $\hat{\varphi}$ as a self-adjoint operator on some Hilbert space $\mathcal{H}$ and we will review the functional integral representation of the C_0 semigroup e^{-tH}.

Let $\mathcal{F} := \bigoplus_{n=0}^{\infty} [\bigotimes_s^n L^2(\mathbb{R}^3 \times \{1,2\})]$ be the Boson Fock space. The state space of one electron minimally coupled with the photon (bose) field is given by

$$\mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathcal{F}.$$

We denote the formal kernels of the annihilation operator and the creation operator on $\mathcal{F}$ by $a(k,j)$ and $a^*(k,j)$, respectively, which satisfy the canonical commutation relations:

$$[a(k,j), a^*(k',j')] = \delta(k-k')\delta_{jj'}, \quad [a(k,j), a(k',j')] = 0 = [a^*(k,j), a^*(k',j')]. \tag{3.1}$$

The free Hamiltonian in $\mathcal{F}$ is defined by

$$H_{\mathrm{f}} := \sum_{j=1,2} \int \omega(k) a^*(k,j) a(k,j) \mathrm{d}^3k.$$

Here dispersion relation ω is given by (2.1). Let us fix a function $\hat{\varphi}$ satisfying Assumption (A) The quantized radiation field $A = (A_1, A_2, A_3)$ with form factor $\hat{\varphi}$ is defined by $A_\mu := \int_{\mathbb{R}^3}^{\oplus} A_\mu(x) \mathrm{d}^3 x$, where we used the isomorphism $\mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} L^2(\mathbb{R}^3)\, \mathrm{d}x$ and

$$A_\mu(x) := \frac{1}{\sqrt{2}} \sum_{j=1,2} \int e_\mu(k,j) \left(e^{-ikx} \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} a^*(k,j) + e^{ikx} \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} a(k,j) \right) \mathrm{d}^3 k.$$

The vectors $e(k,j)$, $j = 1,2$, are the polarization vectors. They satisfy $e(k,i) \cdot e(k,j) = \delta_{ij}$ and $k \cdot e(k,j) = 0$. Note that

$$\sum_{j=1,2} e_\mu(k,j) e_\nu(k,j) = \delta^\perp_{\mu\nu}(k). \tag{3.2}$$

(3.2) is of course independent of the choice of polarization vectors and $k \cdot e(k,j) = 0$ yields that

$$\sum_{\mu=1}^{3} \nabla_{x_\mu} A_\mu(x) = 0. \tag{3.3}$$

The Pauli-Fierz Hamiltonian $H(0)$ is defined by

$$H(0) := \frac{1}{2}(-i\nabla \otimes 1 - \alpha A)^2 + 1 \otimes H_{\mathrm{f}}, \tag{3.4}$$

where $\alpha \in \mathbb{R}$ denotes coupling constant. It is established in [17,19] that $H(0)$ is self-adjoint on $D(-\Delta) \cap D(H_{\mathrm{f}})$ and bounded from below. Moreover $H(0)$ is essentially self-adjoint on any core of $-(1/2)\Delta \otimes 1 + 1 \otimes H_{\mathrm{f}}$. We now introduce a class of external potentials $V : \mathbb{R}^3 \to \mathbb{R}$ that we can add to H_0.

Definition 3.1. $V \in K$ if and only if $V = V_+ - V_-$ such that $V_\pm \geq 0$, $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and V_- relatively form bounded with respect to $-(1/2)\Delta$ with bound strictly smaller than one.

Let $V \in K$. Then we define H as

$$H := H(0) \dot{+} V_+ \otimes 1 \dot{-} V_- \otimes 1, \tag{3.5}$$

where $\dot{\pm}$ denotes the quadratic form sum. To see the weak convergence of μ_T^V, we introduce the assumption below.

Assumption (GS): There exists a ground state φ_{g} of H.

Example 3.1. Let

$$V(x) = -\frac{C}{|x|} + U(x), \tag{3.6}$$

where $C \geq 0$ is a constant, and $U = U_+ - U_- \in L_{\mathrm{loc}}(\mathbb{R}^3)$ such that $U_\pm \geq 0$, $\inf_{x \in \mathbb{R}^3} U(x) > -\infty$, U_- is compactly supported, and $-(1/2)\Delta + U$ has a ground state $\phi > 0$ with ground state energy $-e_0 < 0$ such that $|\phi(x)| \leq \gamma e^{-|x|/\gamma}$ with some constant $\gamma > 0$. Then the ground state of H exists for arbitrary values of α. See [1] and [2,11]. Typical examples are

$$V_{\text{Coulomb}}(x) = -\frac{C}{|x|},$$
$$V_{\text{confining}}(x) = |x|^{2n}, \quad n = 1, 2, \ldots$$

To construct the functional integral representation of e^{-tH} we introduce some probabilistic notation which was already mentioned in Section 1. Let $\{\mathcal{A}^E(f)\}_{f \in \oplus^3 L^2(\mathbb{R}^{3+1})}$, denote the Gaussian random process labeled by $f \in \oplus^3 L^2(\mathbb{R}^{3+1})$ on some probability space (Q_E, Σ_E, μ_E) with mean zero and covariance given by

$$\mathbb{E}_{\mu_E}[\mathcal{A}^E(f)\mathcal{A}^E(g)] = q(f, g),$$

where $q(\cdot, \cdot)$ is defined in (1.7). We define the isometry $j_s : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^{3+1})$ by

$$\widehat{j_s f}(k, k_0) := (e^{-ik_0 t}/\sqrt{\pi})\sqrt{\omega(k)/(\omega(k)^2 + |k_0|^2)}\hat{f}(k)$$

which satisfies (1.9). The crucial identity linking the Pauli-Fierz model to Gibbs measures is

Proposition 3.1.
(1) For arbitrary $f \in L^2(\mathbb{R}^3)$ with $f \geq 0$ but $f \not\equiv 0$, it follows that

$$(\varphi_{\rm g}, f \otimes \Omega)_{\mathcal{H}} > 0 \tag{3.7}$$

(2) Let $f_1, \ldots, f_{n-1} \in L^\infty(\mathbb{R}^3)$. For $-T = t_0 \leq t_1 \leq \cdots \leq t_n = T$, the Euclidean n-point green function is expressed as

$$\frac{\left(\psi \otimes \Omega, e^{-(t_1 - t_0)H}(f_1 \otimes 1) \cdots (f_{n-1} \otimes 1)e^{-(t_n - t_{n-1})H}\psi \otimes \Omega\right)_{\mathcal{H}}}{(\psi \otimes \Omega, e^{-2TH}\psi \otimes \Omega)_{\mathcal{H}}} = \mathbb{E}_{\mu_T^V}\left[\prod_{j=1}^{n-1} f_j(B_{t_j})\right]$$

Proof. See [18] for (1). In [15,20] it is established that

$$\begin{aligned}&(\psi \otimes \Omega, e^{-(t_1 - t_0)H}(f_1 \otimes 1) \cdots (f_{n-1} \otimes 1)e^{-(t_n - t_{n-1})H}\psi \otimes 1)_{\mathcal{H}} \\ &= \mathbb{E}_{\mathcal{W}}\left[\psi(B_{-T})\psi(B_T)\left(\prod_{j=1}^{n-1} f_j(B_{t_j})\right) e^{-\int_{-T}^T V(B_s)ds}\mathbb{E}_{\mu_E}\left[e^{-\mathrm{i}\alpha\int_{-T}^T A_s^E \cdot \mathrm{d}B_s}\right]\right],\end{aligned}$$

where

$$A_{s,\mu}^E := \mathcal{A}^E\left(\oplus_{\nu=1}^3 \delta_{\nu\mu} j_s \lambda(\cdot - B_s)\right), \quad \mu = 1, 2, 3,$$

and $\lambda = (\hat{\varphi}/\sqrt{\omega})^\vee$. Since $Z_T = (\psi \otimes \Omega, e^{-2TH}\psi \otimes \Omega)_{\mathcal{H}}$ and

$$e^{-\alpha^2 S_T} = \mathbb{E}_{\mathcal{G}}[e^{\mathrm{i}\alpha J_T}] = \mathbb{E}_{\mu_T}\left[e^{-\mathrm{i}\alpha\int_{-T}^T A_s^E \cdot \mathrm{d}B_s}\right],$$

the lemma follows. $\square$

Remark 3.1. Formally, (2) of Proposition 3.1 can be deduced from using the Feynman-Kac-Itô formula [9,14,15,20,23,25] but note that integrand A_s^E depends on time s explicitly; although this formula would give the Stratonovitch integral $\int_S^T A_s^E \circ \mathrm{d}B_s = \int_S^T A_s^E \cdot \mathrm{d}B_s - \frac{1}{2}\int_S^T \nabla \cdot A_s^E \mathrm{d}s$ instead of the Itô integral $\int_S^T A_s^E \cdot \mathrm{d}B_s$ above, the Coulomb gauge (3.3) allows us to use the Itô integral instead, since $\nabla_x \cdot \mathcal{A}^E(\lambda(\cdot - x)) = 0$.

By (3.7), we know that the the ground state, φ_{g}, of H is unique if it exists and, in particular, $(\varphi_{\mathrm{g}}, f \otimes \Omega)_{\mathcal{H}} \neq 0$ holds, then we can define the sequence converging to the normalized ground state φ_{g} by

$$\varphi_{\mathrm{g}}^t := \|e^{-tH}(f \otimes \Omega)\|_{\mathcal{H}}^{-1} e^{-tH}(f \otimes \Omega).$$

Actually, by virtue of (3.7), we see that

$$\varphi_{\mathrm{g}} = s - \lim_{t \to \infty} \varphi_{\mathrm{g}}^t. \tag{3.8}$$

One immediate and useful corollary of (3.8) and Proposition 3.1 is as follows.

Corollary 3.1. *Let $\rho, \rho_1, \rho_2 \in L^\infty(\mathbb{R}^3)$. Then for $t > s$,*

$$\lim_{T \to \infty} \mathbb{E}_{\mu_T^V}[\rho(B_0)] = (\varphi_{\mathrm{g}}, (\rho \otimes 1)\varphi_{\mathrm{g}})_{\mathcal{H}},$$
$$\lim_{T \to \infty} \mathbb{E}_{\mu_T^V}[\rho_1(B_s)\rho_2(B_t)] = (\varphi_{\mathrm{g}}, (\rho_1 \otimes 1)e^{-(t-s)H}(\rho_2 \otimes 1)\varphi_{\mathrm{g}})_{\mathcal{H}} e^{(t-s)E(H)},$$

where $E(H) = \inf \sigma(H)$ denotes the ground stare energy of H.

In order to prove the main theorem, we show a more general formula than (2) of Proposition 3.1. Let

$$A(\hat{f}) = \frac{1}{\sqrt{2}} \sum_{\mu=1}^{3} \sum_{j=1,2} \int e_\mu(k,j) \left(\hat{f}_\mu(k) a^*(k,j) + \hat{f}_\mu(-k) a(k,j) \right) dk.$$

Define the isometry $\mathbf{J}_t : \mathcal{F} \to L^2(Q_E)$ by the second quantization of j_s, namely $\mathbf{J}_t : A(\hat{f}_1) \cdots A(\hat{f}_n) : \Omega = :\mathcal{A}^E(j_t f) \cdots \mathcal{A}^E(j_t f_n):$ and $\mathbf{J}_t \Omega = 1$, where $:\xi:$ denotes the Wick product of ξ.

Proposition 3.2. *Let $F, G \in \mathcal{H}$ and $f_1, ..., f_{n-1} \in L^\infty(\mathbb{R}^3)$. For $S = t_0 \leq t_1 \leq \cdots \leq t_n = T$,*

$$(F, e^{-(t_1 - t_0)H}(f_1 \otimes 1) \cdots (f_{n-1} \otimes 1) e^{-(t_n - t_{n-1})H} G)_{\mathcal{H}}$$
$$= \mathbb{E}_{\mathcal{W}}\left[\left(\prod_{j=1}^{n-1} f_j(B_{t_j}) \right) e^{-\int_S^T V(B_s)ds} \mathbb{E}_{\mu_E}\left[\overline{\mathbf{J}_S F(B_S)} e^{-\mathrm{i}\alpha \int_S^T A_s^E \cdot \mathrm{d}B_s} \mathbf{J}_T G(B_T) \right] \right].$$

Proof. See [15,20]. □

We are now ready to state and prove the main theorem of this paper.

Theorem 3.1. *Suppose that Assumption (GS) and (2.7). Then there exists a subsequence T' such that the weak limit of $\mu_{T'}^V$ as $T' \to \infty$ exists.*

Proof. By the Prohorov theorem, it is enough to show two facts:
(1) $\lim_{\Lambda\to\infty}\sup_T \mu_T^V(|B_0|^2>\Lambda)=0$,

(2) for arbitrary $\epsilon>0$, $\lim_{\delta\downarrow 0}\sup_T \mu_T^V\left(\max_{\substack{|t-s|<\delta\\ -T\le s,t\le T}}|B_t-B_s|>\epsilon\right)=0$.

Using Corollary 3.1 we have

$$\mu_T^V(|B_0|^2>\Lambda)=(\varphi_{\rm g}^T,(\chi_{\{|x|^2>\Lambda\}}\otimes 1)\varphi_{\rm g}^T)_{\mathcal H},$$

where χ_D denotes the characteristic function on D. Using the fact that $\varphi_{\rm g}^T\to\varphi_{\rm g}$ strongly as $T\to\infty$ and $\|\chi_{\{|x|^2>\Lambda\}}\varphi_{\rm g}\|_{\mathcal H}\to 0$ as $\Lambda\to\infty$, we get (1). For (2), assume that $|t-s|$ is sufficiently small. It is enough to show that

$$\mathbb{E}_{\mu_T^V}[|B_t-B_s|^{2n}]\le |t-s|^n D \tag{3.9}$$

with some constant D independent of T. To apply Propositions 3.1 and 3.2, we have to truncate the process B_t^μ as

$$(B_t^\mu)_a(w):=\begin{cases} -a,\ B_t^\mu(w)\le -a,\\ B_t^\mu(w),\ |B_t^\mu(w)|<a,\\ a,\ \ B_t^\mu(w)\ge a.\end{cases}$$

and define the truncated multiplication operator h_a^μ, $\mu=1,2,3$, by

$$h_a^\mu f(x)=\begin{cases} -af(x),\ x_\mu\le -a,\\ x_\mu f(x),\ |x_\mu|<a,\\ af(x),\ \ x_\mu\ge a.\end{cases}$$

Note that

$$|h_a(x)-h_a(y)|\le|x-y|\quad x,y\in\mathbb{R}^3, \tag{3.10}$$

for all $a\ge 0$. Since h_a^μ is bounded, we can see that by Proposition (3.1),

$$\begin{aligned}&\mathbb{E}_{\mu_T^V}\left[|(B_t)_a-(B_s)_a|^{2n}\right]\\ &=\sum_{\nu=1}^3\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}(-1)^k\mathbb{E}_{\mu_T^V}\left[(B_s^\nu)_a^k(B_t^\nu)_a^{2n-k}\right]\\ &=\sum_{\nu=1}^3\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}(-1)^k\left((h_a^\nu\otimes 1)^k e^{-sH}\varphi_{\rm g}^T, e^{-(t-s)H}(h_a^\nu\otimes 1)^{2n-k}e^{+tH}\varphi_{\rm g}^T\right)_{\mathcal H},\end{aligned}$$

where $e^{+tH}\varphi_{\rm g}^T$ is well defined for $t<T$. From Proposition (3.2) it follows that

$$\begin{aligned}&=\sum_{\nu=1}^3\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}(-1)^k\mathbb{E}_{\mathcal W}\Big[\left(h_a^\nu(B_0)\right)^k\left(h_a^\nu(B_{t-s})\right)^{2n-k}e^{-\int_0^{t-s}V(B_s){\rm d}s}\\ &\qquad\qquad\times\ \mathbb{E}_{\mu_E}\left[\overline{\mathbf{J}_0e^{-sH}\varphi_{\rm g}^T(B_0)}e^{-i\alpha\int_{-T}^T A_s^E\cdot{\rm d}B_s}\mathbf{J}_{t-s}e^{+tH}\varphi_{\rm g}^T(B_{t-s})\right]\Big]\\ &=\mathbb{E}_{\mathcal W}\Big[|h_a(B_0)-h_a(B_{t-s})|^{2n}e^{-\int_0^{t-s}V(B_s){\rm d}s}\\ &\qquad\qquad\times\ \mathbb{E}_{\mu_E}\left[\overline{\mathbf{J}_0e^{-sH}\varphi_{\rm g}^T(B_0)}e^{-i\alpha\int_{-T}^T A_s^E\cdot{\rm d}B_s}\mathbf{J}_{t-s}e^{+tH}\varphi_{\rm g}^T(B_{t-s})\right]\Big]\end{aligned}$$

By (3.10) and the Schwartz inequality we can estimate above as

$$\begin{aligned}
&\mathbb{E}_{\mu_T^V}\left[|(B_t)_a-(B_s)_a|^{2n}\right]\\
&\leq \mathbb{E}_{\mathcal{W}}\left[\,|B_0-B_{t-s}|^{4n}\|e^{+tH}\varphi_{\rm g}^T(B_{t-s})\|_{\mathcal{H}}^2\right]^{1/2}\\
&\qquad\qquad\times\mathbb{E}_{\mathcal{W}}\left[e^{-2\int_0^{t-s}V(B_s)ds}\|e^{-sH}\varphi_{\rm g}^T\|_{\mathcal{H}}^2\right]^{1/2}\\
&\leq C_V\|e^{+tH}\varphi_{\rm g}^T\|_{\mathcal{H}}\|e^{-sH}\varphi_{\rm g}^T\|_{\mathcal{H}}\mathbb{E}_{P_W^{0,0}}\left[\,|B_0-B_{t-s}|^{4n}\right]^{1/2},
\end{aligned}$$

where $C_V:=\sup_{x\in\mathbb{R}^3}\mathbb{E}_{P_W^x}[e^{-2\int_0^{t-s}V(B_r)dr}]<\infty$. Finally using the fact

$$\mathbb{E}_{P_W^{0,0}}[|B_s-B_t|^{4n}]=C_{4n}|t-s|^{2n}$$

with some constant C_{4n}, we have

$$\mathbb{E}_{\mu_T^V}\left[|(B_t)_a-(B_s)_a|^{2n}\right]\leq|t-s|^n\sqrt{C_{4n}}C_V\|e^{+tH}\varphi_{\rm g}^T\|_{\mathcal{H}}\|e^{-sH}\varphi_{\rm g}^T\|_{\mathcal{H}}.$$

Since $\varphi_{\rm g}^T\to\varphi_{\rm g}$ strongly as $T\to\infty$, we have

$$\|e^{-sH}\varphi_{\rm g}^T\|\to e^{-sE(H)}\|\varphi_{\rm g}\|,\quad \|e^{+tH}\varphi_{\rm g}^T\|\to e^{tE(H)}\|\varphi_{\rm g}\|$$

as $T\to\infty$. Then $D:=\sup_T\sqrt{C_{4n}}C_V\|e^{-sH}\varphi_{\rm g}^T\|_{\mathcal{H}}\|e^{+tH}\varphi_{\rm g}^T\|_{\mathcal{H}}<\infty$ follows. Then we conclude that

$$\mathbb{E}_{\mu_T^V}[|(B_t)_a-(B_s)_a|^{2n}]\leq D|t-s|^n$$

uniformly in a. Since the left-hand side above monotonously increasing as $a\uparrow\infty$, the monotone convergence theorem yields (3.9). Thus (2) follows. □

Definition 3.2. Let $V\in K$ and suppose Assumption (GS). Then the weak limit of the measure $\nu_{T'}^V$ on X is denoted by ν_∞^V.

Using the functional integration of e^{-tH}, it can be show the Carmona type estimate [8], namely $\varphi_{\rm g}$ is spatially localized as follows: if $V(x)=|x|^{2n}$, then $\|\varphi_{\rm g}(x)\|_{\mathcal{F}}\leq C_1e^{-C_2|x|^{n+1}}$, and if $V(x)=-1/|x|$, then $\|\varphi_{\rm g}(x)\|\leq C_3e^{-C_4|x|}$ for some constants C_j. We have a corollary.

Corollary 3.2. *Assume that* $\|\varphi_{\rm g}(x)\|_{\mathcal{F}}\leq Ce^{-c|x|^\gamma}$ *for some positive constants* C,c *and* γ. *Then*

$$\int_X e^{c|B_0|^\gamma}\nu_\infty^V(dw)<\infty. \tag{3.11}$$

Proof. Let $\rho_m(x)=\begin{cases}e^{c|x|^\gamma}, & e^{c|x|^\gamma}\leq m,\\ m, & e^{c|x|^\gamma}>m.\end{cases}$ Then $(\varphi_{\rm g},(\rho_m\otimes 1)\varphi_{\rm g})_{\mathcal{H}}=\int_X\rho_m(B_0)\mu_\infty^V$ follows. By the limiting arguments as $m\to\infty$, we have (3.11). □

3.2. *Concluding remarks*

In this paper we have given one example where we can both make sense of the double stochastic integral and obtain the infinite volume Gibbs measure by coupling Brownian motion to an auxiliary Gaussian measure. The drawback of this particular example is that the Gaussian space is infinite dimensional, and the associated Hamiltonian along with the existence of its ground state is non-trivial, and so we have to rely on a lot of technology. It is conceivable that the same method should work in a much easier case, namely when the auxiliary Gaussian process is just the stationary one-dimensional (or n-dimensional) Ornstein-Uhlenbeck process. However, when trying this approach one notices that on the way we used a lot of special features of the Pauli-Fierz model and its associated functional integral: for example the translation invariance of the coupling ensures that the term arising from the diagonal does not depend on B_t, which is a feature that cannot be reproduced in finite volume. So while we believe that a theory of double stochastic integrals originating from the variance of a Gaussian process could be developed, it is not altogether straightforward and we leave it as a future project.

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