NEAR-CRITICAL BEHAVIOR FOR ONE-PARAMETER FAMILIES OF CIRCLE MAPS

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Past studies of systems showing mixed-mode oscillations have revealed behavior along arbitrarily chosen parameter paths similar to that on the critical surface marking the break-up of invariant tori. Observations of this behavior in a model of the Belousov–Zhabotinskii reaction is presented. Using the theory of circle maps, it is shown that near-critical behavior can arise along one-parameter paths.

Phase locking has been observed in a variety of systems including convective fluids [1], chemical reactions [2], and numerous solid state devices [3,4]. Recent studies have centered on the quasi-periodic transition to chaos associated with the breakup of invariant tori and have served to confirm predictions made from the study of circle maps [5–7]. Of particular importance is the critical line (codimension-one surface) in parameter space on which the underlying torus loses smoothness. On the critical surface, phase locking occurs with full measure. Below this surface, phase-locked tongues are separated by quasiperiodicity; above the critical surface, phase-locked tongues overlap and chaos can occur. Only through the careful variation of two parameters is it generally possible to obtain a path confined to the critical surface. One-parameter paths generically intersect the critical surface transversely and are expected to show, in addition to phase locking, quasiperiodicity or chaos or both with non-zero measure.

Nevertheless, there have been reports of systems showing only periodic states along essentially arbitrary one-parameter paths [2,8–11]. The ordering of these states along the parameter paths is like that of phase-locked states on a torus, i.e. the ordering is naturally described by Farey series. It is this "critical behavior" along one-parameter paths which we addres in this Letter.

We first illustrate the behavior we wish to understand by showing results we have obtained from a model of the Belousov–Zhabotinskii (BZ) reaction (see below). The temporal evolution of the system consists of complicated sequences of large and small amplitude oscillations being associated with nearly-harmonic behavior and the large amplitude oscillations being associated with relaxational behavior. We label a periodic state with L large oscillations followed by S small oscillations by L^S. A more complicated pattern consisting of n transitions between large and small oscillations per period is labeled $L_1^{S_1} ... L_n^{S_n}$.

The ordering of states as a function of a control parameter is governed by the concatenation rule shown in fig. 1a: between two relatively simple parent states is found the daughter state whose pattern is the concatenation of those of the parent states. By assigning a rotation number R to each of the states, this ordering can be described by Farey series. We define R to be $L/(L+S)$, where L and S are, respectively, the total numbers of large and small oscillations per period. Then the concatenation rule implies that between two states with rotation numbers $R_1 = p_1/q_1$ and $R_2 = p_2/q_2$ lies the state with rotation number $R = (p_1+p_2)/(q_1+q_2)$, the Farey mediant of R_1 and R_2.

Over a range of control parameter, only periodic states are observed and these states always obey the concatenation rule. Thus, plotting R as a function of control parameter we obtain a devil's staircase (fig.

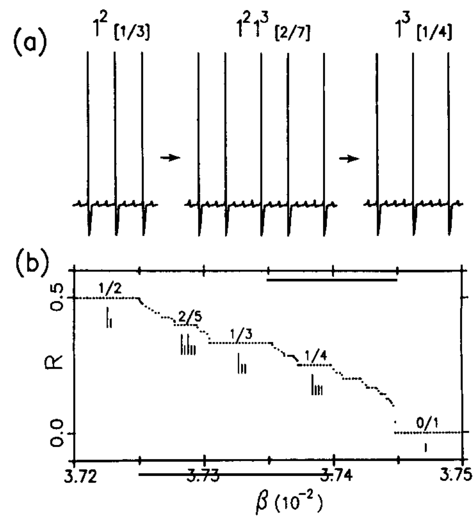

Fig. 1. Representative dynamics. (a) Plots of log U versus time for a model of the BZ reaction at three values of the control parameter β (see text). The variable U represents the concentration of a chemical species. At values of β between the 1^2 and 1^3 states is found the 1^21^3 state whose pattern is the concatenation of the 1^2 and 1^3 patterns. The rotation number R is shown in square brackets for the three states. (b) Devil's staircase of periodic states obtained from the model. Schematic representations of the periodic patterns and the rotation number are given explicitly for five of the steps.

1b). This staircase strongly resembles that expected at criticality (cf. results obtained for the sine circle map [7]) even though only one (arbitrarily chosen) parameter has been varied.

Behavior like that just described has been observed in a variety of systems [8–11], and yet there has been no general explanation of the phenomenon. (Maps of the interval which give rise to this behavior have been studied [12]; certain systems with discontinuities have also been treated [13]. We provide evidence that the mixed-mode oscillations shown in fig. 1 are phase-locked states on a torus, and using the theory of circle maps, we show how near-critical behavior can arise along one-parameter paths. It should be noted that periodic dynamics describable with Farey triangles has also been observed in experiments [8,14], but we shall not address this behavior here.

We have investigated a mathematical model of the BZ reaction given by the following four differential equations [15]:

$$\dot{A} = -AB^2 - \alpha A + g_1 V + \beta ,$$

$$\dot{B} = AB^2 - \gamma B + \delta ,$$

$$\dot{U} = k' U(1-U) - b'(V + g_2 B)\frac{U-a}{U+a} ,$$

$$\dot{V} = k(U - V - g_2 B) .$$

The variables A, B, U, and V represent the concentrations of chemical species in a well-stirred reactor; β is the primary bifurcation parameter. Unless stated otherwise, the remaining model parameters are assigned the following values: $\alpha=0.02$, $\gamma=0.12$, $\delta=2.0\times10^{-3}$, $a=0.01$, $b'=250$, $k'=100$, $k=0.1$, $g_1=0.1$, $g_2=-5.0\times10^{-3}$.

The staircase in fig. 1 has ben obtained by integrating the model equation at approximately 160 values of β. The integration has been performed with a one-step relative error tolerance of at most 10^{-10} (and in some cases as small as 10^{-13} to further verify the periodicity of solutions). States with as many as 95 oscillations per period have been observed. The staircase behavior continues for values of β less than those shown, but non-periodic states are also found (see below); for values of β above those shown, the small amplitude oscillations terminate in a Hopf bifurcation.

To verify that the behavior shown in fig. 1 did not arise by some fortuitous choice of parameter values, we have (1) decreased each of the parameters (except β) by 10% and again varied β, and (2) varied, one at a time, the parameters α, a, and g_2 with β fixed at 3.73×10^{-2}. In each case we have again found "critical behavior".

Shown in fig. 2 is a phase portrait which illustrates an important property of the model dynamics as-

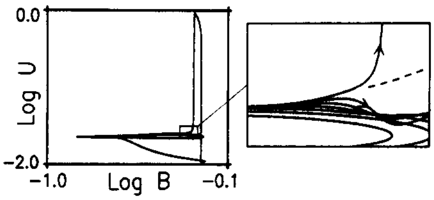

Fig. 2. Phase portrait of the 1^6 state illustrating the local separation of nearby phase-space points by the flow (arrows indicate the direction of increasing time). The dashed curve added in the enlargement represents a portion of the separatrix that trajectories must cross to undergo a relaxation oscillation.

sociated with the dichotomy in the amplitude of the oscillations (large or small amplitude only). There is a region of phase space in which the vector field, particularly the U-component, varies rapidly in magnitude and direction. This gives rise to a separatrix in the phase space which trajectories must cross in order to make a large amplitude relaxation oscillation. As will be important below, this means that there is a region of phase space in which the flow ϕ_t greatly separates, over short times, very nearby phase-space points.

We now provide evidence that the periodic states lie on an invariant torus. We have found it convenient to analyze the dynamics with the next maximum map for the variable B, i.e. the map F: $B_n \mapsto B_{n+1}$ where B_n denotes the nth relative maximum of B. This map is a "one-dimensional projection" of a Poincaré map [#1], and as such, it captures the features of the dynamics important to us here.

Fig. 3 illustrates the evolution of the dynamics as β is varied through the interval corresponding to the 1^2 state. The state is born via a tangent or saddle-node-of-periodic-orbits (SNP) bifurcation: the stable–unstable pair of periodic orbits initially separate, but eventually reunite in another SNP bifurcation at the other end of the 1^2 interval. This is exactly the behavior of phase-locked states along a path through an Arnol'd tongue [16]. In particular, the stable and unstable fixed points of the third maximum map (F^3) pair up differently at the two ends of the 1^2 interval. (We can assume continuity of the underlying Poincaré maps.) This supports the existence of an invariant torus formed by the closure of the unstable manifold of the unstable limit cycle. We have checked many of the periodic states and in each case we have found this same behavior.

Note that the near-infinite slope of the maps in fig. 3 is a reflection of the local separation of trajectories shown in fig. 2; very nearby points can be greatly separated under application of the maps. It is this which has prevented us from numerically visualizing the unstable manifold of the unstable period orbit directly. We have not been able to generate points on

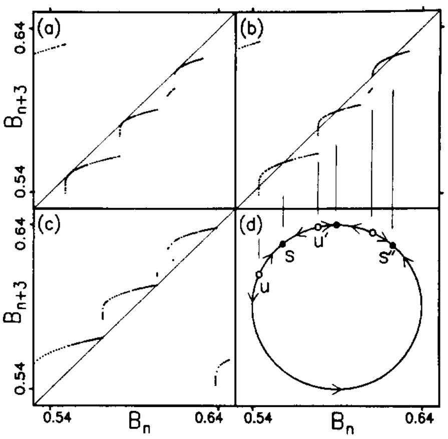

Fig. 3. Evolution of the 1^2 state. The third iterate of the next maximum map is shown at (a) $\beta=3.73528\times10^{-2}$: tangent bifurcation into the period-three state, (b) $\beta=3.7342\times10^{-2}$: stable–unstable pair of periodic orbits, and (c) $\beta=3.73042\times10^{-2}$: tangent bifurcation out of the 1^2 state (enlargement reveals a quadratic tangency). The maps were generated by perturbing the system away from the stable periodic orbit. (d) Sketch of invariant sets for the Poincaré map corresponding to (b). The vertical coordinate is unspecified. Solid (hollow) circles denote stable (unstable) periodic orbits. These orbits pair up u–s etc. in (a) and s–u' etc. in (c). The unstable manifold of the unstable orbit is assumed to be *qualitatively* as shown in (d).

that part of the unstable manifold corresponding to the long arc between u and s'' in fig. 3d. This results in gaps in the maps of fig. 3. While we assume that the actual next maximum maps are well behaved in the gaps, we cannot be certain of this.

We turn to the theory of circle maps to explain why the model dynamics resembles that found on the critical surface. Consider the space of continuous maps from the circle to itself, $C^0(T^1, T^1)$, and the subspace $\mathrm{Diff}(T^1)$ of (orientation preserving) diffeomorphisms of the circle. (See fig. 4.) The parts of the boundary of $\mathrm{Diff}(T^1)$ consisting of maps with a point of zero or infinite slope are labeled S and $\bar{S}$ respectively. In modeling the break-up of invariant tori with circle maps, S represents the critical surface on which the break-up occurs.

We now show that "critical behavior" exists along $\bar{S}$ as well as along S. First note that quasiperiodicity and phase locking are invariant under time reversal. Thus, given any family f_μ in $\mathrm{Diff}(T^1)\cup S\cup\bar{S}$, there is a corresponding family f_μ^{-1} which shows, at any given value of μ, the same dynamics (quasiperiodicity or phase locking) as f_μ. Because maps on $\bar{S}$ are

[#1] Embed the dynamics in $\mathbb{R}^n$ (n sufficiently large) by $B(t)\mapsto(B(t), \ldots, \mathrm{d}^nB(t)/\mathrm{d}t^n)$. Let P be the Poincaré map for the section $\mathrm{d}B/\mathrm{d}t=0$, then the graph of F is a projection of the graph of P. (F is not, in general, single valued and not, strictly speaking, a map.)

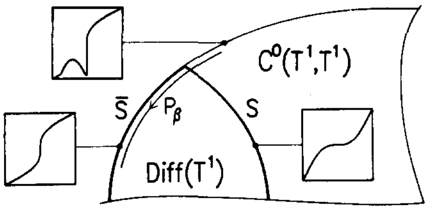

Fig. 4. Schematic diagram showing a portion of the space $C^0(T^1, T^1)$ and the subspace $\mathrm{Diff}(T^1)$. Representative maps are shown at three points in $C^0(T^1, T^1)$. The parts of the boundary of $\mathrm{Diff}(T^1)$ labeled S and $\bar{S}$ consist of all invertible maps in $C^0(T^1, T^1)$ with a point of zero or infinite slope, respectively. P_β represents the family of maps corresponding to the model path. P_β is shown crossing S to illustrate that, at values of β below those in fig. 1, the maps develop a quadratic maximum and more complex dynamics (including chaos) are found.

the inverses of maps on S, critical behavior must exist on both.

The model parameter path corresponds to a family near $\bar{S}$ (indicated by P_β in fig. 4) because along the model path the corresponding maps all have near-infinite slope. *Thus the model dynamics is similar to that along the critical surface, not because the model path lies on S, but instead because it lies extremely close to $\bar{S}$.*

Fig. 4 shows why it is not necessary to simultaneously vary two parameters in order to obtain a path very near $\bar{S}$, and thus why one-parameter families of circle maps can show near-critical behavior. Unlike S, $\bar{S}$ represents an asymptotic limit for differentiable maps of the circle. Thus in families of circle maps, large parameter changes can result in very small changes in the distance to $\bar{S}$.

In a forthcoming publication [17] we show that this asymptotic limit holds in higher dimensional *dissipative* systems. Our investigations of the BZ model (a four-dimensional system) bear this out: we have generated numerous maps but have never observed the nearly vertical slopes to pass through the infinite limit and change sign. Because of this, the BZ reaction exhibits dynamics, along arbitrarily chosen one-parameter paths, like that found on the critical surface. Our results on near-critical behavior are not limited to the BZ reaction, but are applicable to many physical systems having strong local seperation of orbits.

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