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Γ-Limits of Galerkin Discretizations with Quadrature

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Γ-convergence techniques are used to obtain convergence results for Galerkin discretizations of minimization problems for general non-convex functionals on Sobolev spaces. Quadrature approximations to the integrals are analyzed in the case when the stored energy function is convex or polyconvex.

Key words and phrases: Gamma convergence, finite element methods, energy minimization, quadrature, Galerkin discretizations, nonlinear elasticity

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1 Introduction

The problems considered in this paper are classical problems of the calculus of variations that typically arise in the continuum theory of elasticity. In static models, it is usually assumed that the configuration attained is a global minimum of an energy functional defined on a suitable Sobolev space. If the energy is to be physically interesting, then it has to be motion-invariant and therefore cannot be convex [10]. This poses significant difficulties for the analysis of numerical discretizations, as the differential operator of the Euler-Lagrange equation is not monotone and hence Galerkin orthogonality arguments cannot be applied in a straightforward manner. Some results are known in specific cases (see for example [9]), but general results are typically for problems that do not quite justify the use of the term *nonlinear elasticity* as their applicability is usually severely restricted, for example to one dimension in [15] or to small deformations in [7].

In this paper, an entirely different technique is used to obtain convergence of the discrete solutions. This technique, Γ-convergence, is a notion of convergence of nonlinear functionals which is designed to imply the convergence of minimizers. Under standard assumptions, one can obtain the strong convergence of the minimizers of Galerkin discretizations to a solution of the original problem (Section 2.2). If the Hessian in this point exists and is positive definite on a certain subspace of $W^{1,p}(\Omega, \mathbf{R}^m)$, then the classical Galerkin orthogonality argument can be employed again (Section 2.3).

More interesting is the application of this technique to quadrature approximations of the finite element functionals. This can be done without any *a priori* knowledge about the smoothness of the exact or the approximate solutions, which is usually not available for the problems considered here. Usually, quadrature is not expected to cause problems, but few theoretical results are actually known in this setting. Using Γ-convergence, it is possible to formulate sufficient conditions for quadrature rules, so that the discrete minimizers converge to a minimizer of the nonlinear functional (Section 3).

While Γ-convergence has been employed in much more difficult settings, such as the approximation of free-discontinuity problems (see for example [3]), to our knowledge it has never before been applied to the classical problems of the calculus of variations which are considered in this paper. While the results of Section 2.2 are known and only the technique is emphasized, the results about quadrature approximations seem to be novel in this generality.

In the remainder of the introduction, we state a simplified version of the main result of this paper. Let $\Omega \subset \mathbf{R}^n$ be a piecewise polygonal Lipschitz domain, $\Gamma_N \cup \Gamma_D = \partial\Omega$ with Γ_D being relatively open in $\partial\Omega$. For $u \in W^{1,p}(\Omega, \mathbf{R}^m)$, $1 < p < \infty$, define

$$E(u) = \int_\Omega \big(W(x, u, \nabla u) + f(x, u)\big)\,\mathrm{d}x + \int_{\Gamma_N} \phi(x, u)\,\mathrm{d}\mathcal{H}^{n-1},$$

where W is polyconvex (see Section 4 for a definition) in the third variable and W, f and ϕ satisfy the growth conditions (2.6), (2.7), (3.1) and (3.2). In this case it is well-known [11] that there exists a minimizer u of E in

$$\mathcal{A}_0 = \{v \in W^{1,p}(\Omega, \mathbf{R}^m) : v|_{\Gamma_D} = 0\}.$$

For every $h \in \mathbf{N}$, let $\mathcal{T}_h$ be a simplical mesh of Ω and let V_h be the space of continuous piecewise polynomial functions of degree d or less with respect to this mesh. Consider approximations of the integrals appearing in E by quadrature rules given by (not necessarily conforming) piecewise polynomial interpolation operators $J_{b,h}$, $J_{f,h}$ and $J_{s,h}$ (as defined in Section 3), i.e., for $u_h \in V_h$ we define

$$E_h(u_h) = \int_\Omega \big(J_{b,h} W(x, u_h, \nabla u_h) + J_{f,h} f(x,u)\big)\,\mathrm{d}x + \int_{\Gamma_N} J_{s,h}\phi(x, u_h(x))\,\mathrm{d}x.$$

By the degree of an interpolation operator J, we mean the largest integer k so that all polynomials q of degree k or less are interpolated exactly, i.e., $Jq = q$. The degree of the quadrature rule defined by the interpolation operator J is the largest integer k so that all polynomials q of degree k or less are integrated exactly on the domain of interest, i.e., $\int_K Jq\,\mathrm{d}x = \int_K q\,\mathrm{d}x$. For $u \in W^{1,p}(\Omega, \mathbf{R}^m) \setminus V_h$, we set $E_h(u) = +\infty$.

The following theorem follows immediately from Theorem 13 and from the remarks in Section 4.

Theorem 1 *(a) If W is convex in F and the degree of the quadrature rule defined by $J_{b,h}$ is $d-1$ or higher then $(E_h|_{\mathcal{A}_0})$ $\Gamma(\sigma_{1,p})$-converges to $E|_{\mathcal{A}_0}$. If W is polyconvex in F, $p > n$, and the degree of the quadrature rule defined by $J_{b,h}$ is $n(d-1)$ or higher then $(E_h|_{\mathcal{A}_0})$ $\Gamma(\sigma_{1,p})$-converges to $E|_{\mathcal{A}_0}$.*

(b) If the degree of the interpolation operator $J_{b,h}$ is $d-1$ or higher then the family $(E_h|_{\mathcal{A}_0})$ is equicoercive in the weak topology of $W^{1,p}$, i.e., any sequence $(u_h) \subset \mathcal{A}_0$ for which $(E_h(u_h))$ is bounded contains a weakly convergent subsequence in $W^{1,p}(\Omega, \mathbf{R}^m)$.

(c) If the conditions in (a) and (b) are satisfied then any sequence (u_h) of approximate minimizers of $(E_h|_{\mathcal{A}_0})$ contains a subsequence (u_{h_j}) converging weakly in $W^{1,p}$ to a minimizer u of $E|_{\mathcal{A}_0}$ and $E_h(u_h) \to E(u)$. If the minimum is unique then the whole sequence (u_h) converges weakly to u in $W^{1,p}$.

At the end of Section 3, it is mentioned that if in addition to the conditions of Theorem 1 the piecewise defined functions $\nabla^2 u_h$ are uniformly bounded in L^p (note that this does not imply that $u_h \in W^{2,p}$) then $E(u_h) \to E(u)$ in item (c) of the above theorem. In the case of strict convexity or strict polyconvexity this implies strong convergence of u_h to u in $W^{1,p}$.

To put this result into perspective, it should be mentioned that although this method makes it possible to prove convergence of finite element methods for non-convex problems under very general conditions, it does so under the assumption that the global minimizers of the discrete functionals can be computed. As the discrete functionals may be non-convex, this is in general a nontrivial problem which is not addressed in this paper.

1.1 Outline of the technique

For demonstration purposes, we consider an abstract and much simplified minimization problem that highlights the general technique presented in this paper. To this end let X be a Banach space and let $F : X \to [0, \infty]$ be a weakly lower semicontinuous, strongly continuous, and coercive functional. A classical example is presented in Section 2.1. Throughout the paper, F will be said to be coercive if whenever $(u_h)_{h \in \mathbf{N}}$ is a sequence in X such that $(F(u_h))$ is bounded, then (u_h) is weakly precompact in X.

To minimize F *numerically*, we pick a sequence of finite-dimensional subspaces $(X_h)_{h \in \mathbf{N}}$, such that for every $u \in X$ there exist $u_h^* \in X_h$ satisfying $\|u_h^* - u\|_X \to 0$ as $h \to \infty$. Since F is continuous, we have

$$\lim_{h \to \infty} F(u_h) = F(u).$$

For every $h \in \mathbf{N}$, let u_h be a minimizer of F in X_h. If $F(v) < \infty$ for at least on $v \in X$ then it follows from the last paragraph that the sequence $(\min_{X_h} F)_h$ is bounded and, by the coercivity assumption, (u_h) is weakly precompact in X. We can extract a subsequence (u_{h_j}) converging weakly to some $u \in X$. We claim that $F(u) = \inf_X F$. Let v be an arbitrary member of X. From the last paragraph, for each $h \in \mathbf{N}$ we can obtain $v_h^* \in X_h$ such that (v_h^*) converges strongly to v in X and hence

$$F(v) = \lim_{j \to \infty} F(v_{h_j}^*) \geq \liminf_{j \to \infty} F(u_{h_j}) \geq F(u),$$

where we also used the weak lower semicontinuity of F. Furthermore, by setting $v = u$, we obtain $F(u_h) \to F(u)$ as $h \to \infty$. Hence we have found a subsequence of *discrete minimizers*, converging weakly to a solution of the minimization problem. We will see later that under stronger conditions u_h will converge even strongly to u in X as $h \to \infty$.

The technique just described is a special case of Γ-convergence. A fairly general definition is given in Section 1.3.

1.2 Notation

We denote the $(n-1)$-dimensional Hausdorff (surface) measure by $\mathcal{H}^{n-1}$. The Lebesgue measure of a subset A of $\mathbf{R}^n$ is denoted by $|A|$.

Let Ω be a Lipschitz domain in $\mathbf{R}^n$. We denote the Sobolev and Lebesgue spaces of vector-valued functions by $W^{k,p}(\Omega, \mathbf{R}^m)$ and $L^p(\Omega, \mathbf{R}^m)$ respectively. If $m = 1$, we only write $W^{k,p}(\Omega)$ or $L^p(\Omega)$. By ∇u we denote the *Jacobi matrix* of u, if it exists. If ∇u is piecewise differentiable, then $\nabla^2 u$ denotes this piecewise derivative. Note that $\nabla^2 u$ may exist, for example for a finite element function, even if u is not twice (weakly) differentiable.

The norms on the Sobolev and Lebesgue spaces are denoted by $\|\cdot\|_{k,p,\Omega}$ and $\|\cdot\|_{p,\Omega}$ respectively. If it is clear which domain is used, then it is omitted from the notation.

We denote convergence in $W^{k,p}(\Omega, \mathbf{R}^m)$ as follows. (Note that throughout the paper $h \in \mathbf{N}$ denotes a discrete parameter.) If (u_h) converges strongly to u in $W^{k,p}$ or L^p we write

$$u_h \xrightarrow{k,p} u \quad \text{or} \quad u_h \xrightarrow{p} u$$

respectively. If (u_h) converges weakly to u in $W^{k,p}$ or L^p we write

$$u_h \xrightharpoonup{k,p} u \quad \text{or} \quad u_h \xrightharpoonup{p} u.$$

If F is a functional defined on a metric space X and if $\mathcal{A}$ is a subset of X, we define the restriction of F to $\mathcal{A}$ by

$$F|_{\mathcal{A}}(u) = \begin{cases} F(u) & , \text{ if } u \in \mathcal{A}, \text{ and} \\ +\infty & , \text{ if } u \in X \setminus \mathcal{A}. \end{cases}$$

The restriction operator is a useful tool to embed the domain of definition of a functional into a larger space. For example, Dirichlet boundary conditions can be easily taken into account this way.

If X is a Banach space and F is Fréchet differentiable at a point u, we denote the Fréchet derivative at this point by $F'(u)$ and the directional derivative in direction φ by $F'(u;\varphi)$. For second-order derivatives we use the notation $F''(u;\varphi,\psi)$, and so on.

In order to simplify the notation, constants in this paper are generally treated as universal. Normally, we use the symbol c_1 for estimates from above and c_0 for estimates from below; these two constants always satisfy $0 < c_0, c_1 < \infty$.

1.3 Γ-Convergence

The convergence proof presented in Section 1.1 is a special case of the so-called Γ-convergence of $(F|_{X_h})$ to F. Γ-convergence was introduced in 1975 by DeGiorgi and Franzoni [13] to describe the asymptotic behaviour of minimizers of general nonlinear functionals on topological spaces and has since been a versatile tool for describing asymptotic limits and approximations [12].

Let us consider the setting where (X, d) is a metric space on which the functionals F_h and F are defined. Reviewing the arguments in Section 1.1, we identify three main ingredients to describe the convergence of minimizers of F_h to minimizers of F as $h \to \infty$: first, the weak compactness of minimizing sequences; second, the *liminf* condition,

$$\lim_{h\to\infty} d(u_h, u) = 0 \;\Rightarrow\; F(u) \leq \liminf_{h\to\infty} F_h(u_h); \tag{1.1}$$

and third, the *limsup* condition,

$$\forall\, u \in X \;\exists\, (u_h) \subset X \text{ s.t. } \lim_{h\to\infty} d(u_h, u) = 0 \text{ and } \limsup_{h\to\infty} F_h(u_h) \leq F(u). \tag{1.2}$$

We say that the sequence (F_h) $\Gamma(d)$-converges to F if conditions (1.1) and (1.2) hold. If it is clear which topology is meant, we do not state the dependence explicitly. Using the same arguments as in Section 1.1, we obtain the following result.

Theorem 2 *Let (u_h) be an approximate minimizing sequence for (F_h), i.e.*

$$\lim_{h\to\infty} |F_h(u_h) - \inf_X F_h| = 0,$$

and assume that (F_h) $\Gamma(d)$-converges to the functional F. Then the limit point u of any convergent subsequence of (u_h) is a global minimum of F in X and $F_h(u_h) \to F(u)$ as $h \to \infty$.

The proof of Theorem 2 is a repetition of the argument given in Section 1.1. It can also be found in any text book on Γ-convergence (see for example [12]).

The following proposition is trivial but highly useful for technical purposes. It allows us to consider lower order (weakly continuous) terms separately, using simpler methods.

Proposition 3 *If (F_h) $\Gamma(d)$-converges to F and if (G_h) and G satisfy*

$$\lim_{h\to\infty} d(u_h, u) = 0 \;\Rightarrow\; G_h(u_h) \to G(u) \quad \text{as } h \to \infty,$$

then $(F_h + G_h)$ $\Gamma(d)$-converges to $F + G$. In particular, if G is continuous then $(F_h + G)$ $\Gamma(d)$-converges to $F + G$.

For further information on Γ-convergence, see [12].

Remark. With slight abuse of notation, we denote the Γ-convergence in the weak $W^{1,p}$ topology by $\Gamma(\sigma_{1,p})$ although we actually mean Γ-convergence in the metric describing weak convergence of sequences. Similarly, when talking about weakly lower semicontinuous functionals, we always mean sequential weak lower semicontinuity.

2 Conforming Galerkin discretizations for energy minimization problems

2.1 Problem formulation

Fix $p \in (1,\infty)$ and let Ω be a bounded, polygonal Lipschitz domain in $\mathbf{R}^n$. Let $\Gamma_D \subset \partial\Omega$ be relatively open and set $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Since $\mathcal{H}^{n-1}(\Gamma_D) > 0$, the Friedrichs inequality

$$\|u\|_{1,p} \le C_F \|\nabla u\|_p \tag{2.1}$$

holds for all $u \in \mathcal{A}_0 = \{u \in W^{1,p}(\Omega, \mathbf{R}^m) : u|_{\Gamma_D} = 0\}$. Without this condition it would either be necessary that the functional E, defined below, satisfies more stringent growth conditions, or the space of admissible functions has to be otherwise restricted to guarantee coercivity of E.

We consider functionals of the form $E(u) = E_b(u) + E_s(u) + E_f(u)$, where E_b is the *bulk energy*,

$$E_b(u) = \int_\Omega W(x, u, \nabla u)\, \mathrm{d}x\,, \tag{2.2}$$

and E_s is the *surface energy*,

$$E_s(u) = \int_{\Gamma_N} \phi(x, u)\, \mathrm{d}\mathcal{H}^{n-1}, \tag{2.3}$$

where $W : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m\times n} \to \mathbf{R}$ is measurable in x and continuous in u and F, and $\phi : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is $\mathcal{H}^{n-1}$-measurable in x and continuous in u. E_f is the potential of the body force,

$$E_f(u) = \int_\Omega f(x, u)\, \mathrm{d}x, \tag{2.4}$$

where f is measurable in x and continuous in u. This term seems to be included in (2.2), but it is intended to demonstrate how lower order terms can be dealt with more efficiently, for example if W were independent of u.

Given $g \in W^{1,p}(\Omega, \mathbf{R}^m)$, we define the set of trial functions (or admissible functions) to be

$$\mathcal{A} = g + \mathcal{A}_0 = \{u \in W^{1,p}(\Omega, \mathbf{R}^m) : u|_{\Gamma_D} = g|_{\Gamma_D}\}.$$

The topic of this paper is the discretization of the minimization problem

$$\min_{\mathcal{A}} E. \tag{2.5}$$

In order for (2.5) to be well-defined we need to impose several conditions on W, ϕ and f. In principle, we require weak lower semicontinuity, strong continuity, and coercivity of E, for which the following set of conditions is a typical example.

We assume that W is quasiconvex in F, i.e., that for all $\varphi \in C_c^\infty(\Omega, \mathbf{R}^m), y \in \mathbf{R}^n, u \in \mathbf{R}^m$, and $F \in \mathbf{R}^{m\times n}$,

$$W(y,u,F) \leq \fint_\Omega W(y,u,F+\nabla\varphi(x))\,\mathrm{d}x$$

holds, and that there exist constants $c_0, c_1 > 0$ such that

$$c_0(|F|^p - 1) \leq W(x,u,F) \leq c_1(1+|u|^p+|F|^p). \tag{2.6}$$

We assume, furthermore, that there exists r, $1 \leq r < p$ so that

$$\begin{aligned} |f(x,u)| &\leq c_1(|u|^r+1), \text{ and} \\ |\phi(x,u)| &\leq c_1(|u|^r+1). \end{aligned} \tag{2.7}$$

In this case $W^{1,p}(\Omega, \mathbf{R}^m)$ is compactly embedded in $L^r(\Omega, \mathbf{R}^m)$ and $L^r(\partial\Omega, \mathbf{R}^m)$ ([1]) so that E_f and E_s are continuous in the weak topology of $W^{1,p}$ and do not affect the coercivity of E.

It is well known ([11]) that under conditions (2.6) and (2.7), (2.5) admits a (not necessarily unique) solution in $W^{1,p}(\Omega, \mathbf{R}^m)$.

Theorem 4 *If condition* (2.6) *holds, then E_b is weakly lower semicontinuous, strongly continuous and coercive in $W^{1,p}$. If conditions* (2.7) *are satisfied as well then the same is true for E and hence E admits at least one minimizer in the set $\mathcal{A}$.*

2.2 Discretization

In this section, we prove a variant of the known result that the minimizers of Galerkin discretizations converge to a solution of the full problem (2.5). We show that any Galerkin discretization for which the energies can be computed exactly, for example using piecewise affine finite element spaces, converges. In Section 3 we extend this analysis to quadrature approximations of the energies.

Let (V_h) be a family of finite-dimensional subspaces of $W^{1,p}(\Omega, \mathbf{R}^m)$ which satisfy the condition

$$\forall\, u \in \mathcal{A}_0\ \exists\, u_h \in V_h \cap \mathcal{A}_0 : \quad u_h \xrightarrow{1,p} u. \tag{2.8}$$

This guarantees that elements of $\mathcal{A}$ can be approximated by discrete trial functions, i.e., by elements of $\mathcal{A}_h$, where $\mathcal{A}_h$ is defined as

$$\mathcal{A}_h = g_h + V_h \cap \mathcal{A}_0 = \{u_h \in V_h : u_h|_{\Gamma_D} = g_h|_{\Gamma_D}\},$$

where (g_h) is an arbitrary sequence of elements of V_h, converging to g in the strong topology of $W^{1,p}(\Omega, \mathbf{R}^m)$. This result is summarized in the following lemma.

Lemma 5 *(a) If $u \in \mathcal{A}$ then there exists a sequence (u_h) with $u_h \in \mathcal{A}_h$ converging to u in the strong topology of $W^{1,p}(\Omega, \mathbf{R}^m)$.*

(b) If $u \notin \mathcal{A}$ and $u_h \xrightarrow{1,p} u$ then $u_h \notin \mathcal{A}_h$ except for finitely many h.

Proof (a) If $u \in \mathcal{A}$ then $u - g$ vanishes on Γ_D and there exist $v_h \in V_h \cap \mathcal{A}_0$, converging strongly to $u - g$. Hence $u_h = v_h + g_h$ lies in $\mathcal{A}_h$ and converges strongly to u in $W^{1,p}(\Omega, \mathbf{R}^m)$.

(b) If $u_h \in \mathcal{A}_h$ and $v_h = u_h - g_h + g \in \mathcal{A}$ then

$$\|u_h - v_h\|_{1,p} = \|g - g_h\|_{1,p} \to 0. \tag{2.9}$$

Since $\mathcal{A}$ is convex and closed, it is also weakly closed and hence any accumulation point of (v_h), and by (2.9) of (u_h), must lie in $\mathcal{A}$. ■

We are now in the position to prove the first convergence result.

Theorem 6 *(a) $(E|_{\mathcal{A}_h})$ $\Gamma(\sigma^{1,p})$-converges to $E|_{\mathcal{A}}$ and is equicoercive in the same topology. In particular, if (u_h) is a sequence of approximate minimizers of $(E|_{\mathcal{A}_h})$ then there exists a subsequence (u_{h_j}) converging weakly in $W^{1,p}$ to a global minimizer u of $E|_{\mathcal{A}}$ and $E(u_h) \to E(u)$.*

(b) If W is strictly quasiconvex, in the sense that there exists an $\epsilon > 0$ such that for all $\varphi \in C_c^\infty(\Omega, \mathbf{R}^m)$

$$W(y, u, F) + \epsilon \|\nabla\varphi\|_p^p \leq \fint_\Omega W(y, u, F + \nabla\varphi(x))\,\mathrm{d}x$$

holds, then $u_{h_j} \xrightarrow{1,p} u$.

(c) If E has a unique minimizer in $\mathcal{A}$ then in both cases above the complete sequence converges.

Proof 1. We begin by proving the limsup condition (1.2) of Γ-convergence. Fix $u \in W^{1,p}(\Omega, \mathbf{R}^m)$. If $u \notin \mathcal{A}$ then, according to Lemma 5, $u \notin \mathcal{A}_h$ for almost all h. Hence, $E|_{\mathcal{A}}(u) = E|_{\mathcal{A}_h}(u) = +\infty$ for almost all h and (u_h) is a recovery sequence for u. If $u \in \mathcal{A}$, then Lemma 5 provides us with a sequence (u_h), $u_h \in \mathcal{A}_h$, converging strongly to u. The strong continuity of E (compare Theorem 4) implies $E(u_h) \to E(u)$. Hence (u_h) is a recovery sequence for u.

2. To show the liminf condition (1.1), assume that $u_h \xrightarrow{1,p} u$. If $u \notin \mathcal{A}$ then, according to Lemma 5, $u_h \notin \mathcal{A}_h$ for almost all h and hence $E|_{\mathcal{A}}(u) = \liminf_h E|_{\mathcal{A}_h}(u_h) = +\infty$. If $u \in \mathcal{A}$, we can assume without loss of generality that $u_h \in \mathcal{A}_h$ (otherwise we extract a subsequence and if no such subsequence exists the inequality is trivial). Lower semicontinuity of E implies

$$E|_{\mathcal{A}}(u) = E(u) \leq \liminf_h E(u_h) = \liminf_h E|_{\mathcal{A}_h}(u_h).$$

3. Let (u_h) be a sequence of approximate minimizers of $(E|_{\mathcal{A}_h})$. Since E is coercive, (u_h) is precompact and according to Theorem 2 contains a subsequence (u_{h_j}) converging to a global minimum u of $E|_{\mathcal{A}}$ and $E(u_h) \to E(u)$. This concludes the proof of (a).

4. The lower-order terms E_s and E_f are continuous in $\sigma_{1,p}$. Hence $E_s(u_h) \to E_s(u)$ and $E_f(u_h) \to E_f(u)$ which implies also $E_b(u_h) \to E_b(u)$. It is a well-known fact [16] that weak convergence together with convergence of the energy (E_b in this case) implies strong convergence when W is strictly quasiconvex. This proves (b).

5. The argument that shows convergence of the whole sequence for part (c) is a standard uniqueness argument. It is easily seen that in (a) we actually have the stronger result that every subsequence of (u_h) has a further subsequence which converges to a global minimum of E. If that minimum is unique (denoted by u) then every subsequence of (u_h) has a further subsequence converging weakly to u. Now assume that u_h does not converge weakly to u. Then there must exist a subsequence (u_{h_j}) and an element u^* of the dual space of $W^{1,p}$, such that $|\langle u^*, u_{h_j} - u\rangle| \geq \delta > 0$ for all h. But since a subsequence of (u_{h_j}) converges weakly to u, this is a contradiction. ■

Remarks. 1. One can easily consider more general boundary conditions than the Dirichlet boundary conditions in this section. It is sufficient that $\mathcal{A}$ is closed and convex and that $\mathcal{A}_h$ is chosen in such a way that Lemma 5 holds. For example, for the obstacle problem ($m = 1$), an additional condition would be that $u \geq \psi$ in Ω. If we pick an arbitrary $\psi_h \xrightarrow{1,p} \psi$ to restrict the discrete solutions then it is easy to show that Lemma 5 holds again.

2. In the theory of elasticity one is often interested in the convergence of the stress rather than the convergence of the deformation or the energy. In the context of Γ-convergence this can be obtained as a corollary to the strong convergence of minimizers, using growth conditions on the derivatives of W.

2.3 Best approximation error estimates

The strongest result obtainable with Γ-convergence alone is weak convergence of a subsequence of the discrete minimizers (compare Theorem 6 (a)). However, in Theorem 6 (b) and (c) we have seen techniques to strengthen this weak result, using additional information about the problem. In the present section, we give a heuristic argument, based on assumptions which should by valid in many applications but are difficult to justify in theory, suggesting that we can in fact expect optimal convergence rates. To our knowledge, the only results that put the assumptions used in this section on a firm basis are due to Zhang [18]. They are, however, restricted in that only small loads and Dirichlet boundary conditions close to the identity are considered. An analysis of Galerkin discretizations based on these results is given in [7].

Let us assume that we are in the situation where a sequence (u_h) of minimizers of (E_h) converges strongly to a minimizer u of E. If this minimum is essentially a strict local minimum we have a stronger result than mere convergence. We assume that $p = 2$, where this analysis seems to be most natural. Nevertheless, the argument is, in principle, not restricted to this value of p.

Suppose that E is uniformly elliptic in some neighbourhood $B(u, r)$ of u in the sense that there exists a constant $c_0 > 0$ such that for every $\vartheta \in B(u, r)$,

$$\forall u, v \in \mathcal{A} : E''(\vartheta; u - v, u - v) \geq c_0 \|u - v\|_{1,2}^2 \quad .$$

Usually, this property would be of little use as we do not know a priori that our discrete minimizers will lie in $B(u, r)$. Only, by the results of the previous sections we can assume $\mathrm{conv}\{u, u_h\} \subset B(u, r)$ for sufficiently large h.

Now we can use the Galerkin orthogonality argument to obtain a best approximation result. Let $v_h \in \mathcal{A}_h$ and set

$$\tilde{u}_h = u_h - g_h + g \quad \text{and} \quad \tilde{v}_h = v_h - g_h + g.$$

so that $\tilde{u}_h$ and $\tilde{v}_h$ lie in $\mathcal{A}$. Then, for some $\vartheta_1 \in \mathrm{conv}\{u, u_h\}$, we have

$$\begin{aligned} E'(u; u - u_h) - E'(u_h; u - u_h) &= \\ &= E'(u; u - \tilde{u}_h) - E'(u_h; u - u_h) + E'(u; g - g_h) \\ &= E'(u; u - \tilde{v}_h) - E'(u_h; u - v_h) + E'(u; g - g_h) \\ &= E'(u; u - v_h) - E'(u_h; u - v_h) \\ &= E''(\vartheta_1; u - u_h, u - v_h) \\ &\leq c_1 \|u - u_h\|_{1,2} \|u - v_h\|_{1,2}, \end{aligned} \tag{2.10}$$

where c_1 is the continuity constant of E'' in $B(u, r)$. On the other hand, we have, for some $\vartheta_2, \vartheta_3 \in \mathrm{conv}\{u, u_h\}$, due to local uniform ellipticity, that

$$\begin{aligned} E'(u; u - u_h) - E'(u_h; u - u_h) &= \\ &= E'(u; u - \tilde{u}_h) - E'(u_h; u - \tilde{u}_h) \\ &\qquad + E'(u; g - g_h) - E'(u_h; g - g_h) \\ &= E''(\vartheta_2; u - u_h, u - \tilde{u}_h) + E''(\vartheta_3; u - u_h, g - g_h) \\ &= E''(\vartheta_2; u - \tilde{u}_h, u - \tilde{u}_h) + E''(\vartheta_2; g - g_h, u - u_h) \\ &\qquad + E''(\vartheta_3; u - u_h, g - g_h) \\ &\geq c_0 \|u - \tilde{u}_h\|_{1,2}^2 - 2c_1 \|u - u_h\|_{1,2} \, \|g - g_h\|_{1,2} \\ &\geq c_0 \|u - u_h\|_{1,2}^2 - 4c_1 \|u - u_h\|_{1,2} \, \|g - g_h\|_{1,2}. \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) we have the classical best approximation result with a modification for the approximation of the boundary condition,

$$\|u - u_h\|_{1,2} \leq \frac{c_1}{c_0} \left(\min_{v_h \in \mathcal{A}_h} \|u - v_h\|_{1,2} + 4\|g - g_h\|_{1,2} \right) \quad \text{for all } v_h \in \mathcal{A}_h,$$

for all sufficiently large h and with constants depending only on u.

Remark. Unfortunately, for one of the most important classes of stored energy functions, namely those where W is the quasiconvex envelope of a multiwell potential, no strict convexity property can possibly hold and hence one cannot even expect to obtain strong convergence using the methods described in this paper. It is an open problem whether discrete minimizers of such functionals converge strongly in general.

3 Quadrature approximations

In this section, quadrature approximations of the integrals are analyzed using Γ-convergence. The analysis in this section is restricted to the case where $W(x, u, F)$ is convex in F which is the natural assumption in the arguments used. Generalizations to polyconvex integrands are possible and are discussed in Section 4.

We assume furthermore that, for every $h \in \mathbf{N}$, $\mathcal{T}_h$ contains only simplices, that V_h is a finite element space of piecewise polynomials whose degree does not exceed a fixed positive integer d, and that the same quadrature rules are used in every element. Finally, we require that all elements have a chunkiness parameter $\gamma(K) = \operatorname{diam}(K)^n/|K|$ which is uniformly bounded above and below. This condition needs to be imposed as we frequently use interpolation error estimates when estimating the error committed in numerical quadrature.

On the reference simplices $\hat{K}_n$ and $\hat{K}_{n-1}$ in respectively n and $(n-1)$ dimensions, we define quadrature rules by specifying the interpolation nodes

$$\begin{aligned} &\hat{x}_q^{(b)} && (q = 0, \ldots, Q^{(b)}), \\ &\hat{x}_q^{(f)} && (q = 0, \ldots, Q^{(f)}), \text{ and} \\ &\hat{x}_q^{(s)} && (q = 0, \ldots, Q^{(s)}). \end{aligned}$$

The weights in the reference simplices, $|\hat{K}_n|\omega_q^{(b)}, |\hat{K}_n|\omega_q^{(f)}$, and $\mathcal{H}^{n-1}(\hat{K}_{n-1})\omega_q^{(s)}$, are obtained by integrating their nodal basis functions. The affine transformations of the reference nodes $\hat{x}_q^{(b)}, \hat{x}_q^{(f)}, \hat{x}_q^{(s)}$ to an element K (which is a surface in the case of $\hat{x}_q^{(s)}$) are denoted by $x_{q,K}^{(b)}, x_{q,K}^{(f)}$ and $x_{q,K}^{(s)}$. These nodes define, in an obvious way, piecewise polynomial (not necessarily conforming) interpolation operators $J_{b,h}$, $J_{f,h}$, and $J_{s,h}$. The quadrature weights in the element K are $|K|\omega_q^{(b)}$, $|K|\omega_q^{(f)}$, and $\mathcal{H}^{n-1}(K)\omega_q^{(s)}$ respectively. As we are primarily concerned with the quadrature rule for approximating E_b, to avoid unnecessary indices, we set $\hat{x}_q = \hat{x}_q^{(b)}, x_{q,K} = x_{q,K}^{(b)}, \omega_q = \omega_q^{(b)}$ and $Q = Q^{(b)}$.

We approximate the energy E_b by an interpolatory quadrature approximation

$E_{b,h}$ which is defined as

$$\begin{aligned} E_{b,h}(u) &= \int_\Omega J_{b,h} W(x,u,\nabla u)\,\mathrm{d}x \\ &= \sum_{K\in\mathcal{T}_h} |K| \sum_{q=0}^{Q} \omega_q W(x_{q,K}, u(x_{q,K}), \nabla u(x_{q,K})), \end{aligned}$$

if $u \in V_h$ and $E_{b,h}(u) = +\infty$ otherwise. $E_{f,h}$ and $E_{s,h}$ are defined accordingly and we set $E_h = E_{b,h} + E_{f,h} + E_{s,h}$.

In order to estimate the quadrature error we will require the growth conditions

$$\begin{aligned} \left|\frac{\partial W}{\partial x}\right| &\leq c_1(1+|u|^p+|F|^p) \\ \left|\frac{\partial W}{\partial u}\right| &\leq c_1(1+|u|^{p-1}+|F|^{p-1}), \end{aligned} \tag{3.1}$$

$$\begin{aligned} \left|\frac{\partial f}{\partial u}\right| + \left|\frac{\partial \phi}{\partial u}\right| &\leq c_1(1+|u|^{p-1}) \\ \left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial \phi}{\partial x}\right| &\leq c_1(1+|u|^{p}), \text{ and} \end{aligned} \tag{3.2}$$

$$\left|\frac{\partial W}{\partial F}\right| \leq c_1(1+|u|^{p-1}+|F|^{p-1})\,. \tag{3.3}$$

Before we analyze the energy E_h, we need to introduce some terminology. By the *degree of an interpolation operator* J, we mean the largest integer k so that all polynomials q of degree k or less are interpolated exactly, i.e., $Jq = q$. The *degree of the quadrature rule defined by the interpolation operator* J is the largest integer k so that all polynomials q of degree k or less are integrated exactly on the domain of interest, i.e., $\int_K Jq\,\mathrm{d}x = \int_K q\,\mathrm{d}x$.

3.1 Liminf and limsup conditions

In this section we study the Γ-limit of the family (E_h), starting with the liminf condition (1.1) for the leading term.

Lemma 7 *Suppose that W is convex in F, that the integration weights $\omega_q^{(b)}$ are non-negative, that the quadrature rule defined by $J_{b,h}$ is exact on polynomials of degree $d-1$, and that* (3.1) *holds. In this case, if $u_h \in V_h$ and $u_h \xrightarrow{1,p} u$ then*

$$E_b(u) \leq \liminf_h E_{b,h}(u_h).$$

Proof 1. Suppose that $u_h \xrightarrow{1,p} u$. If $W = W(F)$ we have by Jensen's inequality

$$\sum_q \omega_q W(\nabla u_h(x_{q,K})) \;\geq\; W(\sum_q \omega_q \nabla u_h(x_q)) \;=\; W\left(\fint_K \nabla u_h \,\mathrm{d}x \right).$$

This motivates the definition

$$F_h = \sum_{K \in \mathcal{T}_h} \chi_K \fint_K \nabla u_h \,\mathrm{d}x.$$

We claim that $F_h \stackrel{p}{\rightharpoonup} \nabla u$. To see this, pick any $\varphi \in C^1(\bar{\Omega}, \mathbf{R}^{m\times n})$ and consider

$$\begin{aligned}
\int_K (\nabla u_h - F_h) : \varphi \,\mathrm{d}x &= \int_K (\nabla u_h - F_h) : \varphi(x_K) \,\mathrm{d}x + e_K \\
&= \int_K \nabla u_h : \varphi(x_K) \,\mathrm{d}x - |K| F_h : \varphi(x_K) + e_K \\
&= e_K,
\end{aligned}$$

where e_K, the error committed in the piecewise constant approximation of φ, can be estimated by

$$|e_K| \leq \operatorname{diam}(K) \left(\frac{1}{p'} \|\nabla\varphi\|_{p',K}^{p'} + \frac{2}{p} \|\nabla u_h\|_{p,K}^{p} \right).$$

Summing over $K \in \mathcal{T}_h$ and letting $h \to \infty$ shows that $\langle u_h, \varphi \rangle \to 0$. Since (F_h) is bounded in L^p and $C^1(\bar{\Omega}, \mathbf{R}^{m\times n})$ is dense in $L^p(\Omega, \mathbf{R}^{m\times n})$, we obtain $F_h \stackrel{p}{\rightharpoonup} \nabla u$. Using the weak lower semicontinuity of continuous convex functionals, this implies

$$E(u) \leq \liminf_h \int_\Omega W(F_h) \,\mathrm{d}x \leq \liminf_h E_{b,h}(u_h).$$

2. Generalizing this result to the case when W depends also on x and u is a mere technicality. For a fixed K, and for all $x \in K$, we have

$$W(x, u_h(x), F_h|_K) \leq \sum_q \omega_q W(x, u_h(x), \nabla u_h(x_{q,K})).$$

Integrating over K,

$$\int_K W(x, u_h(x), F_h) \,\mathrm{d}x \leq \sum_q \omega_q \int_K W(x, u_h(x), \nabla u_h(x_{q,K})) \,\mathrm{d}x,$$

and replacing the integral on the right-hand side by a one-point quadrature rule with quadrature point in $x_{q,K} \in K$ gives

$$\begin{aligned}
\int_K W(x, u_h(x), F_h) \,\mathrm{d}x &\leq \sum_q |K| \omega_q W(x_{q,K}, u_h(x_{q,K}), \nabla u_h(x_{q,K})) + e_K \\
&= \int_K J_{b,h} W(x, u_h(x), \nabla u_h(x)) \,\mathrm{d}x + e_K,
\end{aligned}$$

where the error e_K is estimated using an interpolation error estimate and the growth conditions;

$$
\begin{aligned}
|e_K| &\leq \sum_q \omega_q \int_K |W(x, u_h(x), \nabla u_h(x_{q,K}) - W(x_{q,K}, u_h(x_{q,K}), \nabla u_h(x_{q,k})| \,\mathrm{d}x \\
&\leq \operatorname{diam}(K) \int_K \left| \frac{\partial W}{\partial x}(x, u_h(x), \nabla u_h(x_{q,K})) \right| \\
&\qquad + \left| \frac{\partial W}{\partial u}(x, u_h(x), \nabla u_h(x_{q,K})) \right| |\nabla u_h(x)| \,\mathrm{d}x \\
&\leq \operatorname{diam}(K)\, c_1(1 + \|u_h\|_{1,p,K}^p). \qquad (3.4)
\end{aligned}
$$

After summing over K, we have

$$\int_\Omega W(x, u_h, F_h) \,\mathrm{d}x \leq E_{b,h}(u_h) + e_h,$$

where the error e_h tends to zero as $h \to \infty$. As $u_h \xrightarrow{p} u$, $F_h \stackrel{p}{\rightharpoonup} \nabla u$, and W is convex in F we obtain

$$E_b(u) \leq \liminf_h \int_\Omega W(x, u_h, F_h) \,\mathrm{d}x \leq \liminf_h E_{b,h}(u_h).$$

■

The proof of the limsup condition (1.2) makes use of the fact that we can pick a piecewise affine approximating sequence for which the highest order terms do not affect the quadrature. We consider two sets of conditions under which (1.2) holds.

We need to make technical assumptions on the approximation spaces V_h and on the boundary condition. First, we assume that V_h contains $\mathcal{S}_h^1$, the set of all continuous, piecewise affine functions. Next, we assume that the piecewise polynomial functions $\nabla^2 g_h$ are uniformly bounded in L^p. (We do not assume here that $g_h \in W^{2,p}(\Omega, \mathbf{R}^m)$.) This assumption is not unnatural. If $g_h \in \mathcal{S}_h^1$ then this condition is automatically satisfied. Choosing a higher polynomial degree $k \geq 2$ only seems to make sense if at least $g \in W^{k,p}(\Omega, \mathbf{R}^m)$ and, in this case, piecewise polynomial interpolants g_h typically satisfy $\|\nabla^2 g_h\|_{p,K} \leq |K| C(\gamma(K), p, n, \|\nabla^2 g\|_{p,K})$ (see, for example, [6]).

Lemma 8 *If $\mathcal{S}_h^1 \subset V_h$, $(\nabla^2 g_h)$ is bounded in L^p, and if* (3.1) *and* (3.3) *hold then for every $u \in \mathcal{A}$ there exists a sequence (u_h), $u_h \in \mathcal{A}_h$, converging strongly to u, such that*

$$E_{b,h}(u_h) \to E_b(u).$$

Proof Let $v_h \in \mathcal{S}_h^1 \cap \mathcal{A}_0$ such that $v_h \xrightarrow{1,p} u - g$ and set $u_h = g_h + v_h$ so that $u_h \in \mathcal{A}_h$ and $u_h \xrightarrow{1,p} u$. Since $\nabla^2 v_h = 0$, $\nabla^2 u_h$ is bounded in L^p, a fact we can use to estimate the quadrature error. For every $K \in \mathcal{T}_h$ we have

$$\begin{aligned}
&\int_K |J_{b,h}W(x,u_h,\nabla u_h) - W(x,u_h,\nabla u_h)|\,\mathrm{d}x \\
&\quad\leq\ c_1\,\mathrm{diam}(K)\int_K \left(\left|\frac{\partial W}{\partial x}\right| + \left|\frac{\partial W}{\partial u}\right|\,|\nabla u_h| + \left|\frac{\partial W}{\partial F}\right|\,|\nabla^2 u_h|\right)\mathrm{d}x \\
&\quad\leq\ c_1\,\mathrm{diam}(K)\int_K \left(\left|\frac{\partial W}{\partial x}\right| + \frac{1}{p'}\left|\frac{\partial W}{\partial u}\right|^{p'} + \frac{1}{p}|\nabla u|^p + \frac{1}{p'}\left|\frac{\partial W}{\partial F}\right|^{p'} + \frac{1}{p}|\nabla^2 u|^p\right)\mathrm{d}x \\
&\quad\leq\ c_1\,\mathrm{diam}(K)\left(\|u_h\|^p_{1,p,K} + \|\nabla^2 g_h\|^p_{p,K} + |K|\right),
\end{aligned}$$

where we used Young's inequality in the second and the growth conditions in the third inequality. We sum over $K \in \mathcal{T}_h$ to obtain $E_{b,h}(u_h) \to E_b(u)$ as $h \to \infty$. ■

If the boundary condition is zero, i.e., if $g_h = g = 0$ for all h then the assumptions to obtain the limsup condition can be weakened. Using a similar proof it can be easily seen that in this case we do not require the differentiability of W with respect to F and the growth condition on $\partial W/\partial F$.

Lemma 9 *If* (3.1) *holds then for every* $u \in \mathcal{A}_0$ *there exists a sequence* (u_h)*,* $u_h \in \mathcal{S}^1_h \cap \mathcal{A}_0$*, converging strongly to* u *such that*

$$E_{b,h}(u_h) \to E_b(u).$$

The next Lemma deals with the lower order terms in the spirit of Proposition 3.

Lemma 10 *If* (2.6) *and* (3.2) *hold and* $u_h \xrightarrow{1,p} u$ *then*

$$E_{f,h}(u_h) \to E_f(u) \quad\text{and}\quad E_{s,h}(u_h) \to E_s(u) \quad\text{as } h \to \infty.$$

Proof 1. We start off by proving the convergence of $E_{f,h}$ to E_f in the above sense. Let $\rho_h = \max_{K\in\mathcal{T}_h}\mathrm{diam}(K)$. For fixed u_h, using a simple interpolation error estimate, Hölder's inequality, and the growth conditions (3.2), we have

$$\begin{aligned}
|E_f(u_h) - E_{f,h}(u_h)| &\leq \int_\Omega |f(x,u) - J_{f,h}f(x,u)|\,\mathrm{d}x \\
&= \sum_{K\in\mathcal{T}_h}\int_K |f(x,u) - J_{f,h}f(x,u)|\,\mathrm{d}x \\
&\leq c_1\sum_{K\in\mathcal{T}_h}\mathrm{diam}(K)\int_K\left(\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial u}\right|\,|\nabla u_h|\right)\mathrm{d}x \\
&\leq \rho_h\,c_1\,(1 + |u_h|_{1,p}).
\end{aligned}$$

If (u_h) converges weakly in $W^{1,p}$ then $\|u_h\|_{1,p}$ is bounded and hence $|E_{b,h}(u_h) - E_b(u_h)| \to 0$ and furthermore $E_{b,h}(u_h) \to E_b(u)$ as $h \to \infty$.

2. The proof of convergence of the surface energy is in principle the same. Instead of the above estimate we obtain

$$|E_s(u_h) - E_{s,h}(u_h)| \le \sum_{e\subset\Gamma_N} \operatorname{diam}(e)(\mathcal{H}^{n-1}(e) + \|u_h\|_{1,p,e}^p). \tag{3.5}$$

As the range of the trace operator is $W^{1-1/p,p}(\partial\Omega)$, $\|u_h\|_{1,p,\Gamma_N}$ may not be bounded. However, we can use the inverse inequality

$$\|u_h\|_{1,p,e}^p \le \operatorname{diam}(e)^{-1/p}\|u_h\|_{1-1/p,p,e}^p \tag{3.6}$$

to control it. Inserting (3.6) into (3.5) yields

$$|E_s(u_h) - E_{s,h}(u_h)| \le C\,\rho_h^{1-1/p}(1 + \|u_h\|_{1-1/p,p,\Omega}^p),$$

which again gives the desired result. ■

3.2 Coercivity and the convergence theorem

Combining Lemmas 7, 8, and 10, it is easy to show that $(E_h|_{\mathcal{A}_h})$ $\Gamma(\sigma_{1,p})$-converges to $E|_{\mathcal{A}}$. We are only left to show that the discrete functionals are coercive.

Lemma 11 *Suppose that on each element $K \in \mathcal{T}_h$, the interpolation operator $J_{b,h}$ is at least of degree $d-1$. Then there exists a constant $c_0 > 0$, depending only on the quadrature rule and the polynomial degree d, such that for all $u_h \in V_h$ and for all $K \in \mathcal{T}_h$,*

$$\int_K J_{b,h} W(x, u_h, \nabla u_h)\,\mathrm{d}x \ge c_0(\|\nabla u_h\|_{p,K}^p - 1).$$

Proof First, we transfer the integration to the reference simplex $\hat{K}_n$;

$$\begin{aligned}\int_K J_{b,h} W(x, u_h, \nabla u_h)\,\mathrm{d}x &= |K|\sum_q \omega_q W(\hat{x}_q, u_h(x(\hat{x}_q)), \nabla u_h(x(\hat{x}_q)))\\ &\ge |K|c_0\left(\sum_q \omega_q |\nabla u_h(x(\hat{x}_q))|^p - 1\right).\end{aligned}$$

Due to norm-equivalence in finite-dimensional spaces (or local compactness) there exists a constant $c_0 = c_0((\omega_q), (\hat{x}_q), d) > 0$ such that for all piecewise polynomials u_h of degree d,

$$\sum_q \omega_q |\nabla u_h(x(\hat{x}_q))|^p \ge c_0 \int_{\hat{K}_n} |\nabla u_h(x(\hat{x}))|^p\,\mathrm{d}\hat{x}.$$

After transforming the integral back to K, this gives

$$\begin{aligned}\int_K J_{b,h} W(x, u_h, \nabla u_h)\,\mathrm{d}x &\ge |K|c_0 \int_{\hat{K}_n} (|\nabla u_h(x(\hat{x}))|^p - 1)\,\mathrm{d}\hat{x}\\ &= c_0 \int_K (|\nabla u_h|^p - 1)\,\mathrm{d}x.\end{aligned}$$

■

In order to see that the lower order terms do not affect the coercivity of E_h we prove growth conditions on $E_{f,h}$ and $E_{s,h}$.

Lemma 12 *There exists a constant $c_1 > 0$ such that, for all $u_h \in V_h$,*

$$\begin{aligned} |E_{f,h}(u_h)| &\leq c_1(\|u_h\|_{r,\Omega}^r + 1), \quad \text{and} \\ |E_{s,h}(u_h)| &\leq c_1(\|u_h\|_{r,\partial\Omega}^r + 1). \end{aligned}$$

Proof The proof follows that of Lemma 11 almost verbatim except that the estimates are from above rather than from below. Instead of norm-equivalence one uses the fact that

$$\left(\sum_q |\omega_q^{(f)}||u_h(x(\hat{x}_q^{(f)}))|^r \right)^{1/r} \quad \text{and} \quad \left(\sum_q |\omega_q^{(s)}||u_h(x(\hat{x}_q^{(s)}))|^r \right)^{1/r}$$

are at least seminorms on the space of polynomials of degree d, which is sufficient to obtain the result. ■

Finally, we can state the convergence theorem for quadrature approximations of Galerkin discretizations.

Theorem 13 *(a) Suppose that W is convex in F, the quadrature rule defined by $J_{b,h}$ is of degree $d-1$ or higher, where d is the maximal polynomial degree in V_h, that $\mathcal{S}_h^1 \subset V_h$, and that (3.1) and (3.2) hold. Suppose also that either $(\nabla^2 g_h)$ is uniformly bounded in L^p and that (3.3) holds or that $g_h = g = 0$ for all h. Then the family $(E_h|_{\mathcal{A}_h})$ $\Gamma(\sigma_{1,p})$-converges to $E|_{\mathcal{A}}$.*

(b) If the degree of the interpolation operator $J_{b,h}$ is $d-1$ or higher then the family $(E_h|_{\mathcal{A}_h})$ is equi-coercive in the weak topology of $W^{1,p}(\Omega, \mathbf{R}^m)$ and hence any sequence of approximate minimizers contains a subsequence which converges weakly to a minimizer of $E|_{\mathcal{A}}$.

Proof The proof of Γ-convergence is completely contained in the Lemmas of Section 3.1. To prove coercivity, note that Lemmas 11 and 12 imply that

$$\begin{aligned} E_h(u_h) &\geq c_0(\|\nabla u_h\|_p^p - 1) - c_1(\|u_h\|_r^r + \|u_h\|_{r,\partial\Omega}^r + 1) \\ &\geq c_0\|u_h\|_{1,p}^p - c_1(\|u_h\|_{1,r}^r + 1), \end{aligned}$$

using also Friedrichs' inequality (2.1) and the embedding of $W^{1,r}(\Omega, \mathbf{R}^m)$ in $L^r(\partial\Omega, \mathbf{R}^m)$. It is now easy, using Young's inequality, to obtain

$$\|u_h\|_{1,p}^p \leq c_1(E_h(u_h) + 1),$$

where c_1 does not depend on h or u_h. ■

Remarks. 1. Unfortunately, it is not easy to establish strong convergence in this case, since using the techniques from the previous sections we would require an estimate on $|E_h(u_h) - E(u_h)|$ which does not seem easy to obtain. It is not clear whether minimizers of E_h are approximate minimizers of E unless additional regularity of the discrete minimizers is known. Compare also with the remarks at the end of Section 4.

2. Lemma 11 is sharp in the sense that if $J_{b,h}$ is of a lower degree then there exist discrete functions for which E_h remains bounded but which do not contain weakly convergent subsequences. This can be seen, for example, on easily constructed counterexamples in one dimension, using a P2 finite element space and a midpoint rule to approximate the Dirichlet integral, where such functions can be chosen to be minimizers of E_h.

4 The polyconvex case

4.1 Hyperelasticity

Assume for the sake of notational simplicity that W depends only on F. One of the conditions for a physically realistic stored energy function of an elastic material is that

$$\begin{aligned} W(F) &\to \infty \quad \text{as} \quad \det F \to 0+, \text{ and} \\ W(F) &= +\infty \quad \text{if} \quad \det F \leq 0. \end{aligned} \tag{4.1}$$

Typical examples of stored energy functions which satisfy this condition but retain the weak lower semicontinuity property of the energy functional are polyconvex stored energy functions. W is polyconvex if there exists a convex function

$$\Phi : \mathbf{R}^D \to \mathbf{R},$$

where D is the dimension of $\mathbf{M}(\mathbf{R}^{m\times n})$, $\mathbf{M}(F)$ being the vector of the minors of F, so that

$$W(x, u, F) = \Phi(x, u, \mathbf{M}(F)).$$

There are several sets of conditions on Φ that guarantee the well-posedness of the minimization problem. We consider here a relatively simple case; we assume that $p > n$ and that

$$c_0(|F|^p + \gamma(\det F) - 1) \leq W(F) \leq c_1(|F|^p + \gamma(\det F) + 1) \tag{4.2}$$

where γ is convex and bounded below. In this case, Theorem 4 remains true [17].

However, E is not continuous and hence the approximation problem (compare Lemma 5) is a much more difficult one. We would need to be able to show that any $u \in \mathcal{A}$ for which $E(u)$ is finite can be approximated by $u_h \in \mathcal{A}_h$ so that $E(u_h)$ remains bounded as well. Under these conditions, E is continuous along the sequence (u_h). It is anything but straightforward, however, to construct such a sequence. If u_h is any sort of approximation of u, it is not clear whether $E(u_h)$ is even finite.

For further information on polyconvex energies, see for example [2, 10, 17].

4.2 Quadrature approximations of polyconvex energies

If we neglect condition (4.1) and assume that W satisfies the usual growth conditions then the theory of Section 3 can be carried through for polyconvex W with only minor modifications. The proofs of the limsup condition (Lemmas 8 and 9) and the coercivity (Lemma 11) remain unchanged (as they would for any stored energy function satisfying the growth conditions). After replacing W by its convex representation Φ, we can repeat the arguments which prove the Γ-convergence of (E_h) to E. The crucial difference is that the order of the quadrature rule has to be raised so that all minors of ∇u_h are integrated exactly as well. If the order of the quadrature rule defined by $J_{b,h}$ is at least $n(d-1)$ then for a polyconvex stored energy function W, Lemma 7 and Theorem 13 remain true as well. This might not be a great restriction in two dimensions, as to achieve coercivity of the discrete functionals the number of quadrature points is the same as what is required to integrate the determinant exactly. In higher dimensions, however, this seems unnecessarily expensive.

Let us review the crucial part of the argument, which fails for the polyconvex case when a lower order quadrature rule is employed. As $\sum_q \omega_q = 1$, we can use the convexity of Φ to obtain

$$\Phi(\sum_q \omega_q \mathbf{M}(\nabla u_h(x_{q,K}))) \leq \sum_q \omega_q \Phi(\mathbf{M}(\nabla u_h(x_{q,K}))) = \sum_q \omega_q W(\nabla u_h(x_{q,K}).$$

$\mathbf{M}$ is not linear, but a polynomial of degree n; for example in two dimensions, it contains the determinant $\det \nabla u_h$. To proceed with the argument, we would have to show that

$$(\delta_h - \det \nabla u_h) \xrightarrow{p/2} 0, \tag{4.3}$$

where

$$\delta_h = \sum_{K \in \mathcal{T}_h} \chi_K \left(\sum_q \omega_q \det \nabla u_h(x_{q,K}) \right).$$

It is easy to find weakly convergent sequences (u_h) for which (4.3) fails if the quadrature rule is of order strictly less than $2(d-1)$.

Condition (4.3) seems to be also necessary to obtain the liminf condition (1.1). For the same examples for which (4.3) fails the quadrature approximations of the quasi-affine functional

$$E(u) = \int_\Omega \det \nabla u \, \mathrm{d}x$$

do not Γ-converge to E. It is also straightforward to find counterexamples to the liminf condition, when γ is strictly convex; for example if

$$E(u) = \int_\Omega \left(\epsilon |\det \nabla u|^{p/2} + \det \nabla u \right) \mathrm{d}x$$

then for sufficiently small ϵ and the same examples as before, the liminf condition fails again.

Note, however, that this does not imply that minimizers of $E_h|_{\mathcal{A}_h}$ do not converge to a minimizer of $E_{\mathcal{A}}$. In fact, if we sharpen the proof of Theorem 2, we find that it is sufficient that the liminf condition (1.1) holds for minimizers of the discrete functionals $E_h|_{\mathcal{A}_h}$. A regularity result for minimizers of the discrete functionals, for example stating that the sequence $(\nabla^2 u_h)$ is uniformly bounded in L^p would be sufficient to extend the convergence result to polyconvex energies. In this case, we would also obtain an estimate on $|E_h(u_h) - E(u_h)|$ which could be used to obtain the strong convergence of minimizers. Unfortunately, such a theory seems to be lacking at present.

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