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Existence of global weak solutions to kinetic models for dilute polymers

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We study the existence of global-in-time weak solutions to a coupled microscopic-macroscopic bead-spring model which arises from the kinetic theory of dilute solutions of polymeric liquids with noninteracting polymer chains. The model consists of the unsteady incompressible Navier–Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, for the velocity and the pressure of the fluid, with an elastic extra-stress tensor as right-hand side in the momentum equation. The extra-stress tensor stems from the random movement of the polymer chains and is defined through the associated probability density function which satisfies a Fokker–Planck type parabolic equation, a crucial feature of which is the presence of a centre-of-mass diffusion term. The anisotropic Friedrichs mollifiers, which naturally arise in the course of the derivation of the model in the Kramers expression for the extra stress tensor and in the drag term in the Fokker–Planck equation, are replaced by isotropic Friedrichs mollifiers. We establish the existence of global-in-time weak solutions to the model for a general class of spring-force-potentials including in particular the widely used FENE (Finitely Extensible Nonlinear Elastic) potential. We justify also, through a rigorous limiting process, certain classical reductions of this model appearing in the literature which exclude the centre-of-mass diffusion term from the Fokker–Planck equation on the grounds that the diffusion coefficient is small relative to other coefficients featuring in the equation. In the case of a corotational drag term we perform a rigorous passage to the limit as the Friedrichs mollifiers in the Kramers expression and the drag term converge to identity operators.

Key words and phrases: Polymeric flow models, existence of weak solutions, Navier–Stokes equations, Fokker–Planck equations, FENE.

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1 Introduction

This paper is concerned with the question of existence of global weak solutions to a system of nonlinear partial differential equations which arises from the kinetic theory of dilute polymer solutions. The solvent is an incompressible, viscous, isothermal Newtonian fluid confined to a open set $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, with boundary $\partial\Omega$. For the sake of simplicity of presentation we shall suppose that Ω has solid boundary $\partial\Omega$; the velocity field $\underset{\sim}{u}$ will then satisfy the no-slip boundary condition $\underset{\sim}{u} = \underset{\sim}{0}$ on $\partial\Omega$. The polymer chains which are suspended in the solvent are assumed not to interact with each other. The conservation of momentum and mass equations for the solvent then have the form of the incompressible Navier–Stokes equations in which the elastic *extra-stress* tensor $\underset{\approx}{\tau}$ (i.e. the polymeric part of the Cauchy stress tensor), appears as a source term:
Find $\underset{\sim}{u} : (\underset{\sim}{x}, t) \in \mathbb{R}^{d+1} \mapsto \underset{\sim}{u}(\underset{\sim}{x}, t) \in \mathbb{R}^d$ and $p : (\underset{\sim}{x}, t) \in \mathbb{R}^{d+1} \mapsto p(\underset{\sim}{x}, t) \in \mathbb{R}$ such that

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u} - \nu\, \Delta_x \underset{\sim}{u} + \underset{\sim}{\nabla}_x p = \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau} \qquad \text{in } \Omega \times (0, T], \tag{1.1a}$$

$$\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} = 0 \qquad \text{in } \Omega \times (0, T], \tag{1.1b}$$

$$\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{on } \partial\Omega \times (0, T], \tag{1.1c}$$

$$\underset{\sim}{u}(\underset{\sim}{x}, 0) = \underset{\sim}{u}_0(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega; \tag{1.1d}$$

where $\underset{\sim}{u}$ is the velocity field, p is the pressure of the fluid, and $\nu \in \mathbb{R}_{>0}$ is the viscosity of the solvent. For the sake of simplicity we shall assume that there are no body forces present: the presence of a body force $\underset{\sim}{f} \in L^2(\mathbb{R}, [H^{-1}(\Omega)]^d)$ on the right-hand side of (1.1a) would not cause any particular technical complications. The extra stress tensor $\underset{\approx}{\tau}$ is defined as the second moment of ψ, the probability density function of the (random) conformation vector of the polymer molecules. As will be seen below, the Kolmogorov equation satisfied by ψ is a Fokker–Planck type second-order parabolic equation whose transport coefficients depend on the velocity field $\underset{\sim}{u}$.

Polymer solutions exhibit a range of non-Newtonian flow properties: in particular, the stress endured by a fluid element depends upon the history of deformations experienced by that element. Thereby, rheological properties of non-Newtonian fluids are governed by the flow-induced evolution of their internal microstructure. Following Keunings [16], a relevant feature of the microstructure is the *conformation* of the macromolecules, i.e. their orientation and the degree of stretching they experience. From the macroscopic viewpoint it is only the statistical distribution of conformations that matters: for, the macroscopic stress carried by each fluid element is governed by the distribution of polymer conformations within that element. Motivated by this observation, kinetic theories of polymeric fluids ignore quantum mechanical and atomistic effects, and focus on 'coarse-grained' models of the polymeric conformations. Depending on the level of coarse-graining, one may arrive at a hierarchy of kinetic models. For example, a dilute solution of linear polymers in a Newtonian solvent can be described in some detail by the freely jointed bead-rod

Kramers chain, which comprises a number of beads (of the order of 100) connected by rigid linear segments. A coarser model of the same polymer is the freely jointed bead-spring chain, a *Rouse chain*, consisting of a smaller number of beads (of the order of 10) connected linearly by entropic springs. A coarser model still is the dumbbell model which involves two beads connected by a spring [5]. As has been emphasized by Keunings [16], such coarse-grained models are not meant to capture the detailed structure of the polymer. Rather, they are intended to describe, in more or less detail, the evolution of polymer conformations in a macroscopic flow.

Many of the interesting properties of dilute polymer solutions can be understood by modelling them as suspensions of simple coarse-grained objects (viz. dumbbells) in a Newtonian fluid. This paper is devoted to the mathematical analysis of dumbbell models which are nonlinearly coupled Navier–Stokes–Fokker–Planck systems of partial differential equations: from the technical viewpoint these relatively simple models already exemplify many of the analytical difficulties which are encountered in the study of more complex models.

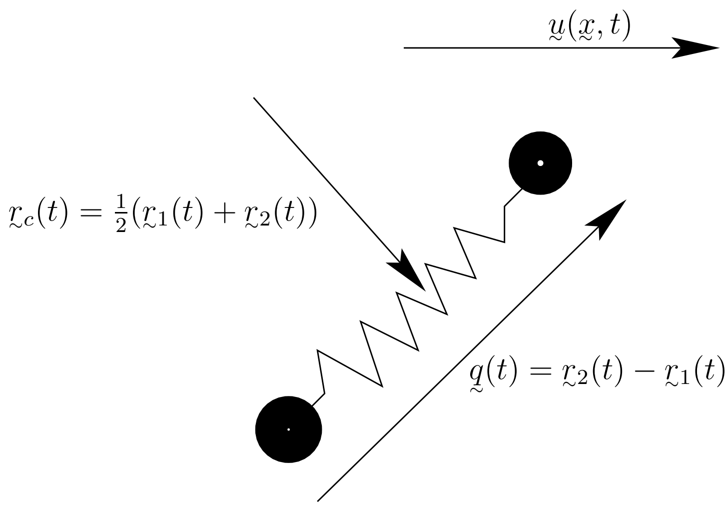

Figure 1: Noninteracting polymer chains, immersed into an incompressible Newtonian solvent with flow velocity $\underset{\sim}{u}$, are modelled by using dumbbells, each dumbbell representing a polymer chain. A dumbbell is a pair of beads, with centres of mass located, respectively, at $\underset{\sim}{r}_1(t)$ and $\underset{\sim}{r}_2(t)$ at time $t \geq 0$, connected with an elastic spring. The dumbbell is characterized by the position $\underset{\sim}{r}_c(t) = \frac{1}{2}(\underset{\sim}{r}_1(t) + \underset{\sim}{r}_2(t))$ of its centre of mass and its elongation vector $\underset{\sim}{q}(t) = \underset{\sim}{r}_2(t) - \underset{\sim}{r}_1(t)$.

Since our model problem differs in slight yet crucial details from classical bead spring models, we provide here a brief overview of the derivation of the model. Some of the key steps have been stimulated by the arguments put forward in Schieber [26] and in Chapter 1 of the recent doctoral thesis of Lozinski [19]; see also Lozinski, Owens & Fang [21].

Let us denote by $\underset{\sim}{r}_i(t) \in \mathbb{R}^d$, $i = 1, 2$, the position vectors of the centres of mass of the two beads in the dumbbell at time $t \geq 0$. The centre of mass of the dumbbell is then located at $\underset{\sim}{r}_c(t) := \frac{1}{2}(\underset{\sim}{r}_1(t) + \underset{\sim}{r}_2(t))$. We define the *elongation vector* (or end-to-end vector) of the molecule at time t by $\underset{\sim}{q}(t) = \underset{\sim}{r}_2(t) - \underset{\sim}{r}_1(t)$; see Figure 1. The elongation vector $\underset{\sim}{q}(t)$ is assumed to be confined to a balanced convex open set $D \subset \mathbb{R}^d$; the term *balanced* means that $\underset{\sim}{0} \in D$, and $-\underset{\sim}{q} \in D$ whenever $\underset{\sim}{q} \in D$. Typically, D is an open d-dimensional ball of fixed radius $Q_{\max}$, an ellipse with fixed half-axes, or the whole of $\mathbb{R}^d$.

Assuming that each bead has mass m, the spring is massless and in the absence of external forces, Langevin's equation from statistical mechanics states that

$$\begin{aligned} m\,\mathrm{d}\underset{\sim}{v}_1 + \zeta\,\underset{\sim}{v}_1\,\mathrm{d}t &= \underset{\sim}{B}_1\,\mathrm{d}t + \underset{\sim}{F}(\underset{\sim}{r}_2 - \underset{\sim}{r}_1)\,\mathrm{d}t, \\ m\,\mathrm{d}\underset{\sim}{v}_2 + \zeta\,\underset{\sim}{v}_2\,\mathrm{d}t &= \underset{\sim}{B}_2\,\mathrm{d}t + \underset{\sim}{F}(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)\,\mathrm{d}t, \end{aligned}$$

where $\underset{\sim}{v}_i$ is the velocity of the i^{th} bead, ζ is a friction coefficient, k is the Boltzmann constant and μ is the absolute temperature. Further, $\underset{\sim}{B}_i(t)$ denotes the Brownian force acting on bead i at time t, defined by a d-component vectorial Wiener process $\underset{\sim}{W}_i(t)$; thus, $\underset{\sim}{B}_i(t)\,\mathrm{d}t = \sqrt{2k\,\mu\,\zeta}\;\mathrm{d}\underset{\sim}{W}_i(t)$, with

$$\langle \underset{\sim}{W}_i \rangle = \underset{\sim}{0}, \quad i = 1, 2, \quad \text{and} \quad \langle \underset{\sim}{W}_i(t)\,[\underset{\sim}{W}_j(t')]^{\top} \rangle = \delta_{ij}\,\min(t, t')\,\underset{\approx}{I}, \quad i, j = 1, 2,$$

where $\underset{\approx}{I}$ is the $d \times d$ identity matrix, the angle-brackets $\langle\cdot\rangle$ denote (in this section of the paper only) the ensemble average over the *phase-space* $\{(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2) \in \mathbb{R}^{4d}\}$ of possible realizations. The elastic force $\underset{\sim}{F} : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ of the spring connecting the two beads is defined by a (sufficiently smooth) potential $U : \mathbb{R}_{\geq 0} \to \mathbb{R}$ through

$$\underset{\sim}{F}(\underset{\sim}{q}) = H\,U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\,\underset{\sim}{q}, \tag{1.2}$$

where $H \in \mathbb{R}_{>0}$ is a spring constant. It follows from the definition of $\underset{\sim}{F}$ that

$$\underset{\sim}{F}(\underset{\sim}{q}) + \underset{\sim}{F}(-\underset{\sim}{q}) = \underset{\sim}{0} \quad \text{and} \quad \underset{\sim}{F}(\underset{\sim}{q}) - \underset{\sim}{F}(-\underset{\sim}{q}) = 2\,\underset{\sim}{F}(\underset{\sim}{q}). \tag{1.3}$$

Example 1 Typical examples include the Hookean (linear) spring force $\underset{\sim}{F}(\underset{\sim}{q}) = H\underset{\sim}{q}$, where $\underset{\sim}{q} \in D = \mathbb{R}^d$, corresponding to $U(s) = s$; and the Finitely Extensible Nonlinear Elastic (FENE) spring force

$$\underset{\sim}{F}(\underset{\sim}{q}) = \frac{H\,\underset{\sim}{q}}{1 - |\underset{\sim}{q}|^2/Q_{\max}^2}, \qquad \underset{\sim}{q} \in D = \{\underset{\sim}{q} \in \mathbb{R}^d \,:\, |\underset{\sim}{q}| < Q_{\max}\},$$

corresponding to $U(s) = -\frac{1}{2}\,Q_{\max}^2\,\ln\left(1 - \frac{2s}{Q_{\max}^2}\right)$, where $Q_{\max}$ is the maximum length of the spring. ◇

The velocity $\underset{\sim}{v}_i$ of bead i, is governed by the differential equation

$$\dot{\underset{\sim}{r}}_i = \underset{\sim}{v}_i(t) + \underset{\sim}{u}(\underset{\sim}{r}_i(t), t), \qquad i = 1, 2,$$

where $\underset{\sim}{u}(\underset{\sim}{r}_i(t),t)$ is the solvent velocity at the point with position vector $\underset{\sim}{r}_i(t)$ at time t. On neglecting the acceleration terms in Langevin's equation, as m is small, the equations of motion of the beads become

$$\zeta\,(\mathrm{d}\underset{\sim}{r}_1(t) - \underset{\sim}{u}(\underset{\sim}{r}_1(t),t)\,\mathrm{d}t) = \underset{\sim}{B}_1(t)\,\mathrm{d}t + \underset{\sim}{F}(\underset{\sim}{r}_2 - \underset{\sim}{r}_1)\,\mathrm{d}t, \tag{1.4a}$$
$$\zeta\,(\mathrm{d}\underset{\sim}{r}_2(t) - \underset{\sim}{u}(\underset{\sim}{r}_2(t),t)\,\mathrm{d}t) = \underset{\sim}{B}_2(t)\,\mathrm{d}t + \underset{\sim}{F}(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)\,\mathrm{d}t. \tag{1.4b}$$

Let $f : (\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2, t) \mapsto f(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2, t)$ denote the phase-space probability density function, defined as the nonnegative function f such that

$$\int_{\mathcal{A}} f(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2, t)\,\mathrm{d}\underset{\sim}{r}_1\,\mathrm{d}\underset{\sim}{r}_2\,\mathrm{d}\dot{\underset{\sim}{r}}_1\,\mathrm{d}\dot{\underset{\sim}{r}}_2$$

is the expected number of dumbbells at time t having bead positions and velocities in the Borel set $\mathcal{A}$ of the phase-space $\mathbb{R}^{4d}$. We define the velocity-space average $\langle\!\langle A \rangle\!\rangle$ of a function $A : (\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2) \mapsto A(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2)$ by

$$\langle\!\langle A \rangle\!\rangle(\underset{\sim}{r}_1, \underset{\sim}{r}_2, t) = \frac{1}{\psi^{12}} \int_{\dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2} A(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2)\, f(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2, t)\,\mathrm{d}\dot{\underset{\sim}{r}}_1\,\mathrm{d}\dot{\underset{\sim}{r}}_2,$$

where the contracted configuration distribution function $\psi^{12}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, t)$ is defined as the marginal distribution of f, that is

$$\psi^{12}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, t) = \int_{\dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2} f(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dot{\underset{\sim}{r}}_1, \dot{\underset{\sim}{r}}_2, t)\,\mathrm{d}\dot{\underset{\sim}{r}}_1\,\mathrm{d}\dot{\underset{\sim}{r}}_2.$$

With these definitions, $\langle\!\langle 1 \rangle\!\rangle = 1$. By virtue of Liouville's theorem from statistical mechanics, ψ^{12} satisfies the following *continuity equation*:

$$\frac{\partial \psi^{12}}{\partial t} + \underset{\sim}{\nabla}_{r_1} \cdot \left(\langle\!\langle \dot{\underset{\sim}{r}}_1 \rangle\!\rangle\, \psi^{12}\right) + \underset{\sim}{\nabla}_{r_2} \cdot \left(\langle\!\langle \dot{\underset{\sim}{r}}_2 \rangle\!\rangle\, \psi^{12}\right) = 0. \tag{1.5}$$

On applying the velocity-space average $\langle\!\langle \cdot \rangle\!\rangle$ to (1.4a,b), we obtain

$$\zeta\left(\langle\!\langle \dot{\underset{\sim}{r}}_1(t) \rangle\!\rangle - \underset{\sim}{u}(\underset{\sim}{r}_1(t),t)\right) = \langle\!\langle \underset{\sim}{B}_1(t) \rangle\!\rangle + \underset{\sim}{F}(\underset{\sim}{r}_2 - \underset{\sim}{r}_1), \tag{1.6a}$$
$$\zeta\left(\langle\!\langle \dot{\underset{\sim}{r}}_2(t) \rangle\!\rangle - \underset{\sim}{u}(\underset{\sim}{r}_2(t),t)\right) = \langle\!\langle \underset{\sim}{B}_2(t) \rangle\!\rangle + \underset{\sim}{F}(\underset{\sim}{r}_1 - \underset{\sim}{r}_2). \tag{1.6b}$$

Now, by adopting the Curtiss–Bird–Hassager hypothesis of *equilibration in momentum space* [7], the velocity-averaged Brownian force $\langle\!\langle \underset{\sim}{B}_i \rangle\!\rangle$, $i = 1, 2$, satisfies

$$\langle\!\langle \underset{\sim}{B}_i(t) \rangle\!\rangle = -k\,\mu\,\underset{\sim}{\nabla}_{r_i} \ln \psi^{12}.$$

We note in passing that Schieber & Öttinger highlight in [27] that *equilibration in momentum space* and neglecting the acceleration term in Langevin's equation are a consequence of a single hypothesis: — that one can formally let the bead-mass $m \to 0_+$.

Thus, (1.6a,b) yield

$$\langle\!\langle \dot{\underset{\sim}{r}}_1 \rangle\!\rangle\, \psi^{12} \;=\; -\frac{k\,\mu}{\zeta}\,\underset{\sim}{\nabla}_{r_1}\psi^{12} + \underset{\sim}{u}(\underset{\sim}{r}_1,t)\,\psi^{12} + \frac{1}{\zeta}\,\underset{\sim}{F}(\underset{\sim}{r}_2-\underset{\sim}{r}_1)\,\psi^{12},$$
$$\langle\!\langle \dot{\underset{\sim}{r}}_2 \rangle\!\rangle\, \psi^{12} \;=\; -\frac{k\,\mu}{\zeta}\,\underset{\sim}{\nabla}_{r_2}\psi^{12} + \underset{\sim}{u}(\underset{\sim}{r}_2,t)\,\psi^{12} + \frac{1}{\zeta}\,\underset{\sim}{F}(\underset{\sim}{r}_1-\underset{\sim}{r}_2)\,\psi^{12}.$$

On substituting these into (1.5) we obtain the following Fokker–Planck equation for ψ^{12}:

$$\frac{\partial\psi^{12}}{\partial t} + \underset{\sim}{\nabla}_{r_1}\cdot\left[\underset{\sim}{u}(\underset{\sim}{r}_1,t)\,\psi^{12} + \frac{1}{\zeta}\,\underset{\sim}{F}(\underset{\sim}{r}_2-\underset{\sim}{r}_1)\,\psi^{12}\right]$$
$$+\,\underset{\sim}{\nabla}_{r_2}\cdot\left[\underset{\sim}{u}(\underset{\sim}{r}_2,t)\,\psi^{12} + \frac{1}{\zeta}\,\underset{\sim}{F}(\underset{\sim}{r}_1-\underset{\sim}{r}_2)\,\psi^{12}\right] = \frac{k\,\mu}{\zeta}\,\Delta_{r_1}\psi^{12} + \frac{k\,\mu}{\zeta}\,\Delta_{r_2}\psi^{12}. \quad (1.7)$$

Recalling (1.3) and defining

$$\psi(\underset{\sim}{x},\underset{\sim}{q},t) = \psi^{12}(\underset{\sim}{x}-\tfrac{1}{2}\underset{\sim}{q},\underset{\sim}{x}+\tfrac{1}{2}\underset{\sim}{q},t),$$

based on changing to centre-of-mass co-ordinates, $\underset{\sim}{r}_c(t) = \frac{1}{2}\,(\underset{\sim}{r}_1(t)+\underset{\sim}{r}_2(t))$, we deduce from (1.7) that

$$\frac{\partial\psi}{\partial t} \;+\; \underset{\sim}{\nabla}_x\cdot\left(\frac{\underset{\sim}{u}(\underset{\sim}{x}-\frac{1}{2}\underset{\sim}{q},t)+\underset{\sim}{u}(\underset{\sim}{x}+\frac{1}{2}\underset{\sim}{q},t)}{2}\,\psi\right)$$
$$+\;\underset{\sim}{\nabla}_q\cdot\left([\underset{\sim}{u}(\underset{\sim}{x}+\tfrac{1}{2}\underset{\sim}{q},t)-\underset{\sim}{u}(\underset{\sim}{x}-\tfrac{1}{2}\underset{\sim}{q},t)]\,\psi - \frac{2}{\zeta}\,\underset{\sim}{F}(\underset{\sim}{q})\,\psi\right) = \frac{k\,\mu}{2\zeta}\,\Delta_x\psi + \frac{2k\,\mu}{\zeta}\,\Delta_q\psi.$$

In order to ensure that the definition of $\psi(\underset{\sim}{x},\underset{\sim}{q},t)$ is meaningful for all $\underset{\sim}{x}\in\Omega$, we shall suppose that $\underset{\sim}{q}\in D(\underset{\sim}{x})$, where

$$D(\underset{\sim}{x}) = \{\underset{\sim}{q}\in D \,:\, \underset{\sim}{x}\pm s\,\underset{\sim}{q}\in\Omega, \text{ for all } s\in[-\tfrac{1}{2},\tfrac{1}{2}]\}.$$

Hence, the set $D(\underset{\sim}{x})$ of admissible end-to-end vectors depends on the choice of $\underset{\sim}{x}\in\Omega$. Since D has been assumed to be balanced, the same is true of $D(\underset{\sim}{x})$. Note, in particular, that since the macroscopic domain Ω is, by hypothesis, bounded, necessarily $|\underset{\sim}{q}|\le \operatorname{diam}(\Omega)$ for any $\underset{\sim}{q}\in D(x)$.

Now, we can express

$$\underset{\sim}{u}(\underset{\sim}{x}+\tfrac{1}{2}\underset{\sim}{q},t)-\underset{\sim}{u}(\underset{\sim}{x}-\tfrac{1}{2}\underset{\sim}{q},t) = (\underset{\approx}{\nabla}_x\,\underset{\sim}{\mathcal{J}}^{x}_{1,q}\underset{\sim}{u}(\underset{\sim}{x},t))\,\underset{\sim}{q}, \qquad \underset{\sim}{q}\in D(\underset{\sim}{x}),$$

where $\underset{\sim}{\mathcal{J}}^{x}_{\alpha,q}\,:\,\underset{\sim}{w}\mapsto\underset{\sim}{\mathcal{J}}^{x}_{\alpha,q}\underset{\sim}{w}$ is the directional Friedrichs mollifier with respect to $\underset{\sim}{x}$, over an interval of length $\alpha\,|\underset{\sim}{q}|$, in the direction $\underset{\sim}{q}$, defined by

$$(\underset{\sim}{\mathcal{J}}^{x}_{\alpha,q}\underset{\sim}{w})(\underset{\sim}{x}) = \frac{1}{\alpha}\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}}\underset{\sim}{w}(\underset{\sim}{x}+\theta\,\underset{\sim}{q})\,\mathrm{d}\theta, \qquad \underset{\sim}{q}\in D(\underset{\sim}{x}), \quad \underset{\sim}{x}\in\Omega.$$

Thus,

$$\frac{\partial\psi}{\partial t} + \underset{\sim}{\nabla}_x \cdot \left(\frac{\underset{\sim}{u}(\underset{\sim}{x} - \frac{1}{2}\underset{\sim}{q}, t) + \underset{\sim}{u}(\underset{\sim}{x} + \frac{1}{2}\underset{\sim}{q}, t)}{2}\, \psi \right) + \underset{\sim}{\nabla}_q \cdot \left((\underset{\sim}{\nabla}_x \, \underset{\sim}{\mathcal{J}}^x_{1,q}\underset{\sim}{u})\, \underset{\sim}{q}\, \psi - \frac{2}{\zeta}\, \underset{\sim}{F}(\underset{\sim}{q})\, \psi \right) = \frac{k\,\mu}{2\zeta}\, \Delta_x \psi + \frac{2k\,\mu}{\zeta}\, \Delta_q \psi,$$

for all $\underset{\sim}{q} \in D(\underset{\sim}{x})$ and $\underset{\sim}{x} \in \Omega$.

The fraction appearing in the second term on the left-hand side of this equation can be written as follows

$$\frac{\underset{\sim}{u}(\underset{\sim}{x} - \frac{1}{2}\underset{\sim}{q}, t) + \underset{\sim}{u}(\underset{\sim}{x} + \frac{1}{2}\underset{\sim}{q}, t)}{2} = \underset{\sim}{u}(\underset{\sim}{x}, t) + \frac{1}{2}\, \Delta^2_{\frac{1}{2}\underset{\sim}{q}}\, \underset{\sim}{u}(\underset{\sim}{x}, t), \qquad \underset{\sim}{q} \in D(\underset{\sim}{x}), \quad \underset{\sim}{x} \in \Omega,$$

where

$$\Delta^2_{\frac{1}{2}\underset{\sim}{q}}\, \underset{\sim}{u}(\underset{\sim}{x}, t) = \underset{\sim}{u}(\underset{\sim}{x} + \tfrac{1}{2}\underset{\sim}{q}, t) - 2\, \underset{\sim}{u}(\underset{\sim}{x}, t) + \underset{\sim}{u}(\underset{\sim}{x} - \tfrac{1}{2}\underset{\sim}{q}, t)$$

is the second difference of $\underset{\sim}{u}$. Now, assuming that $\underset{\sim}{u}(\cdot, t)$ belongs to the Zygmund class $\underset{\sim}{\mathcal{C}}^1_{\rm loc}$ (cf. [29], for example), we have that

$$|\Delta^2_{\frac{1}{2}\underset{\sim}{q}}\, \underset{\sim}{u}(\underset{\sim}{x}, t)| \leq \frac{1}{2}\, |\underset{\sim}{q}|\, |\underset{\sim}{u}(\cdot, t)|_{\mathcal{C}^1(B(\underset{\sim}{x}, \frac{1}{2}|\underset{\sim}{q}|))}, \qquad \underset{\sim}{q} \in D(\underset{\sim}{x}), \quad \underset{\sim}{x} \in \Omega,$$

and hence

$$\left| \frac{\underset{\sim}{u}(\underset{\sim}{x} - \frac{1}{2}\underset{\sim}{q}, t) + \underset{\sim}{u}(\underset{\sim}{x} + \frac{1}{2}\underset{\sim}{q}, t)}{2} - \underset{\sim}{u}(\underset{\sim}{x}, t) \right| \leq \frac{1}{4}\, |\underset{\sim}{q}|\, |\underset{\sim}{u}(\cdot, t)|_{\mathcal{C}^1(B(\underset{\sim}{x}, \frac{1}{2}|\underset{\sim}{q}|))}, \quad \underset{\sim}{q} \in D(\underset{\sim}{x}), \quad \underset{\sim}{x} \in \Omega.$$

The requirement $\underset{\sim}{u} \in \underset{\sim}{\mathcal{C}}^1_{\rm loc}$ is a very weak hypothesis on the regularity of $\underset{\sim}{u}$; in particular, $\underset{\sim}{u} \in \underset{\sim}{\mathcal{C}}^1_{\rm loc}$ may be nowhere differentiable on Ω.

We proceed by adopting the *local homogeneity assumption*, $\ell_0\, |\underset{\sim}{u}(\cdot, t)|_{\mathcal{C}^1(B(\underset{\sim}{x}, \frac{1}{2}\ell_0))} \approx 0$, where $\ell_0 \ll \mathrm{diam}(\Omega)$ is the characteristic microscopic length-scale (of the characteristic dumbbell size). The validity of this assumption rests on the premise that, while the velocity field may exhibit wide variation with respect to $\underset{\sim}{x}$ over distances comparable to the size of an ensemble of dumbbells, its variation over the length-scale $|\underset{\sim}{q}| \approx \ell_0$ of a single dumbbell is small. Under this hypothesis, the arithmetic mean $\frac{1}{2}(\underset{\sim}{u}(\underset{\sim}{x} - \frac{1}{2}\underset{\sim}{q}, t) + \underset{\sim}{u}(\underset{\sim}{x} + \frac{1}{2}\underset{\sim}{q}, t))$ is simply replaced by $\underset{\sim}{u}(\underset{\sim}{x}, t)$. Hence, we arrive at the Fokker–Planck equation

$$\frac{\partial\psi}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\underset{\sim}{u}(\underset{\sim}{x}, t)\, \psi) + \underset{\sim}{\nabla}_q \cdot \left((\underset{\sim}{\nabla}_x \, \underset{\sim}{\mathcal{J}}^x_{1,q}\underset{\sim}{u})\, \underset{\sim}{q}\, \psi - \frac{2}{\zeta}\, \underset{\sim}{F}(\underset{\sim}{q})\, \psi \right) = \frac{k\,\mu}{2\zeta}\, \Delta_x \psi + \frac{2k\,\mu}{\zeta}\, \Delta_q \psi,$$

for $\underset{\sim}{x} \in \Omega$, $\underset{\sim}{q} \in D(\underset{\sim}{x})$, $|\underset{\sim}{q}| = \mathcal{O}(\ell_0)$ and $t > 0$. The equation is supplemented by an initial condition $\psi(\underset{\sim}{x}, \underset{\sim}{q}, 0) = \psi_0(\underset{\sim}{x}, \underset{\sim}{q}) \geq 0$ and boundary conditions which will be specified later.

Next we define the extra stress tensor $\underset{\approx}{\tau}$ in terms of ψ. Taking an arbitrary surface in the dumbbell solution we consider the contribution to $\underset{\approx}{\tau}$ at a point P with position vector $\underset{\sim}{x}$ due to: (a) the spring tension in the dumbbells straddling the surface at P; and (b) changes in momentum brought about by beads passing through the surface at P. Following Biller & Petruccione [4, 24], we then have that

$$\begin{aligned}\underset{\approx}{\tau}(\underset{\sim}{x},t) &= -2k\,\mu\,\rho^{+}(\underset{\sim}{x},t)\,\underset{\approx}{I} + \int_{D(\underset{\sim}{x})}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{q}\,[\underset{\sim}{F}(\underset{\sim}{q})]^{\top}\,\psi(\underset{\sim}{x}+\theta\,\underset{\sim}{q},\underset{\sim}{q},t)\,\mathrm{d}\theta\,\mathrm{d}\underset{\sim}{q} \\ &= -2k\,\mu\,\rho^{+}(\underset{\sim}{x},t)\,\underset{\approx}{I} + \int_{D(\underset{\sim}{x})} \underset{\sim}{q}\,[\underset{\sim}{F}(\underset{\sim}{q})]^{\top}\,\mathcal{J}^{x}_{1,q}\psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}, \end{aligned}\tag{1.8}$$

where $\mathcal{J}^{x}_{\alpha,q}$ is the scalar version of the operator $\underset{\sim}{\mathcal{J}}^{x}_{\alpha,q}$, and

$$\rho^{+}(\underset{\sim}{x},t) = \int_{D(\underset{\sim}{x})} \psi(\underset{\sim}{x}+\tfrac{1}{2}\underset{\sim}{q},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}, \qquad \underset{\sim}{q}\in D(\underset{\sim}{x}), \quad |\underset{\sim}{q}| = \mathcal{O}(\ell_0), \quad \underset{\sim}{x}\in\Omega. \tag{1.9}$$

We nondimensionalize $\underset{\sim}{q}$ by performing the change of variables $\widehat{\underset{\sim}{q}} = \underset{\sim}{q}/\ell_0$. On noting that

$$(\underset{\sim}{\mathcal{J}}^{x}_{1,q}\underset{\sim}{w})(\underset{\sim}{x}) = (\underset{\sim}{\mathcal{J}}^{x}_{\ell_0,\widehat{q}}\,\underset{\sim}{w})(\underset{\sim}{x}) = \frac{1}{\ell_0}\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \underset{\sim}{w}(\underset{\sim}{x}+\theta\,\widehat{\underset{\sim}{q}})\,\mathrm{d}\theta, \qquad \underset{\sim}{q} = \ell_0\,\widehat{\underset{\sim}{q}}\in D(\underset{\sim}{x}), \quad \underset{\sim}{x}\in\Omega,$$

the Fokker–Planck equation (1.8) becomes

$$\frac{\partial\widehat{\psi}}{\partial t} \;+\; \underset{\sim}{\nabla}_x\cdot\left(\underset{\sim}{u}\,\widehat{\psi}\right) + \underset{\sim}{\nabla}_{\widehat{q}}\cdot\left((\underset{\approx}{\nabla}_x\,\underset{\sim}{\mathcal{J}}^{x}_{\ell_0,\widehat{q}}\,\underset{\sim}{u})\,\widehat{\underset{\sim}{q}}\,\widehat{\psi} - \frac{1}{2\lambda}\,\widehat{\underset{\sim}{F}}(\widehat{\underset{\sim}{q}})\,\widehat{\psi}\right) = \frac{\ell_0^2}{8\lambda}\,\Delta_x\widehat{\psi} + \frac{1}{2\lambda}\,\Delta_{\widehat{q}}\,\widehat{\psi},$$

where $\widehat{\psi}(\underset{\sim}{x},\widehat{\underset{\sim}{q}},t) = \ell_0^d\,\psi(\underset{\sim}{x},\underset{\sim}{q},t)$, $\underset{\sim}{q} = \ell_0\,\widehat{\underset{\sim}{q}}$, $\ell_0 = \sqrt{k\,\mu/H}$, $\lambda = \zeta/4H$,

$$\widehat{\underset{\sim}{F}}(\widehat{\underset{\sim}{q}}) = \widehat{U}'\left(\tfrac{1}{2}|\widehat{\underset{\sim}{q}}|^2\right)\widehat{\underset{\sim}{q}}, \qquad \text{with} \qquad \widehat{U}\,(s) = \ell_0^{-2}\,U(\ell_0^2\,s).$$

We define $\widehat{D} = \ell_0^{-1}D$ and $\widehat{D}(\underset{\sim}{x}) = \ell_0^{-1}D(\underset{\sim}{x})$.

Example 2 For the Hookean spring force, $\widehat{\underset{\sim}{F}}(\widehat{\underset{\sim}{q}}) = \widehat{\underset{\sim}{q}}$ with $\widehat{\underset{\sim}{q}}\in\widehat{D} = \mathbb{R}^d$, corresponding to $\widehat{U}(s) = s$. For the FENE spring force,

$$\widehat{\underset{\sim}{F}}(\widehat{\underset{\sim}{q}}) = \frac{1}{1-|\widehat{\underset{\sim}{q}}|^2/b}\,\widehat{\underset{\sim}{q}}, \qquad \underset{\sim}{q}\in\widehat{D} = \{\underset{\sim}{q}\in\mathbb{R}^d \,:\, |\widehat{\underset{\sim}{q}}|^2 < b\},$$

where $b = Q_{\max}^2/\ell_0^2$, corresponding to $\widehat{U}(s) = -\frac{b}{2}\ln\left(1-\frac{2s}{b}\right)$, $|s| < \frac{b}{2}$. ⋄

On changing variables in the expression for ρ^{+} we deduce that

$$\rho^{+}(\underset{\sim}{x},t) = \int_{\widehat{D}(\underset{\sim}{x})} \widehat{\psi}(\underset{\sim}{x}+\tfrac{\ell_0}{2}\widehat{\underset{\sim}{q}},\widehat{\underset{\sim}{q}},t)\,\mathrm{d}\widehat{\underset{\sim}{q}}.$$

By an identical argument, on noting that $H\ell_0^2 = k\,\mu$,

$$\begin{aligned}\underset{\approx}{\tau}(\underset{\sim}{x},t) &= -2k\,\mu\,\rho^+(\underset{\sim}{x},t)\,\underset{\approx}{I} + H\ell_0^2\int_{\widehat{D}(\underset{\sim}{x})} \widehat{\underset{\sim}{q}}\,\widehat{\underset{\sim}{q}}^\top\,\widehat{U}'\left(\tfrac{1}{2}|\widehat{\underset{\sim}{q}}|^2\right)\mathcal{J}^x_{\ell_0,\widehat{q}}\,\widehat{\psi}(\underset{\sim}{x},\widehat{\underset{\sim}{q}},t)\,\mathrm{d}\widehat{\underset{\sim}{q}}\\ &= p_0(\underset{\sim}{x},t)\,\underset{\approx}{I} + k\,\mu\left(\int_{\widehat{D}(\underset{\sim}{x})}\widehat{\underset{\sim}{q}}\,\widehat{\underset{\sim}{q}}^\top\,\widehat{U}'\left(\tfrac{1}{2}|\widehat{\underset{\sim}{q}}|^2\right)\mathcal{J}^x_{\ell_0,\widehat{q}}\,\widehat{\psi}(\underset{\sim}{x},\widehat{\underset{\sim}{q}},t)\,\mathrm{d}\widehat{\underset{\sim}{q}} - \rho(\underset{\sim}{x},t)\,\underset{\approx}{I}\right),\end{aligned}$$

where

$$p_0(\underset{\sim}{x},t) = k\,\mu\left(\rho(\underset{\sim}{x},t) - 2\rho^+(\underset{\sim}{x},t)\right) \qquad\text{and}\qquad \rho(\underset{\sim}{x},t) = \int_{\widehat{D}(\underset{\sim}{x})}\widehat{\psi}(\underset{\sim}{x},\widehat{\underset{\sim}{q}},t)\,\mathrm{d}\widehat{\underset{\sim}{q}}.$$

Since the right-hand side of the equation (1.1a) is equal to

$$\begin{aligned}\underset{\sim}{\nabla}_x\cdot\underset{\approx}{\tau}(\underset{\sim}{x},t) &= \underset{\sim}{\nabla}_x\,p_0(\underset{\sim}{x},t)\\ &\quad+\underset{\sim}{\nabla}_x\cdot k\,\mu\left(\int_{\widehat{D}(\underset{\sim}{x})}\widehat{\underset{\sim}{q}}\,\widehat{\underset{\sim}{q}}^\top\,\widehat{U}'\left(\tfrac{1}{2}|\widehat{\underset{\sim}{q}}|^2\right)\mathcal{J}^x_{\ell_0,\widehat{q}}\,\widehat{\psi}(\underset{\sim}{x},\widehat{\underset{\sim}{q}},t)\,\mathrm{d}\widehat{\underset{\sim}{q}} - \rho(\underset{\sim}{x},t)\,\underset{\approx}{I}\right),\end{aligned}$$

after transferring $\underset{\sim}{\nabla}_x\,p_0(\underset{\sim}{x},t)$ to the left-hand side of (1.1a) and redefining the pressure $p(\underset{\sim}{x},t)$ as $p(\underset{\sim}{x},t) - p_0(\underset{\sim}{x},t)$, the extra-stress tensor $\underset{\approx}{\tau}(\underset{\sim}{x},t)$ becomes

$$\underset{\approx}{\tau}(\underset{\sim}{x},t) = k\,\mu\left(\int_{\widehat{D}(\underset{\sim}{x})}\widehat{\underset{\sim}{q}}\,\widehat{\underset{\sim}{q}}^\top\,\widehat{U}'\left(\tfrac{1}{2}|\widehat{\underset{\sim}{q}}|^2\right)\mathcal{J}^x_{\ell_0,\widehat{q}}\,\widehat{\psi}(\underset{\sim}{x},\widehat{\underset{\sim}{q}},t)\,\mathrm{d}\widehat{\underset{\sim}{q}} - \rho(\underset{\sim}{x},t)\,\underset{\approx}{I}\right),$$

which is the Kramers expression for the elastic extra stress tensor $\underset{\approx}{\tau}$, except that here, in the case of a general heterogeneous solvent-velocity field $\underset{\sim}{u}(\underset{\sim}{x},t)$, we have the directional Friedrichs mollification $\mathcal{J}^x_{\ell_0,\widehat{q}}\widehat{\psi}$ of $\widehat{\psi}$ instead of $\widehat{\psi}$ itself appearing in the classical Kramers expression.

For the sake of simplicity of exposition we drop the $\widehat{\ }$s from our notation: in what follows we shall write ψ, $\underset{\sim}{q}$, $\underset{\sim}{F}$, U, D instead of $\widehat{\psi}$, $\widehat{\underset{\sim}{q}}$, $\widehat{\underset{\sim}{F}}$, $\widehat{U}$, $\widehat{D}$ throughout. Thus, the governing equations become (1.1a–d), where $\underset{\approx}{\tau}$ is defined by

$$\underset{\approx}{\tau}(\underset{\sim}{x},t) = k\,\mu\left(\int_{D(\underset{\sim}{x})}\underset{\sim}{q}\,\underset{\sim}{q}^\top\,U'\left(\tfrac{1}{2}|\underset{\sim}{q}|^2\right)\mathcal{J}^x_{\ell_0,q}\psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q} - \rho(\underset{\sim}{x},t)\,\underset{\approx}{I}\right), \tag{1.10}$$

with

$$\rho(\underset{\sim}{x},t) = \int_{D(\underset{\sim}{x})}\psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}, \tag{1.11}$$

and, on defining $\varepsilon = \ell_0^2/8\lambda$, we see that $\psi(\underset{\sim}{x},\underset{\sim}{q},t)$ is a solution to the Fokker–Planck equation

$$\frac{\partial\psi}{\partial t} + (\underset{\sim}{u}\cdot\nabla_x)\psi + \underset{\sim}{\nabla}_q\cdot((\underset{\approx}{\nabla}_x\,\mathcal{J}^x_{\ell_0,q}\underset{\sim}{u})\,\underset{\sim}{q}\,\psi) = \varepsilon\,\Delta_x\psi + \frac{1}{2\lambda}\,\underset{\sim}{\nabla}_q\cdot(\underset{\sim}{\nabla}_q\,\psi + U'\,\underset{\sim}{q}\,\psi). \tag{1.12}$$

This model has two noteworthy features compared to classical Fokker–Planck equations for bead-spring models appearing in the literature. The first of these is the presence of the $\underset{\sim}{x}$-dissipative centre-of-mass diffusion term $\varepsilon\,\Delta_x\psi \equiv (\ell_0^2/8\lambda)\,\Delta_x\psi$ on the right-hand side of the Fokker–Planck equation (1.12). In standard derivations of bead-spring models the centre-of-mass diffusion term is routinely omitted, on the grounds that it is several orders of magnitude smaller than the other terms in the equation. Indeed, when $L \approx 1$ is a characteristic macroscopic length-scale (such as, for example, diam(Ω)), Bhave, Armstrong & Brown, [3] estimate the ratio ℓ_0^2/L^2 to be in the range of about 10^{-9} to 10^{-7}. However, the omission of the term $\varepsilon\,\Delta_x\psi$ from (1.12) in the case of a heterogeneous solvent-velocity $\underset{\sim}{u}(\underset{\sim}{x},t)$ is a mathematically counterproductive model-reduction. When $\varepsilon\,\Delta_x\psi$ is absent, (1.12) becomes a degenerate parabolic equation exhibiting hyperbolic behaviour with respect to $(\underset{\sim}{x},t)$. Since the study of weak solutions to the coupled problem requires one to work with velocity fields $\underset{\sim}{u}$ that have very limited Sobolev regularity (typically $\underset{\sim}{u} \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{H}^1_0(\Omega))$), one is then forced into the technically unpleasant framework of hyperbolically degenerate parabolic equations with rough transport coefficients [1]. The resulting difficulties are further exacerbated by the fact that, when D is bounded, a typical spring force $\underset{\sim}{F}(\underset{\sim}{q})$ for a finitely extensible model (such as FENE) explodes as $\underset{\sim}{q}$ approaches ∂D; see Example 2 above. For these reasons, here we shall retain the centre-of-mass diffusion term in (1.12). In fact, one of the objectives of this paper is to give mathematical foundation to the model-reduction $\varepsilon = 0$ by rigorously justifying the limiting process $\varepsilon \to 0_+$.

The second noteworthy feature of the model is the presence of the directional Friedrichs mollifier $\mathcal{J}^x_{\ell_0,q}$ in the Kramers expression (1.10) and in the Fokker–Planck equation (1.12). In standard derivations of these, upon postulating that ψ and $\underset{\sim}{u}$ are sufficiently smooth, the local homogeneity assumption is used (in its classical form, expressed as the requirement that $\underset{\sim}{u}$ is 'approximately linear' on the characteristic microscopic length-scale ℓ_0,) to approximate $\mathcal{J}^x_{\ell_0,q}\psi(\underset{\sim}{x},\underset{\sim}{q},t)$ by $\psi(\underset{\sim}{x},\underset{\sim}{q},t)$ and $\underset{\sim}{\mathcal{J}}^x_{\ell_0,q}\underset{\sim}{u}(\underset{\sim}{x},t)$ by $\underset{\sim}{u}(\underset{\sim}{x},t)$ to simplify the model. We shall refrain from performing such approximations and will retain the Friedrichs mollifiers in the Kramers expression and in the Fokker–Planck equation, given that they naturally arise in the derivation of the model. Instead, we shall make a different, apparently more reasonable, simplification which does not necessitate the imposition of additional smoothness requirements on $\underset{\sim}{u}$ or ψ: we shall replace, in both (1.10) and (1.12), the directional Friedrichs mollifiers $\mathcal{J}^x_{\ell_0,q}$ and $\underset{\sim}{\mathcal{J}}^x_{\ell_0,q}$ by their isotropic counterparts J^x_α and $\underset{\sim}{J}^x_\alpha$, where $0 < \alpha \le \ell_0$. In addition, we shall also suppose, for the sake of simplicity, that $D(\underset{\sim}{x}) = D$ for all $\underset{\sim}{x} \in \Omega$ in (1.10) and (1.11). In the simplified case, when the drag term in (1.12) is corotational; that is, the tensor $\underset{\approx}{\nabla}_x\,\underset{\sim}{u}$ is replaced by its skew-symmetric part $\frac{1}{2}([\underset{\approx}{\nabla}_x\,\underset{\sim}{u}] - [\underset{\approx}{\nabla}_x\,\underset{\sim}{u}]^\top))$; we shall rigorously justify, by passing to the limit $\alpha \to 0_+$, the model-reduction $\alpha = 0$ which corresponds to replacing the Friedrichs mollifiers by identity operators.

We conclude this introduction with a brief survey of developments on the analysis of classical bead-spring models, all of which correspond to formally letting $\varepsilon = 0$ in

(1.12), i.e. omitting the centre-of-mass diffusion term, and formally letting $\alpha = 0$, i.e. replacing the Friedrichs mollifiers $\mathcal{J}^{x}_{\alpha,q}$ and $\underset{\sim}{\mathcal{J}}{}^{x}_{\alpha,q}$ by identity operators.

An early effort to show the existence and uniqueness of local-in-time solutions to a family of bead-spring type polymeric flow models is due to Renardy [25]. While the class of potentials $\underset{\sim}{F}(\underset{\sim}{q})$ considered by Renardy [25] (cf. hypotheses (F) and (F′) on pp. 314–315) does include the case of Hookean dumbbells, it excludes the practically relevant case of the FENE model (see Example 2 above). More recently, E, Li & Zhang [10] and Li, Zhang & Zhang [18] have revisited the question of local existence of solutions for dumbbell models.

Constantin [6] has considered the Navier–Stokes equations coupled to nonlinear Fokker–Planck equations describing the evolution of the probability distribution of the particles interacting with the fluid. He described, in the case when D is a Riemannian manifold, relations determining the coefficients of the stresses added in the fluid by the particles; these relations link the extra stresses to the kinematic effect of the fluid-velocity on the particles and to the inter-particle interaction potential. In equations (of Type 1, in the terminology of [6],) where the extra stresses depend linearly on the particle distribution density, as is the case in the present paper, the energy balance requires a response potential. In equations (of Type 2) where the added stresses depend quadratically on the particle distribution, it is shown that energy balance can be achieved without a dynamic response potential, and global existence of smooth solutions is shown if inertial effects are neglected. The necessary relationship (eq. (2.14) in [6]) for the existence of a Lyapunov function in the sense of Theorem 2.2 of [6] does not hold for the polymer models considered in the present paper.

Otto & Tzavaras [22] have investigated the Doi model (which is similar to a Hookean model considered here, except that $D = S^2$) for suspensions of rod-like molecules in the dilute regime. For certain parameter values, the velocity gradient vs. stress relation defined by the stationary and homogeneous flow is not rank-one monotone. They considered the evolution of possibly large perturbations of stationary flows and proved that, even in the absence of a microscopic cut-off, discontinuities in the velocity gradient cannot occur in finite time.

In a recent paper Jourdain, Lelièvre & Le Bris [15] studied the existence of solutions to the FENE model in the case of a simple Couette flow; by using tools from the theory of stochastic differential equations, they established the existence of a unique local-in-time solution to the FENE model in two space dimensions ($d = 2$) when the velocity field $\underset{\sim}{u}$ is unidirectional and of the particular form $\underset{\sim}{u}(x_1, x_2) = (u_1(x_2), 0)^\top$. The notion of solution for which existence is proved in the paper of Jourdain, Lelièvre & Le Bris [15] is mixed *deterministic-stochastic* in the sense that it is deterministic in the 'macroscopic' variable $\underset{\sim}{x}$, but stochastic in the 'microscopic' variable $\underset{\sim}{q}$. In contrast, our notion of solution (cf. Sec. 3 below) is deterministic both macroscopically and microscopically, since the microscales are modelled here by the probability density function $\psi(\underset{\sim}{x}, \underset{\sim}{q}, t)$. The choice between these different notions of solution has far-reaching consequences on computational simulation: mixed deterministic-stochastic notions of solution necessitate the use of Monte Carlo-type algorithms for the numerical approximation of polymer configurations, as proposed in the monograph of Öttinger [23] and, for example, in the paper of Jourdain, Lelièvre & Le Bris [14]; whereas weak solutions in the sense considered in the present paper can be approximated by entirely deterministic (e.g. Galerkin-type) schemes, as was done, for example, in Lozinski, Chauvière, Fang & Owens [20].

In the case of Hookean dumbbells, and assuming $\varepsilon = 0$ and $\alpha = 0$, the coupled microscopic-macroscopic model described above yields, formally, on taking the second moment of $\underset{\sim}{q} \mapsto \psi(\underset{\sim}{q}, \underset{\sim}{x}, t)$, the fully macroscopic, Oldroyd-B model of viscoelastic flow (cf. Sec. 2.2 below). Lions & Masmoudi [17] have shown the existence of global-in-time weak solutions to the Oldroyd-B model in the simplified corotational case, as described above. The argument of Lions & Masmoudi [17] is based on exploiting the propagation in time of the compactness of the solution and the DiPerna–Lions [8] theory of renormalized solutions to linear hyperbolic equations with nonsmooth transport coefficients. It is not known if an identical global existence result for the Oldroyd-B model also holds in the absence of the crucial assumption that the drag term is corotational. We note in passing that, on assuming $\varepsilon > 0$ and $\alpha = 0$, the coupled microscopic-macroscopic model above yields, on taking the second moment in the case of Hookean dumbbells. In this sense, the Hookean dumbbell model has a macroscopic closure: it is the Oldroyd-B model when $\varepsilon = 0$, and a dissipative version of Oldroyd-B (cf. (2.18) below) when $\varepsilon > 0$. In contrast, the FENE model does not have an exact closure at the macroscopic level, though Du, Yu & Liu [9] and Yu, Du & Liu [30] have recently considered the analysis of approximate closures of the FENE model. Previously, El-Kareh and Leal [11] had proposed a macroscopic model, with added dissipation in the equation which governs the evolution of the conformation tensor

$$\underset{\approx}{A}(\underset{\sim}{x}, t) := \int_D \underset{\sim}{q}\, \underset{\sim}{q}^{\top} \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\, \mathrm{d}\underset{\sim}{q}$$

in order to account for Brownian motion across streamlines; the model can be thought of as an approximate macroscopic closure of a FENE-type micro-macro model with centre-of-mass diffusion.

Simultaneously, Barrett, Schwab & Süli [2] established the existence of global-in-time weak solutions to the coupled microscopic-macroscopic model (1.1a–d) and (1.12) with $\varepsilon = 0$, an $\underset{\sim}{x}$-mollified velocity-gradient in the Fokker–Planck equation and an $\underset{\sim}{x}$-mollified probability density function ψ in the Kramers expression, — admitting a large class of potentials U (including the Hookean dumbbell model as well as general FENE-type models); in addition to these mollifications, $\underset{\sim}{u}$ in the $\underset{\sim}{x}$-convective term $(\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\psi$ in the Fokker–Planck equation was also mollified. Unlike Lions & Masmoudi [17], the arguments in [2] did not require the assumption that the drag term was corotational in the FENE case. The mollification $S_\alpha \underset{\sim}{u}$ of the velocity field $\underset{\sim}{u}$ that was considered in [2] was stimulated by the Leray-α model of the incompressible Navier–Stokes equations (the viscous Camassa–Holm equations), proposed by Foias, Holm & Titi [12] and was defined as follows: the mollified velocity field $S_\alpha \underset{\sim}{u} = \underset{\sim}{v}$ is the solution of the following Helmholtz-Stokes problem

$$\underset{\sim}{v} - \alpha\,\Delta_x\,\underset{\sim}{v} + \underset{\sim}{\nabla}_x\,\pi = \underset{\sim}{u} \quad \text{in } \Omega, \qquad \underset{\sim}{\nabla}_x \cdot \underset{\sim}{v} = 0 \quad \text{in } \Omega, \qquad \underset{\sim}{v} = \underset{\sim}{0} \quad \text{on } \partial\Omega; \tag{1.13}$$

where π is a pressure-like auxiliary variable (with no particular physical meaning). This definition ensures that the mollified velocity field $S_\alpha \underset{\sim}{u} = \underset{\sim}{v}$ remains divergence-free and satisfies the same boundary condition as $\underset{\sim}{u}$. In [2] the motivation for introducing the mollification was of purely technical nature: the need to rigorously justify the passage to the limit in the proof of the existence of weak solutions, based on a compactness argument. It is interesting to observe on reflection that, when starting from first principles, the derivation of the coupled Navier–Stokes–Fokker–Planck model does, in fact, include a mollification of ψ in the Kramers formula for the extra-stress tensor as well as of the velocity gradient in the Fokker–Planck equation, just as in [2], albeit the mollifiers are directional Friedrichs mollifiers rather than Helmholtz–Stokes mollifiers. In classical derivations of the model the mollifiers are approximated by identity operators, on the grounds that the functions to which they are applied are smooth enough to justify such a model-reduction; absurdly, in the proof of existence of weak solutions to the reduced model, the mollifiers then have to be reinstated since the requisite smoothness hypotheses which were used to justify the model-reduction are absent. Thus, in this paper we chose to retain the Friedrichs mollifiers which naturally arise in the derivation of the model, — our only modelling approximation being to replace the directional Friedrichs mollifiers by their isotropic counterparts J^x_α and $\underset{\sim}{J}^x_\alpha$, $0 < \alpha \leq \ell_0$; in particular, unlike the argument presented in [2], here we do not mollify the $\underset{\sim}{x}$-convective term $(\underset{\sim}{u} \cdot \nabla_x)\psi$ in the Fokker–Planck equation. For the same reason, instead of formally neglecting the centre-of-mass diffusion term $\varepsilon\Delta_x\psi$ in the Fokker–Planck equation (1.12) on the grounds that $\varepsilon \ll 1$, we consciously retain this term in our model, at least initially. We shall then rigorously justify the model-reduction $\varepsilon = 0$.

Our first objective is to show the existence of weak solutions to the complete model, corresponding to $\alpha > 0$ and $\varepsilon > 0$. This is accomplished in Sec. 3 after formulating carefully in Sec. 2 the class of potentials U considered. In particular, we show in Sec. 2 how the Hookean dumbbell model and the FENE model fit into the general setting. In addition, we introduce a family of weighted Sobolev spaces which represent the natural functional-analytic framework for the problem; we also recall from [2] some crucial density and trace results in these weighted spaces. These weighted space are closely related to the Fokker–Planck equation under consideration: the weight of the space is the Maxwellian induced by the potential U appearing in the Fokker–Planck equation. We then rigorously justify the model-reduction $\varepsilon = 0$. In particular, we rigorously pass to the limit $\varepsilon \to 0_+$ in both the corotational and the general noncorotational case; in the latter case we confine ourselves to the physically relevant situation when D is a bounded domain. We also justify the model-reduction $\alpha = 0$ in the corotational case by rigorously passing to the limit $\alpha \to 0_+$. The rigorous justification of the model-reduction $\alpha = 0$ remains open in the general noncorotational case; under our weak smoothness requirements on the data, the justification of the simultaneous model-reduction $(\alpha, \varepsilon) = (0, 0)$ also remains open. While our 'macroscopic' energy estimate, which bounds the velocity field $\underset{\sim}{u}$ in terms of the data, is completely uniform with respect to both ε and α in both the corotational and the general noncorotation case, the resulting compactness results are, unfortunately, of insufficient strength to admit rigorous passage to the simultaneous limit $(\alpha, \varepsilon) \to (0_+, 0_+)$.

2 Polymer Models

We term polymer models under consideration here microscopic-macro- scopic type models, since the continuum mechanical *macroscopic* equations of incompressible fluid flow are coupled to a *microscopic* model: the Fokker–Planck equation describing the statistical properties of particles in the continuum. We first present these equations and collect assumptions on the parameters in the model.

2.1 Microscopic–Macroscopic Polymer Models

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz-continuous boundary $\partial\Omega$, and suppose that the set $D \subseteq \mathbb{R}^d$, $d = 2$ or 3, of admissible elongation vectors $\underset{\sim}{q}$ in (1.12) is an open set which may be bounded or unbounded. For the sake of simplicity of presentation, we shall suppose that D is either a bounded open ball in $\mathbb{R}^d$, or $D = \mathbb{R}^d$; these two cases cover all practically relevant scenarios involving the microscopic-macroscopic models discussed here. Our arguments in the case when the configuration domain D is a bounded open ball can be extended, with only minimal changes, to situations when D is any bounded open domain in $\mathbb{R}^d$ with smooth boundary (e.g. an ellipse, to account for anisotropy in the molecule's configuration).

Our system of equations involves the following Friedrichs mollifier with respect to $\underset{\sim}{x}$. Let $j \in W^{1,\infty}(\mathbb{R}^d)$ with compact support in the closed unit ball $B(\underset{\sim}{0}, 1)$, such that

$$\int_{B(\underset{\sim}{0},1)} j(\underset{\sim}{x})\,\mathrm{d}\underset{\sim}{x} = 1 \qquad \text{and} \qquad j(\underset{\sim}{x}) \geq 0, \quad j(-\underset{\sim}{x}) = j(\underset{\sim}{x}) \quad \text{for all } \underset{\sim}{x} \in B(\underset{\sim}{0}, 1).$$

Then for any $\alpha \in (0, 1]$, let $(J^x_\alpha \eta)(\underset{\sim}{x}) : L^1(\Omega) \to W^{1,\infty}(\Omega)$ be such that

$$(J^x_\alpha \eta)(\underset{\sim}{x}) = \int_\Omega j_\alpha(\underset{\sim}{x} - \underset{\sim}{y})\,\eta(\underset{\sim}{y})\,\mathrm{d}\underset{\sim}{y} \qquad \forall \underset{\sim}{x} \in \Omega\,, \tag{2.1}$$

where $j_\alpha(\underset{\sim}{x}) = \alpha^{-d}\,j(\alpha^{-1}\,\underset{\sim}{x})$. In addition, we extend J^x_α in the natural way to vector and tensor functions to obtain $\underset{\sim}{J}^x_\alpha : \underset{\sim}{L}^1(\Omega) \to \underset{\sim}{W}^{1,\infty}(\Omega)$ and $\underset{\approx}{J}^x_\alpha : \underset{\approx}{L}^1(\Omega) \to \underset{\approx}{W}^{1,\infty}(\Omega)$.

Gathering (1.1a–d), (1.10) and (1.12) together, we then consider the following initial-boundary-value problem dependent on parameters $\alpha, \varepsilon \in (0, 1]$.

$(\mathbf{P}_{\alpha,\varepsilon})$ Find functions

$$\underset{\sim}{u}_{\alpha,\varepsilon} : (\underset{\sim}{x}, t) \in \mathbb{R}^{d+1} \mapsto \underset{\sim}{u}_{\alpha,\varepsilon}(\underset{\sim}{x}, t) \in \mathbb{R}^d \quad \text{and} \quad p_{\alpha,\varepsilon} : (\underset{\sim}{x}, t) \in \mathbb{R}^{d+1} \mapsto p_{\alpha,\varepsilon}(\underset{\sim}{x}, t) \in \mathbb{R}$$

such that

$$\begin{aligned} \frac{\partial \underset{\sim}{u}_{\alpha,\varepsilon}}{\partial t} &+ (\underset{\sim}{u}_{\alpha,\varepsilon} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{u}_{\alpha,\varepsilon} \\ -\nu \, \Delta_x \, \underset{\sim}{u}_{\alpha,\varepsilon} + \underset{\sim}{\nabla}_x \, p_{\alpha,\varepsilon} &= \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau}^{(\alpha)}(\psi_{\alpha,\varepsilon}) \quad \text{in } \Omega \times (0, T], && \text{(2.2a)} \\ \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_{\alpha,\varepsilon} &= 0 \quad \text{in } \Omega \times (0, T], && \text{(2.2b)} \\ \underset{\sim}{u}_{\alpha,\varepsilon} &= \underset{\sim}{0} \quad \text{on } \partial\Omega \times (0, T], && \text{(2.2c)} \\ \underset{\sim}{u}_{\alpha,\varepsilon}(\underset{\sim}{x}, 0) &= \underset{\sim}{u}_0(\underset{\sim}{x}) \quad \forall \underset{\sim}{x} \in \Omega; && \text{(2.2d)} \end{aligned}$$

where $\nu \in \mathbb{R}_{>0}$ is the viscosity and $\underset{\approx}{\tau}^{(\alpha)}(\psi_{\alpha,\varepsilon}) : (\underset{\sim}{x}, t) \in \mathbb{R}^{d+1} \mapsto \underset{\approx}{\tau}^{(\alpha)}(\psi_{\alpha,\varepsilon})(\underset{\sim}{x}, t) \in \mathbb{R}^{d\times d}$ is the symmetric extra-stress tensor, dependent on a probability density function $\psi_{\alpha,\varepsilon} : (\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}^{2d+1} \mapsto \psi_{\alpha,\varepsilon}(\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}$, defined as

$$\underset{\approx}{\tau}^{(\alpha)}(\psi_{\alpha,\varepsilon}) = k \, \mu \, (\underset{\approx}{C}(J^x_\alpha \psi_{\alpha,\varepsilon}) - \rho(\psi_{\alpha,\varepsilon}) \, \underset{\approx}{I}) \equiv k \, \mu \, (\underset{\approx}{J}^x_\alpha[\underset{\approx}{C}(\psi_{\alpha,\varepsilon})] - \rho(\psi_{\alpha,\varepsilon}) \, \underset{\approx}{I}). \qquad (2.3)$$

Here k, $\mu \in \mathbb{R}_{>0}$ are, respectively, the Boltzmann constant and the absolute temperature, $\underset{\approx}{I}$ is the unit $d \times d$ tensor, and

$$\begin{aligned} \underset{\approx}{C}(\psi_{\alpha,\varepsilon})(\underset{\sim}{x}, t) &= \int_D \psi_{\alpha,\varepsilon}(\underset{\sim}{x}, \underset{\sim}{q}, t) \, U'(\tfrac{1}{2}|\underset{\sim}{q}|^2) \, \underset{\sim}{q} \, \underset{\sim}{q}^\top \, \mathrm{d}\underset{\sim}{q} && \text{(2.4a)} \\ \text{and} \qquad \rho(\psi_{\alpha,\varepsilon})(\underset{\sim}{x}, t) &= \int_D \psi_{\alpha,\varepsilon}(\underset{\sim}{x}, \underset{\sim}{q}, t) \, \mathrm{d}\underset{\sim}{q}. && \text{(2.4b)} \end{aligned}$$

In addition, the real-valued, continuous, nonnegative and strictly monotonic increasing function U, defined on a relatively open subset of $[0, \infty)$, is an elastic potential which gives the elastic force $\underset{\sim}{F} : D \to \mathbb{R}^d$ on the springs *via* (1.2).

The probability density $\psi_{\alpha,\varepsilon}(\underset{\sim}{x}, \underset{\sim}{q}, t)$ represents the probability at time t of finding the centre of mass of a dumbbell in the volume element $\underset{\sim}{x} + \mathrm{d}\underset{\sim}{x}$ and having the endpoint of its elongation vector within the volume element $\underset{\sim}{q} + \mathrm{d}\underset{\sim}{q}$. Hence $\rho(\psi_{\alpha,\varepsilon})(\underset{\sim}{x}, t)$ is the density of the polymer chains located at $\underset{\sim}{x}$ at time t. It follows from (1.12) that $\psi_{\alpha,\varepsilon}$ satisfies the Fokker–Planck equation, together with suitable boundary and initial conditions:

$$\begin{aligned} \frac{\partial \psi_{\alpha,\varepsilon}}{\partial t} &+ (\underset{\sim}{u}_{\alpha,\varepsilon} \cdot \underset{\sim}{\nabla}_x) \psi_{\alpha,\varepsilon} + \underset{\sim}{\nabla}_q \cdot (\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon}) \, \underset{\sim}{q} \, \psi_{\alpha,\varepsilon}) \\ &= \frac{1}{2\,\lambda} \underset{\sim}{\nabla}_q \cdot (\underset{\sim}{\nabla}_q \, \psi_{\alpha,\varepsilon} + U' \, \underset{\sim}{q} \, \psi_{\alpha,\varepsilon}) + \varepsilon \, \Delta_x \, \psi_{\alpha,\varepsilon} \quad \text{in } \Omega \times D \times (0, T], && \text{(2.5a)} \\ \psi_{\alpha,\varepsilon} &= 0 \quad \text{on } \Omega \times \partial D \times (0, T], && \text{(2.5b)} \\ \varepsilon \, \underset{\sim}{\nabla}_x \, \psi_{\alpha,\varepsilon} \cdot \underset{\sim}{n} &= 0 \quad \text{on } \partial\Omega \times D \times (0, T], && \text{(2.5c)} \\ \psi_{\alpha,\varepsilon}(\underset{\sim}{x}, \underset{\sim}{q}, 0) = \psi_0(\underset{\sim}{x}, \underset{\sim}{q}) &\geq 0 \quad \forall (\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega \times D; && \text{(2.5d)} \end{aligned}$$

where $\underset{\sim}{n}$ is normal to $\partial\Omega$. When $D = \mathbb{R}^d$, the boundary condition (2.5b) on ∂D, the boundary of D, is replaced by a decay condition at infinity which demands that $|\psi|$ converges to 0 sufficiently fast as $|\underset{\sim}{q}|$ tends to ∞; we shall be more specific about this later.

In (2.5a) the parameter $\lambda \in \mathbb{R}_{>0}$ characterizes the elastic relaxation property of the fluid, and $\underset{\approx}{\sigma}(\underset{\sim}{v})$ is related to $\underset{\approx}{\nabla}_x \underset{\sim}{v}$, where $(\underset{\approx}{\nabla}_x \underset{\sim}{v})(\underset{\sim}{x}, t) \in \mathbb{R}^{d\times d}$ and $\{\underset{\approx}{\nabla}_x \underset{\sim}{v}\}_{ij} = \dfrac{\partial v_i}{\partial x_j}$. We will be interested in two possible choices:

$$\begin{array}{llll} & \text{(i)} & \text{the noncorotational case} & \underset{\approx}{\sigma}(\underset{\sim}{v}) = \underset{\approx}{\nabla}_x \underset{\sim}{v}, \qquad (2.6a) \\ \text{or} & \text{(ii)} & \text{the corotational case} & \underset{\approx}{\sigma}(\underset{\sim}{v}) = \underset{\approx}{\omega}(\underset{\sim}{v})\,; \qquad (2.6b) \end{array}$$

where

$$\underset{\approx}{\nabla}_x \underset{\sim}{v} = \underset{\approx}{D}(\underset{\sim}{v}) + \underset{\approx}{\omega}(\underset{\sim}{v}),\ \ \underset{\approx}{D}(\underset{\sim}{v}) = \frac{1}{2}\,[\underset{\approx}{\nabla}_x \underset{\sim}{v} + (\underset{\approx}{\nabla}_x \underset{\sim}{v})^\top\,],\ \ \underset{\approx}{\omega}(\underset{\sim}{v}) = \frac{1}{2}\,[\underset{\approx}{\nabla}_x \underset{\sim}{v} - (\underset{\approx}{\nabla}_x \underset{\sim}{v})^\top\,]\,. \tag{2.7}$$

In the corotational case no smoothing is necessary in the extra stress tensor, so we will replace (2.3) by

$$\underset{\approx}{\tau}^{(\alpha)}(\psi_{\alpha,\varepsilon}) = \begin{cases} k\,\mu\,(\underset{\approx}{J}{}^x_\alpha[\underset{\approx}{C}(\psi_{\alpha,\varepsilon})] - \rho(\psi_{\alpha,\varepsilon})\,\underset{\approx}{I}) & \text{if } \underset{\approx}{\sigma}(\cdot) = \underset{\approx}{\nabla}_x\,\cdot\,, \\ k\,\mu\,(\underset{\approx}{C}(\psi_{\alpha,\varepsilon}) - \rho(\psi_{\alpha,\varepsilon})\,\underset{\approx}{I}) & \text{if } \underset{\approx}{\sigma}(\cdot) = \underset{\approx}{\omega}(\cdot)\,. \end{cases} \tag{2.8}$$

On introducing the (normalized) Maxwellian

$$M(\underset{\sim}{q}) = \frac{\mathrm{e}^{-U(\frac{1}{2}\,|\underset{\sim}{q}|^2)}}{\displaystyle\int_D \mathrm{e}^{-U}\,\mathrm{d}\underset{\sim}{q}}\,,$$

we have that

$$M\,\underset{\sim}{\nabla}_q\,M^{-1} = -M^{-1}\,\underset{\sim}{\nabla}_q\,M = U'\,\underset{\sim}{q}. \tag{2.9}$$

In addition, the following identities hold:

$$\underset{\sim}{\nabla}_q\,U = U'\,\underset{\sim}{q}, \qquad \underset{\sim}{\nabla}_q\,U' = U''\,\underset{\sim}{q} \qquad \text{and} \qquad \Delta_q\,U = U''\,|\underset{\sim}{q}|^2 + U'\,d. \tag{2.10}$$

Thus, the Fokker–Planck equation (2.5a) can be rewritten as

$$\frac{\partial\psi_{\alpha,\varepsilon}}{\partial t} + (\underset{\sim}{u}_{\alpha,\varepsilon}\cdot\underset{\sim}{\nabla}_x\,)\psi_{\alpha,\varepsilon} + \underset{\sim}{\nabla}_q\,\cdot(\underset{\approx}{\sigma}(J^x_\alpha\underset{\sim}{u}_{\alpha,\varepsilon})\,\underset{\sim}{q}\,\psi_{\alpha,\varepsilon})$$
$$= \frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q\,\cdot\left(M\,\underset{\sim}{\nabla}_q\left(\frac{\psi_{\alpha,\varepsilon}}{M}\right)\right) + \varepsilon\,\Delta_x\,\psi_{\alpha,\varepsilon} \quad \text{in } \Omega\times D\times(0,T]. \tag{2.11}$$

2.2 Two Examples

1. FENE-type models. A widely used model is the FENE (Finitely Extensible Nonlinear Elastic) model, where

$$D = B(\underset{\sim}{0}, b^{\frac{1}{2}}) \quad \text{and} \quad U(s) = -\frac{b}{2} \ln\left(1 - \frac{2\,s}{b}\right),$$

$$\text{and hence} \quad \mathrm{e}^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)} = \left(1 - \frac{|\underset{\sim}{q}|^2}{b}\right)^{\frac{b}{2}}. \tag{2.12}$$

Here $B(\underset{\sim}{0}, s)$ is the bounded open ball of radius $s > 0$ in $\mathbb{R}^d$ centred at the origin, and $b > 0$ is an input parameter. Hence the length $|\underset{\sim}{q}|$ of the elongation vector $\underset{\sim}{q}$ cannot exceed $b^{\frac{1}{2}}$.

2. Hookean dumbbells. Letting $b \to \infty$ in (2.12) leads to the so-called Hookean dumbbell model where

$$D = \mathbb{R}^d \qquad \text{and} \qquad U(s) = s, \qquad \text{and therefore} \qquad \mathrm{e}^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)} = \mathrm{e}^{-\frac{1}{2}|\underset{\sim}{q}|^2}. \tag{2.13}$$

This particular kinetic model for $\varepsilon, \alpha \in (0, 1]$, with $\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon}) = \underset{\sim}{\nabla}_x (\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon})$, corresponds formally to an Oldroyd-B type model, or with $\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon}) = \underset{\approx}{\omega}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon})$ to a corotational Oldroyd-B type model. Indeed, on multiplying (2.5a) by $\underset{\sim}{q}\,\underset{\sim}{q}^{\top}$, integrating over D, performing integration by parts (assuming that $\psi_{\alpha,\varepsilon}$ and $|\nabla_q \psi_{\alpha,\varepsilon}|$ decay to zero sufficiently fast as $|\underset{\sim}{q}| \to \infty$), and noting (2.4a) and for any $\underset{\sim}{r} \in \mathbb{R}^d$ that

$$(\underset{\sim}{r} \cdot \underset{\sim}{\nabla}_q)\, \underset{\sim}{q}\,\underset{\sim}{q}^{\top} = \underset{\sim}{r}\,\underset{\sim}{q}^{\top} + \underset{\sim}{q}\,\underset{\sim}{r}^{\top} \qquad \text{and} \qquad \Delta_q\,(\underset{\sim}{q}\,\underset{\sim}{q}^{\top}) = 2\,\underset{\approx}{I}\,, \tag{2.14}$$

yields that $\underset{\approx}{C}_{\alpha,\varepsilon}(\underset{\sim}{x}, t) \equiv \underset{\approx}{C}(\psi_{\alpha,\varepsilon}(\underset{\sim}{x}, t))$ satisfies

$$\lambda\,(\frac{\delta \underset{\approx}{C}_{\alpha,\varepsilon}}{\delta t} + \varepsilon\,\Delta_x\,\underset{\approx}{C}_{\alpha,\varepsilon}) + \underset{\approx}{C}_{\alpha,\varepsilon} = \rho_{\alpha,\varepsilon}\,\underset{\approx}{I} \qquad \text{in } \Omega \times (0, T], \tag{2.15}$$

where $\rho_{\alpha,\varepsilon}(\underset{\sim}{x}, t) \equiv \rho(\psi_{\alpha,\varepsilon}(\underset{\sim}{x}, t))$ and

$$\frac{\delta \underset{\approx}{C}}{\delta t} = \frac{\partial \underset{\approx}{C}}{\partial t} + (\underset{\sim}{u}_{\alpha,\varepsilon} \cdot \underset{\sim}{\nabla}_x)\,\underset{\approx}{C} - [\,\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon})\,\underset{\approx}{C} + \underset{\approx}{C}\,[\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon})]^{\top}\,] \tag{2.16}$$

is the upper-convected time derivative. Similarly, on integrating (2.5a) over D and noting (2.4b) yields that $\rho_{\alpha,\varepsilon}$ satisfies

$$\frac{\partial \rho_{\alpha,\varepsilon}}{\partial t} + \varepsilon\,\Delta_x\,\rho_{\alpha,\varepsilon} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\rho_{\alpha,\varepsilon} = 0 \qquad \text{in } \Omega \times (0, T]. \tag{2.17}$$

Hence in the Hookean case, the probability density function $\psi_{\alpha,\varepsilon}$ can be eliminated leading to a closed model for $\underset{\sim}{u}_{\alpha,\varepsilon}, \underset{\approx}{C}_{\alpha,\varepsilon}$ and $\rho_{\alpha,\varepsilon}$. Moreover, if either $\alpha = 0$ or

$\underset{\approx}{\sigma}(\underset{\sim}{v}) = \underset{\approx}{\omega}(\underset{\sim}{v})$ then (2.15) and (2.17) can be combined, on noting (2.8), to yield that the extra-stress $\underset{\approx}{\tau}_{\alpha,\varepsilon}(\underset{\sim}{x}, t) \equiv \underset{\approx}{\tau}^{(\alpha)}(\psi_{\alpha,\varepsilon})$ satisfies

$$\lambda\,(\frac{\delta \underset{\approx}{\tau}_{\alpha,\varepsilon}}{\delta t} + \varepsilon\,\Delta_x\,\underset{\approx}{\tau}_{\alpha,\varepsilon}) + \underset{\approx}{\tau}_{\alpha,\varepsilon} = k\,\mu\,\lambda\,\rho_{\alpha,\varepsilon}\,[\,\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon}) + [\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon})]^\top\,] \quad \text{in } \Omega \times (0,T], \tag{2.18}$$

which, in the case of formally setting $\varepsilon = \alpha = 0$, is the Oldroyd-B constitutive equation if $\underset{\approx}{\sigma}(\underset{\sim}{u}) = \underset{\approx}{\nabla}_x\,\underset{\sim}{u}$ or the corotational Oldroyd-B constitutive equation if $\underset{\approx}{\sigma}(\underset{\sim}{u}) = \underset{\approx}{\omega}(\underset{\sim}{u})$; in the latter case, the right-hand side of (2.18) is identically equal to 0.

2.3 General Structural Assumptions on the Potential

Suppose that D is a bounded open ball in $\mathbb{R}^d$ or $D = \mathbb{R}^d$. We assume that $\underset{\sim}{q} \mapsto U(\frac{1}{2}|\underset{\sim}{q}|^2) \in C^\infty(D)$ with $\underset{\sim}{q} \mapsto U(\frac{1}{2}|\underset{\sim}{q}|^2)$ nonnegative and $\underset{\sim}{q} \mapsto U'(\frac{1}{2}|\underset{\sim}{q}|^2)$ positive on D, and that there exist constants $c_i > 0$, $i = 1, 2$, such that

$$(U')^2 - U'' \geq c_1 \quad \forall \underset{\sim}{q} \in D \quad \text{and} \quad (U')^2 - U'' \geq 2c_2\,U' \quad \forall \underset{\sim}{q}\;:\; |\underset{\sim}{q}|^2 \geq \frac{d}{c_2}, \tag{2.19}$$

where $B(\underset{\sim}{0}, \left(\frac{d}{c_2}\right)^{\frac{1}{2}}) \subset\subset D$.

The above assumptions hold for the Hookean case, (2.13), with $c_1 = 2c_2 = 1$; and the FENE case, (2.12), on assuming that $b > 2$, with $c_1 = \frac{b-2}{b}$ and $c_2 = \frac{b+2d-2}{2b}$.

We shall also suppose that there exist positive constants c_i, $i = 3, \ldots, 7$, and $\kappa > 0$, such that the Maxwellian M and the associated elastic potential U satisfy

$$c_3\,[\text{dist}(\underset{\sim}{q},\,\partial D)]^\kappa \leq M(\underset{\sim}{q})\; \leq c_4\,[\text{dist}(\underset{\sim}{q},\,\partial D)]^\kappa \quad \forall \underset{\sim}{q} \in D, \tag{2.20a}$$

$$c_5 \leq [\text{dist}(\underset{\sim}{q},\,\partial D)]\,U'(\tfrac{1}{2}|\underset{\sim}{q}|^2) \leq c_6, \quad [\text{dist}(\underset{\sim}{q},\,\partial D)]^2\,|U''(\tfrac{1}{2}|\underset{\sim}{q}|^2)| \leq c_7 \quad \forall \underset{\sim}{q} \in D; \tag{2.20b}$$

when $D = \mathbb{R}^d$, then $[\text{dist}(\underset{\sim}{q},\,\partial D)]^\kappa$ in (2.20a) is replaced by $\exp(-|\underset{\sim}{q}|^2)$, and the expressions $[\text{dist}(\underset{\sim}{q},\,\partial D)]$ and $[\text{dist}(\underset{\sim}{q},\,\partial D)]^2$ in (2.20b) are omitted.

It is an easy matter to show that the Maxwellian M and the elastic potential U of the FENE model and of the Hookean dumbbell model satisfy conditions (2.20a,b), — with $D = B(\underset{\sim}{0}, b^{\frac{1}{2}})$ and $\kappa = \frac{b}{2}$ in the case of the FENE model; and $D = \mathbb{R}^d$ for the Hookean dumbbell model.

We shall also require that

$$\int_D \Big[1 + (1 + |\underset{\sim}{q}|^2)\,((U)^2 + |\underset{\sim}{q}|^2\,(U')^2)\Big]\,M\;\mathrm{d}\underset{\sim}{q} < \infty. \tag{2.21}$$

For the Hookean model (2.13) and the FENE model (2.12), with $b > 2$, (2.21) is easily shown to hold. For example, we have that

$$\mathcal{M} := \int_D M\,(U')^2\,|\underset{\sim}{q}|^4\,\mathrm{d}\underset{\sim}{q} < \infty \tag{2.22}$$

for both models. In the Hookean case, (2.22) follows since

$$\int_0^\infty \mathrm{e}^{-s}\, s^{\frac{d+2}{2}}\, \mathrm{d}s < \infty, \tag{2.23}$$

while in the FENE case, (2.22) follows since

$$\int_0^b \left(1 - \frac{s}{b}\right)^{\frac{b-4}{2}} s^{\frac{d+2}{2}}\, \mathrm{d}s < \infty \quad \text{if } b > 2. \tag{2.24}$$

More generally, it follows from (2.20a,b), on noting that $U(\frac{1}{2}|\underset{\sim}{q}|^2) = -\ln M(\underset{\sim}{q}) + Const.$, that (2.21) holds provided that either: (i) $\kappa > 1$ when D is a bounded open ball in $\mathbb{R}^d$; or (ii) when $D = \mathbb{R}^d$.

3 Existence of Global Weak Solutions

Let

$$\underset{\sim}{H} := \{\underset{\sim}{w} \in \underset{\sim}{L}^2(\Omega) : \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} = 0\} \quad \text{and} \quad \underset{\sim}{V} := \{\underset{\sim}{w} \in \underset{\sim}{H}^1_0(\Omega) : \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} = 0\}; \tag{3.1}$$

where the divergence operator $\underset{\sim}{\nabla}_x \cdot$ is to be understood in the sense of vector-valued distributions on Ω. Let $\underset{\sim}{V}'$ be the dual of $\underset{\sim}{V}$. Then for any $\gamma \in (0,1]$, let $\underset{\sim}{S}_\gamma : \underset{\sim}{V}' \to \underset{\sim}{V}$ be such that $\underset{\sim}{S}_\gamma\, \underset{\sim}{v}$ is the unique solution to the Helmholtz-Stokes problem

$$\int_\Omega \underset{\sim}{S}_\gamma\, \underset{\sim}{v} \cdot \underset{\sim}{w}\, \mathrm{d}\underset{\sim}{x} + \gamma \int_\Omega \underset{\approx}{\nabla}_x\, \underset{\sim}{S}_\gamma\, \underset{\sim}{v} : \underset{\approx}{\nabla}_x\, \underset{\sim}{w}\, \mathrm{d}\underset{\sim}{x} = \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle \qquad \forall \underset{\sim}{w} \in \underset{\sim}{V}, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\underset{\sim}{V}'$ and $\underset{\sim}{V}$. We note that

$$\langle \underset{\sim}{v}, \underset{\sim}{S}_\gamma \underset{\sim}{v} \rangle = \int_\Omega \left[\gamma\, |\underset{\approx}{\nabla}_x\, [\underset{\sim}{S}_\gamma \underset{\sim}{v}]|^2 + |\underset{\sim}{S}_\gamma \underset{\sim}{v}|^2\right]\, \mathrm{d}\underset{\sim}{x} \qquad \forall \underset{\sim}{v} \in \underset{\sim}{V}' \supset (\underset{\sim}{H}^1_0(\Omega))', \tag{3.3}$$

and $\|\underset{\sim}{S}_\gamma \cdot \|_{H^1(\Omega)}$ is a norm on V'. In addition, we have from (3.2) that

$$\|\underset{\sim}{S}_\gamma \underset{\sim}{v}\|^2_{L^2(\Omega)} + \gamma\, \|\underset{\approx}{\nabla}_x\, [\underset{\sim}{S}_\gamma \underset{\sim}{v}]\|^2_{L^2(\Omega)} \;\; \leq \;\; \|\underset{\sim}{v}\|^2_{L^2(\Omega)} \quad \forall \underset{\sim}{v} \in \underset{\sim}{L}^2(\Omega); \tag{3.4a}$$

$$\begin{aligned} &\|(\underset{\sim}{I} - \underset{\sim}{S}_\gamma)\underset{\sim}{v}\|^2_{L^2(\Omega)} + \gamma\, \|\nabla_x\, (\underset{\sim}{I} - \underset{\sim}{S}_\gamma)\underset{\sim}{v}\|^2_{L^2(\Omega)} \\ &\quad = \gamma \int_\Omega \nabla_x\, \underset{\sim}{v} \cdot \nabla_x\, (I - \underset{\sim}{S}_\gamma)\, \underset{\sim}{v}\, \mathrm{d}\underset{\sim}{x} \leq \begin{cases} \gamma\, \|\nabla_x\, \underset{\sim}{v}\|^2_{L^2(\Omega)} & \forall \underset{\sim}{v} \in \underset{\sim}{V}, \\ \gamma^2\, \|\Delta_x \underset{\sim}{v}\|^2_{L^2(\Omega)} & \forall \underset{\sim}{v} \in \underset{\sim}{V} \cap \underset{\sim}{H}^2(\Omega). \end{cases} \end{aligned} \tag{3.4b}$$

Hence it follows from (3.4b) that

$$\|(\underset{\sim}{I} - \underset{\sim}{S}_\gamma)\underset{\sim}{v}\|_{H^1(\Omega)} \leq \gamma^{\frac{1}{2}}\, \|\Delta_x \underset{\sim}{v}\|_{L^2(\Omega)} \leq (\gamma d)^{\frac{1}{2}} \|\underset{\sim}{v}\|_{H^2(\Omega)} \quad \forall \underset{\sim}{v} \in \underset{\sim}{V} \cap \underset{\sim}{H}^2(\Omega). \tag{3.5}$$

Furthermore, for $\partial\Omega \in C^2$ and $r > d$ (c.f. Girault & Raviart [13], p.88) we have that

$$\underset{\sim}{S}_\gamma : \underset{\sim}{L}^r(\Omega) \subset \underset{\sim}{V}' \to \underset{\sim}{V} \cap \underset{\sim}{W}^{2,r}(\Omega) \subset \underset{\sim}{V} \cap \underset{\sim}{C}^1(\overline{\Omega}) \text{ is a bounded linear operator;} \tag{3.6a}$$

and hence Sobolev embedding yields that

$$\|\underset{\sim}{S}_\gamma \underset{\sim}{v}\|_{W^{1,\infty}(\Omega)} \leq C\, \|\underset{\sim}{S}_\gamma \underset{\sim}{v}\|_{W^{2,r}(\Omega)} \leq C(\gamma)\, \|\underset{\sim}{v}\|_{L^r(\Omega)} \quad \forall \underset{\sim}{v} \in \underset{\sim}{L}^r(\Omega). \tag{3.6b}$$

The aims of this paper are to prove existence of a (global-in-time) solution of a weak formulation of: (i) the problem $(\mathrm{P}_{\alpha,\varepsilon})$ for any fixed parameters $\alpha, \varepsilon \in (0,1]$; (ii) the problem (P_α), obtained by formally setting $\varepsilon = 0$ in $(\mathrm{P}_{\alpha,\varepsilon})$ (cf. (3.91a,b)), for any fixed parameter $\alpha \in (0,1]$ for both the corotational and general noncorotational cases under the following assumptions on the data

$$\partial\Omega \in C^{0,1}, \quad \underset{\sim}{u}_0 \in \underset{\sim}{H} \quad \text{and} \quad M^{-\frac{1}{2}}\,\psi_0 \in L^2(\Omega\times D) \text{ with } \psi_0 \geq 0 \text{ a.e. in } \Omega\times D. \tag{3.7}$$

In addition, if $\partial\Omega \in C^2$ we prove existence of a (global-in-time) solution of a weak formulation of the problem (P_ε), obtained by formally setting $\alpha = 0$ in $(\mathrm{P}_{\alpha,\varepsilon})$ (cf. (3.98a,b)), for any fixed parameter $\varepsilon \in (0,1]$, in the corotational case only.

The following results for J^x_α are easily established:

$$\begin{aligned}
&\|J^x_\alpha \eta\|_{L^2(\Omega)} \leq \|\eta\|_{L^2(\Omega)} && \forall \eta \in L^2(\Omega)\,, && (3.8a)\\
&\|(I - J^x_\alpha)\eta\|_{L^2(\Omega)} \to 0 \quad \text{as } \alpha \to 0, && \forall \eta \in L^2(\Omega)\,, && (3.8b)\\
&\int_\Omega (J^x_\alpha \eta_1)\,\eta_2 \,\mathrm{d}\underset{\sim}{x} = \int_\Omega \eta_1\,(J^x_\alpha \eta_2)\,\mathrm{d}\underset{\sim}{x} && \forall \eta_1,\, \eta_2 \in L^2(\Omega)\,, && (3.8c)\\
&\frac{\partial}{\partial x_i}(J^x_\alpha \eta) = J^x_\alpha\left(\frac{\partial \eta}{\partial x_i}\right), \quad i = 1 \to d, && \forall \eta \in H^1_0(\Omega)\,. && (3.8d)
\end{aligned}$$

It follows from (3.8a) and (3.8d) that $\underset{\sim}{J}^x_\alpha$ satisfies

$$\begin{aligned}
&\|\underset{\sim}{J}^x_\alpha \underset{\sim}{v}\|_{H^1(\Omega)} \leq \|\underset{\sim}{v}\|_{H^1(\Omega)} && \forall \underset{\sim}{v} \in \underset{\sim}{H}^1_0(\Omega)\,, && (3.9a)\\
&\|\underset{\sim}{J}^x_\alpha \underset{\sim}{v}\|_{W^{1,\infty}(\Omega)} \leq C(\alpha)\, \|\underset{\sim}{v}\|_{L^1(\Omega)} && \forall \underset{\sim}{v} \in \underset{\sim}{L}^1(\Omega)\,. && (3.9b)
\end{aligned}$$

We note that the results (3.8a–d) and (3.9a) hold if $j \in L^\infty(\mathbb{R}^d)$, with compact support in the closed unit ball $B(\underset{\sim}{0},1)$, $j(\underset{\sim}{x}) \geq 0$ for $\underset{\sim}{x} \in \mathbb{R}^d$, $\int_{B(\underset{\sim}{0},1)} j(\underset{\sim}{x})\,\mathrm{d}\underset{\sim}{x} = 1$, and $j(-\underset{\sim}{x}) = j(\underset{\sim}{x})$ for all $\underset{\sim}{x} \in B(\underset{\sim}{0},1)$, e.g. a constant multiple of the characteristic function of $B(\underset{\sim}{0},1)$. However, we require the result (3.9b) for our existence proof, and hence our restriction to $j \in W^{1,\infty}(\mathbb{R}^d)$ instead of $j \in L^\infty(\mathbb{R}^d)$.

On introducing

$$\|\varphi\|_{H^{0,1}(\Omega\times D;M)} := \left\{\int_{\Omega\times D} M\,\left[|\varphi|^2 + |\underset{\sim}{\nabla}_q\,\varphi|^2\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right\}^{\frac{1}{2}}, \tag{3.10a}$$

$$\text{and} \quad \|\varphi\|_{H^1(\Omega\times D;M)} := \left\{\int_{\Omega\times D} M\,\left[|\varphi|^2 + |\underset{\sim}{\nabla}_x\,\varphi|^2 + |\underset{\sim}{\nabla}_q\,\varphi|^2\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right\}^{\frac{1}{2}}, \tag{3.10b}$$

we then set

$$\begin{aligned} H^{0,1}(\Omega \times D; M) &:= \left\{\varphi \in L^1_{\rm loc}(\Omega \times D) : \|\varphi\|_{H^{0,1}(\Omega\times D;M)} < \infty\right\}, && (3.11a)\\ \text{and} \quad H^{1}(\Omega \times D; M) &:= \left\{\varphi \in L^1_{\rm loc}(\Omega \times D) : \|\varphi\|_{H^{1}(\Omega\times D;M)} < \infty\right\}. && (3.11b) \end{aligned}$$

We then define

$$\begin{aligned} X &:= M\, H^1(\Omega \times D; M), && (3.12a)\\ X_q &:= \{\varphi \in X : \int_{\Omega\times D} |\underset{\sim}{q}|^2 \frac{|\varphi|^2}{M} \, {\rm d}\underset{\sim}{q} \, {\rm d}\underset{\sim}{x} < \infty \}, && (3.12b)\\ X^+ &:= \{\varphi \in X : \varphi(\underset{\sim}{x}, \underset{\sim}{q}) \geq 0 \text{ for a.e. } (\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega \times D \}, && (3.12c)\\ \text{and} \qquad X_q^+ &:= X_q \cap X^+. && (3.12d) \end{aligned}$$

Clearly, if D is bounded then $X_q = X$ and $X_q^+ = X^+$. We remark, in particular, that due to the structural hypotheses on U (specifically, (2.21) and (2.20a)), both M and $M\,U$ belong to X_q^+. Similarly to above, we define X^0, X_q^0, $X^{0,+}$ and $X_q^{0,+}$, where $X^0 := M\, H^{0,1}(\Omega \times D; M)$.

We note for future reference that (2.4a) and (2.22) yield that, for $\varphi \in X^0$,

$$\begin{aligned} \int_\Omega |\underset{\approx}{C}(\varphi)|^2 \, {\rm d}\underset{\sim}{x} &= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \left(\int_D \varphi\, U'\, q_i\, q_j \, {\rm d}\underset{\sim}{q} \right)^2 {\rm d}\underset{\sim}{x} \\ &\leq d \left(\int_D M\, (U')^2\, |\underset{\sim}{q}|^4 \, {\rm d}\underset{\sim}{q} \right) \left(\int_{\Omega\times D} \frac{|\varphi|^2}{M} \, {\rm d}\underset{\sim}{q}\, {\rm d}\underset{\sim}{x} \right) \\ &= d\, \mathcal{M} \left(\int_{\Omega\times D} \frac{|\varphi|^2}{M} \, {\rm d}\underset{\sim}{q}\, {\rm d}\underset{\sim}{x} \right). \end{aligned} \qquad (3.13)$$

On recalling Lemma 3.1 in Barrett, Schwab & Süli [2], we have that

$$\mathcal{K}^0 := \begin{cases} M \cdot C^\infty(\overline{\Omega \times D}) & \text{if } D \text{ is a bounded open ball in } \mathbb{R}^d, \\ C_0^\infty(\Omega \times D) & \text{if } D \equiv \mathbb{R}^d \end{cases} \qquad (3.14)$$

is dense in X_q^0. It is a simple matter to adapt the proof there to show that

$$\mathcal{K} := \begin{cases} M \cdot C^\infty(\overline{\Omega \times D}) & \text{if } D \text{ is a bounded open ball in } \mathbb{R}^d, \\ C^\infty(\overline{\Omega}; C_0^\infty(D)) & \text{if } D \equiv \mathbb{R}^d \end{cases} \qquad (3.15)$$

is dense in X_q. In addition, on recalling Lemma 3.2 in Barrett, Schwab & Süli [2], we have that
(a) If D is a bounded open ball in $\mathbb{R}^d$ and the elastic potential U and the associated Maxwellian M satisfy (2.20a) with $\kappa \geq 5$ and (2.20b); then,

$$U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\, \varphi = 0 \quad \text{on } \Omega \times \partial D, \qquad \forall \varphi \in X^0. \qquad (3.16a)$$

(b) If $D = \mathbb{R}^d$; then, all for $\varphi \in X^0$,

$$\lim_{R\to\infty} R^\beta \int_{\Omega\times\partial B(\underset{\sim}{0},R)} U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\,|\varphi|\;\mathrm{d}S(\underset{\sim}{q})\;\mathrm{d}\underset{\sim}{x} = 0 \qquad \text{for all } \beta \geq 0. \tag{3.16b}$$

On recalling Lemma 3.3 in Barrett, Schwab & Süli [2], we have for any constant $L \geq 0$ that

$$\varphi \in X_q^0 \quad \Rightarrow \quad [\varphi - L\,M]_+,\ [\varphi + L\,M]_- \in X_q^0. \tag{3.17}$$

Of course, this result remains true if X_q^0 is replaced by X_q.

We note that

$$\underset{\approx}{\omega}(\underset{\sim}{v}) = -[\underset{\approx}{\omega}(\underset{\sim}{v})]^\top \ \text{ and hence } \ \underset{\sim}{q}^\top\,\underset{\approx}{\omega}(\underset{\sim}{v})\,\underset{\sim}{q} = 0 \qquad \forall \underset{\sim}{q} \in \mathbb{R}^d. \tag{3.18}$$

On recalling (4.15a,b) in Barrett, Schwab & Süli [2], it follows for all $\underset{\sim}{v} \in \underset{\sim}{W}^{1,\infty}(\Omega)$ that

$$\int_{\Omega\times D} \varphi\,(\underset{\approx}{\omega}(\underset{\sim}{v})\,\underset{\sim}{q})\cdot\underset{\sim}{\nabla}_q\left(\frac{\varphi}{M}\right)\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \;=\; 0, \tag{3.19a}$$

$$\int_{\Omega\times D} M\,(\underset{\approx}{\omega}(\underset{\sim}{v})\,\underset{\sim}{q})\cdot\underset{\sim}{\nabla}_q\left(\frac{\varphi}{M}\right)\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \;=\; 0 \qquad \forall \varphi \in X_q^0. \tag{3.19b}$$

As no smoothing is required of the extra stress tensor on the right-hand side of the Navier–Stokes equations in the corotational case, we introduce

$$\underset{\sim}{I}_\alpha := \begin{cases} \underset{\sim}{J}_\alpha^x & \text{if} \quad \underset{\approx}{\sigma}(\cdot) = \underset{\sim}{\nabla}_x\,\cdot, \\ \underset{\sim}{I} & \text{if} \quad \underset{\approx}{\sigma}(\cdot) = \underset{\approx}{\omega}(\cdot). \end{cases} \tag{3.20}$$

3.1 Existence for $(\mathbf{P}_{\alpha,\varepsilon})$

In this subsection we will prove existence of a solution to the following weak formulation of $(\mathrm{P}_{\alpha,\varepsilon})$ for given parameters $\alpha, \varepsilon \in (0,1]$:

$(\mathbf{P}_{\alpha,\varepsilon})$ Find functions

$$\underset{\sim}{u}_{\alpha,\varepsilon} \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}') \quad \text{and} \quad \psi_{\alpha,\varepsilon} \in L^2(0,T;X),$$

with

$$\underset{\sim}{J}_\alpha^x \underset{\sim}{u}_{\alpha,\varepsilon} \in L^\infty(0,T;\underset{\sim}{W}^{1,\infty}(\Omega)), \quad M^{-\frac{1}{2}}\,\psi_{\alpha,\varepsilon} \in L^\infty(0,T;L^2(\Omega\times D))$$

and

$$\underset{\approx}{C}(\psi_{\alpha,\varepsilon}) \in L^\infty(0,T;\underset{\approx}{L}^2(\Omega)),$$

such that $\underset{\sim}{u}_{\alpha,\varepsilon}(\cdot,0)=\underset{\sim}{u}_0(\cdot)$ and

$$\int_0^T \left\langle \frac{\partial \underset{\sim}{u}_{\alpha,\varepsilon}}{\partial t}, \underset{\sim}{w} \right\rangle \mathrm{d}t + \int_0^T \int_\Omega \left[[(\underset{\sim}{u}_{\alpha,\varepsilon} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u}_{\alpha,\varepsilon}] \cdot \underset{\sim}{w} + \nu \underset{\approx}{\nabla}_x \underset{\sim}{u}_{\alpha,\varepsilon} : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$= -k\,\mu \int_0^T \int_\Omega \underset{\approx}{C}(\psi_{\alpha,\varepsilon}) : \underset{\approx}{\nabla}_x (\underset{\sim}{I}_\alpha \underset{\sim}{w})\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \qquad \forall \underset{\sim}{w} \in L^{\frac{4}{4-d}}(0,T;\underset{\sim}{V}); \tag{3.21a}$$

$$-\int_0^T \int_{\Omega\times D} \frac{\psi_{\alpha,\varepsilon}}{M} \frac{\partial \varphi}{\partial t}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t - \int_{\Omega\times D} \frac{\psi_0(\cdot,\cdot)}{M}\,\varphi(\cdot,\cdot,0)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$+ \int_0^T \int_{\Omega\times D} \left[\frac{M}{2\lambda} \underset{\sim}{\nabla}_q \left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) - [\underset{\approx}{\sigma}(\underset{\sim}{J}_\alpha^x \underset{\sim}{u}_{\alpha,\varepsilon})\,\underset{\sim}{q}]\,\psi_{\alpha,\varepsilon} \right] \cdot \underset{\sim}{\nabla}_q \left(\frac{\varphi}{M}\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+ \int_0^T \int_{\Omega\times D} \left[\varepsilon\, M\, \underset{\sim}{\nabla}_x \left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) - \underset{\sim}{u}_{\alpha,\varepsilon}\,\psi_{\alpha,\varepsilon} \right] \cdot \underset{\sim}{\nabla}_x \left(\frac{\varphi}{M}\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0 \quad \forall \varphi \in \mathcal{X}; \tag{3.21b}$$

where $\mathcal{X}$ is the completion of $C_0^\infty((-T,T);\mathcal{K})$ in the norm $\|\cdot\|_{\mathcal{X}}$ defined by

$$\|\varphi\|_{\mathcal{X}} := \|\varphi\|_{L^2(0,T;X_q)} \quad + \quad \left\| M^{-\frac{1}{2}} \frac{\partial \varphi}{\partial t} \right\|_{L^1(0;T;L^2(\Omega\times D))}. \tag{3.22}$$

This, in particular, implies that each $\varphi \in \mathcal{X}$ satisfies $\varphi(\cdot,\cdot,T)=0$.

Remark 3 *If $d=2$, then $\underset{\sim}{u}_{\alpha,\varepsilon} \in C([0,T];\underset{\sim}{H})$ (cf. Lemma 1.2 on p.176 of Temam [28]), whereas if $d=3$, then $\underset{\sim}{u}_{\alpha,\varepsilon}$ is only weakly continuous as a mapping from $[0,T]$ into $\underset{\sim}{H}$ (similarly as in Theorem 3.1 on p.191 in Temam [28]). It is in the latter, weaker sense that the imposition of the initial condition to the $\underset{\sim}{u}_{\alpha,\varepsilon}$-equation will be understood for $d=2,3$: that is $\lim_{t\to 0}(\underset{\sim}{u}_{\alpha,\varepsilon}(\cdot,t),\underset{\sim}{v}(\cdot)) = (\underset{\sim}{u}_0(\cdot),\underset{\sim}{v}(\cdot))$ for all $\underset{\sim}{v} \in \underset{\sim}{H}$.*

Throughout we will assume that (2.20a,b) hold, with $\kappa \geq 5$ if D is a bounded open ball in $\mathbb{R}^d$ so that (3.16a) holds. In addition, we assume that (2.21) and (3.7) hold. In order to prove existence of these weak solutions to $(\mathrm{P}_{\alpha,\varepsilon})$, we consider a time semidiscretization. To this end, for any $T>0$, let $N\,\Delta t = T$ and $t_n = n\,\Delta t$, $n = 0 \to N$.

In order to prove existence of weak solutions under minimal smoothness requirements on the initial data, we introduce $\underset{\sim}{u}^0 \in \underset{\sim}{V}$ and $\psi^0 \in L^2(\Omega\times D; M^{-1})$ as follows:

$$\underset{\sim}{u}^0 \quad = \quad \underset{\sim}{S}_{\Delta t}\,\underset{\sim}{u}_0 \qquad \text{and} \qquad \psi^0 = \frac{\psi_0}{1+\Delta t\,|\underset{\sim}{q}|^2}. \tag{3.23}$$

It immediately follows that $\psi^0 \geq 0$ a.e. in $\Omega\times D$, and from (3.7) and (3.4a) that

$$\int_\Omega [\,|\underset{\sim}{u}^0|^2 + \Delta t\,|\underset{\approx}{\nabla}_x \underset{\sim}{u}^0|^2\,]\,\mathrm{d}\underset{\sim}{x} + \int_{\Omega\times D} (1+\Delta t\,|\underset{\sim}{q}|^2)\,\frac{|\psi^0|^2}{M}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \leq C\,. \tag{3.24}$$

In addition, we have that $\underset{\sim}{u}^0$ converges to $\underset{\sim}{u}_0$ weakly in $\underset{\sim}{H}$ and ψ^0 converges to ψ_0 weakly in $L^2(\Omega \times D; M^{-1})$ as $\Delta t \to 0$.

As seen in [2], the noncorotational case is harder to analyze than the corotational case. Therefore in some places in the analysis below, we have to distinguish between these cases. For all $\underset{\sim}{v} \in \underset{\sim}{W}^{1,\infty}(\Omega)$, we introduce

$$A(\underset{\sim}{v}) := \begin{cases} \|\underset{\approx}{\nabla}_x \underset{\sim}{v}\|^2_{L^\infty(\Omega)} & \text{if} \quad \underset{\approx}{\sigma}(\cdot) = \underset{\approx}{\nabla}_x \cdot, \\ 0 & \text{if} \quad \underset{\approx}{\sigma}(\cdot) = \underset{\approx}{\omega}(\cdot). \end{cases} \tag{3.25}$$

It follows from (3.9b), that $A(\underset{\sim}{J}^x_\alpha \underset{\sim}{v})$ is well defined for all $\underset{\sim}{v} \in \underset{\sim}{L}^1(\Omega)$.

Let $\underset{\sim}{u}^0_\alpha = \underset{\sim}{u}^0$ and $\psi^0_\alpha = \psi^0$. Then, for $n = 1 \to N$, given $\{\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}, A^{n-1}_{\alpha,\varepsilon}, \psi^{n-1}_{\alpha,\varepsilon}\} \in \underset{\sim}{V} \times \mathbb{R}^+ \times X^+_q$, where $A^{n-1}_{\alpha,\varepsilon} = A(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon})$; find $\{\underset{\sim}{u}^n_{\alpha,\varepsilon}, A(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon}), \psi^n_{\alpha,\varepsilon}, \underset{\approx}{C}(\psi^n_{\alpha,\varepsilon})\} \in \underset{\sim}{V} \times \mathbb{R}^+ \times X^+_q \times \underset{\approx}{L}^2(\Omega)$ such that

$$\int_\Omega \left[\frac{\underset{\sim}{u}^n_{\alpha,\varepsilon} - \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}}{\Delta t} + (\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{u}^n_{\alpha,\varepsilon} \right] \cdot \underset{\sim}{w} \, \mathrm{d}\underset{\sim}{x} + \nu \int_\Omega \underset{\approx}{\nabla}_x \underset{\sim}{u}^n_{\alpha,\varepsilon} : \underset{\approx}{\nabla}_x \underset{\sim}{w} \, \mathrm{d}\underset{\sim}{x}$$
$$= -k\,\mu \int_\Omega \underset{\approx}{C}(\psi^n_{\alpha,\varepsilon}) : \underset{\approx}{\nabla}_x (\underset{\sim}{I}_\alpha \underset{\sim}{w}) \, \mathrm{d}\underset{\sim}{x} \quad \forall \underset{\sim}{w} \in \underset{\sim}{V}, \tag{3.26a}$$

$$\int_{\Omega \times D} \frac{\psi^n_{\alpha,\varepsilon} - \psi^{n-1}_{\alpha,\varepsilon}}{\Delta t} \frac{\varphi}{M} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$
$$+ \int_{\Omega \times D} |\underset{\sim}{q}|^2 \left[(1 + \lambda\, A(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon}))\, \psi^n_{\alpha,\varepsilon} - (1 + \lambda\, A^{n-1}_{\alpha,\varepsilon})\, \psi^{n-1}_{\alpha,\varepsilon} \right] \frac{\varphi}{M} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$
$$+ \int_{\Omega \times D} \left[\frac{M}{2\lambda} \underset{\sim}{\nabla}_q \left(\frac{\psi^n_{\alpha,\varepsilon}}{M} \right) - [\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon})\, \underset{\sim}{q}]\, \psi^n_{\alpha,\varepsilon} \right] \cdot \underset{\sim}{\nabla}_q \left(\frac{\varphi}{M} \right) \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$
$$+ \int_{\Omega \times D} \left[\varepsilon\, M\, \underset{\sim}{\nabla}_x \left(\frac{\psi^n_{\alpha,\varepsilon}}{M} \right) - \underset{\sim}{u}^n_{\alpha,\varepsilon}\, \psi^n_{\alpha,\varepsilon} \right] \cdot \underset{\sim}{\nabla}_x \left(\frac{\varphi}{M} \right) \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} = 0 \quad \forall \varphi \in X_q. \tag{3.26b}$$

It is convenient to rewrite (3.26a) as

$$b(\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon})(\underset{\sim}{u}^n_{\alpha,\varepsilon}, \underset{\sim}{w}) = \int_\Omega \left[\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon} \cdot \underset{\sim}{w} - \Delta t\, k\, \mu\, \underset{\approx}{C}(\psi^n_{\alpha,\varepsilon}) : \underset{\approx}{\nabla}_x (\underset{\sim}{I}_\alpha \underset{\sim}{w}) \right] \mathrm{d}\underset{\sim}{x} \qquad \forall \underset{\sim}{w} \in \underset{\sim}{V}, \tag{3.27}$$

where for all $\underset{\sim}{v} \in \underset{\sim}{V}$, $\underset{\sim}{w}_i \in \underset{\sim}{H}^1_0(\Omega)$, $i = 1, 2$,

$$b(\underset{\sim}{v})(\underset{\sim}{w}_1, \underset{\sim}{w}_2) := \int_\Omega [\underset{\sim}{w}_1 + \Delta t\, (\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{w}_1] \cdot \underset{\sim}{w}_2 \, \mathrm{d}\underset{\sim}{x} + \Delta t\, \nu \int_\Omega \underset{\approx}{\nabla}_x \underset{\sim}{w}_1 : \underset{\approx}{\nabla}_x \underset{\sim}{w}_2 \, \mathrm{d}\underset{\sim}{x}. \tag{3.28}$$

It follows from (3.28) that

$$\int_\Omega [(\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{w}_1] \cdot \underset{\sim}{w}_2 \, \mathrm{d}\underset{\sim}{x} = -\int_\Omega [(\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{w}_2] \cdot \underset{\sim}{w}_1 \, \mathrm{d}\underset{\sim}{x} \qquad \forall \underset{\sim}{v} \in V, \quad \forall \underset{\sim}{w}_1, \underset{\sim}{w}_2 \in \underset{\sim}{H}^1_0(\Omega); \tag{3.29}$$

and hence $b(\underset{\sim}{v})(\cdot,\cdot)$ is a continuous and coercive bilinear functional on $\underset{\sim}{V} \times \underset{\sim}{V}$.

For $r > d$, let

$$\underset{\sim}{Y}^r := \{\underset{\sim}{v} \in \underset{\sim}{L}^r(\Omega) : \int_\Omega \underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x \, \underset{\sim}{w} \, \mathrm{d}\underset{\sim}{x} = 0 \quad \forall \underset{\sim}{w} \in \underset{\sim}{W}^{1,\frac{r}{r-1}}(\Omega)\}\,. \tag{3.30}$$

It is also convenient to rewrite (3.26b) as

$$a_{\alpha,\varepsilon}(\underset{\sim}{u}^n_{\alpha,\varepsilon})(\psi^n_{\alpha,\varepsilon},\varphi) = \ell^n_{\alpha,\varepsilon}(\varphi) \qquad \forall \varphi \in X_q; \tag{3.31}$$

where, for all φ_1, φ_2, $\varphi \in X_q$ and $\underset{\sim}{v} \in \underset{\sim}{Y}^r$,

$$\begin{aligned} a_{\alpha,\varepsilon}(\underset{\sim}{v})(\varphi_1,\varphi_2) := \int_{\Omega\times D} \Bigg(& W(\underset{\sim}{J}^x_\alpha \underset{\sim}{v})\,\varphi_1\,\varphi_2 + \Delta t \left[\varepsilon\, M\, \underset{\sim}{\nabla}_x \left(\frac{\varphi_1}{M}\right) - \underset{\sim}{v}\,\varphi_1\right] \cdot \underset{\sim}{\nabla}_x \left(\frac{\varphi_2}{M}\right) \\ & + \Delta t \left[\frac{M}{2\,\lambda}\,\underset{\sim}{\nabla}_q \left(\frac{\varphi_1}{M}\right) - [\,\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{v})\,\underset{\sim}{q}\,]\,\varphi_1\right] \cdot \underset{\sim}{\nabla}_q \left(\frac{\varphi_2}{M}\right)\Bigg) \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}, \end{aligned} \tag{3.32a}$$

$$\ell^n_{\alpha,\varepsilon}(\varphi) := \int_{\Omega\times D} W^{n-1}_{\alpha,\varepsilon}\,\psi^{n-1}_{\alpha,\varepsilon}\,\varphi \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}, \tag{3.32b}$$

and

$$W(\underset{\sim}{v}) := \frac{1 + \Delta t\,|\underset{\sim}{q}|^2\,(1 + \lambda\, A(\underset{\sim}{v})\,)}{M}, \qquad W^{n-1}_{\alpha,\varepsilon} = \frac{1 + \Delta t\,|\underset{\sim}{q}|^2\,(1 + \lambda\, A^{n-1}_{\alpha,\varepsilon}\,)}{M}\,. \tag{3.32c}$$

As $L^s(\Omega; L^2(D; M)) \subset H^1(\Omega; L^2(D; M))$, for $s \in [1,\infty)$ if $d = 2$ and $s \in [1, 6]$ if $d = 3$, it follows from (3.30) that for $r > d$

$$\int_{\Omega\times D} \underset{\sim}{v}\,\varphi \cdot \underset{\sim}{\nabla}_x \left(\frac{\varphi}{M}\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = 0 \qquad \forall \underset{\sim}{v} \in \underset{\sim}{Y}^r, \quad \forall \varphi \in X_q\,. \tag{3.33}$$

In addition, it is easily deduced from (3.25), (3.19a) and (3.12b) that $a_{\alpha,\varepsilon}(\underset{\sim}{v})(\cdot,\cdot)$ is a continuous non-symmetric bilinear functional on $X_q \times X_q$ and $\ell^n_\alpha(\cdot)$ is a linear functional on X_q. Moreover, on noting (3.33), and either (3.19a) in the corotational case or noting (3.25) and applying a Young's inequality in the noncorotational case, we see that

$$a_{\alpha,\varepsilon}(\underset{\sim}{v})(\varphi,\varphi) \geq \int_{\Omega\times D} \left[W_c\,|\varphi|^2 + \Delta t\,\varepsilon\, M\, \left|\underset{\sim}{\nabla}_x \left(\frac{\varphi}{M}\right)\right|^2 + \frac{\Delta t\, M}{4\,\lambda} \left|\underset{\sim}{\nabla}_q \left(\frac{\varphi}{M}\right)\right|^2\right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \qquad \forall \varphi \in X_q; \tag{3.34}$$

where

$$W_c := \frac{1 + \Delta t\,|\underset{\sim}{q}|^2}{M}. \tag{3.35}$$

Hence $a_{\alpha,\varepsilon}(\underset{\sim}{v})(\cdot,\cdot)$ is coercive on $X_q \times X_q$.

In order to prove existence of a solution to (3.26a,b), we consider a fixed point argument. Given $\widehat{\underset{\sim}{u}} \in \underset{\sim}{Y}^r$ with $r > d$; let $\{\psi^\star, \underset{\sim}{u}^\star\} \in X_q \times \underset{\sim}{V}$ be such that

$$a_{\alpha,\varepsilon}(\widehat{\underset{\sim}{u}})(\psi^\star, \varphi) = \ell^n_{\alpha,\varepsilon}(\varphi) \qquad \forall \varphi \in X_q, \tag{3.36a}$$

$$b(\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon})(\underset{\sim}{u}^\star, \underset{\sim}{w}) = \int_\Omega \left[\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon} \cdot \underset{\sim}{w} - \Delta t\, k\, \mu\, \underset{\approx}{C}(\psi^\star) : \underset{\approx}{\nabla}_x (\underset{\sim}{I}_\alpha \underset{\sim}{w})\right] \mathrm{d}\underset{\sim}{x} \quad \forall \underset{\sim}{w} \in \underset{\sim}{V}; \tag{3.36b}$$

where, on recalling (3.9b), $\underset{\sim}{J}^x_\alpha \widehat{\underset{\sim}{u}} \in \underset{\sim}{W}^{1,\infty}(\Omega)$.

On noting (3.34), the Lax–Milgram theorem yields the existence of a unique solution to (3.36a). On noting (3.29), there exists a unique solution to (3.36b). Therefore the overall procedure (3.36a,b) is well-posed.

Lemma 4 *Let $\underset{\sim}{G} : \underset{\sim}{Y}^r \to \underset{\sim}{V} \subset \underset{\sim}{Y}^r$, $r \in (d, 6)$, denote the nonlinear map that takes $\widehat{\underset{\sim}{u}}$ to $\underset{\sim}{u}^\star = \underset{\sim}{G}(\widehat{\underset{\sim}{u}})$ via the procedure (3.36a,b). Then $\underset{\sim}{G}$ has a fixed point. Hence there exists a solution $\{\underset{\sim}{u}^n_{\alpha,\varepsilon}, A(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon}), \psi^n_{\alpha,\varepsilon}, \underset{\approx}{C}(\psi^n_{\alpha,\varepsilon})\} \in \underset{\sim}{V} \times \mathbb{R}^+ \times X^+_q \times \underset{\approx}{L}^2(\Omega)$ to (3.26a,b).*

Proof Clearly, a fixed point of $\underset{\sim}{G}$ yields a solution of (3.26a,b). In order to show that $\underset{\sim}{G}$ has a fixed point, we apply Schauder's fixed point theorem; that is, we need to show that (i) $\underset{\sim}{G} : \underset{\sim}{Y}^r \to \underset{\sim}{Y}^r$, $r \in (d, 6)$, is continuous, (ii) compact, and (iii) there exists a $C_\star \in \mathbb{R}^+$ such that

$$\|\widehat{\underset{\sim}{u}}\|_{L^r(\Omega)} \leq C_\star \tag{3.37}$$

for every $\widehat{\underset{\sim}{u}} \in \underset{\sim}{Y}^r$ and $\beta \in (0, 1]$ satisfying $\widehat{\underset{\sim}{u}} = \beta\, \underset{\sim}{G}(\widehat{\underset{\sim}{u}})$.

Let $\{\widehat{\underset{\sim}{u}}^{(i)}\}_{i \geq 0}$ be such that

$$\widehat{\underset{\sim}{u}}^{(i)} \in \underset{\sim}{Y}^r \to \widehat{\underset{\sim}{u}} \in \underset{\sim}{Y}^r \quad \text{strongly in } \underset{\sim}{L}^r(\Omega) \quad \text{as } i \to \infty. \tag{3.38}$$

We need to show that

$$\widehat{\underset{\sim}{v}}^{(i)} := \underset{\sim}{G}(\widehat{\underset{\sim}{u}}^{(i)}) \to \underset{\sim}{G}(\widehat{\underset{\sim}{u}}) \quad \text{strongly in } \underset{\sim}{L}^r(\Omega) \quad \text{as } i \to \infty, \tag{3.39}$$

in order to prove (i) above. We have from the definition of $\underset{\sim}{G}$, see (3.36a,b), that, for all $i \geq 0$,

$$b(\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon})(\widehat{\underset{\sim}{v}}^{(i)}, \underset{\sim}{w}) = \int_\Omega \left[\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon} \cdot \underset{\sim}{w} - \Delta t\, k\, \mu\, \underset{\approx}{C}(\widehat{\psi}^{(i)}) : \underset{\approx}{\nabla}_x (\underset{\sim}{I}_\alpha \underset{\sim}{w})\right] \mathrm{d}\underset{\sim}{x} \quad \forall \underset{\sim}{w} \in \underset{\sim}{V}; \tag{3.40a}$$

where $\widehat{\psi}^{(i)} \in X_q$ satisfies

$$a_{\alpha,\varepsilon}(\widehat{\underset{\sim}{u}}^{(i)})(\widehat{\psi}^{(i)}, \varphi) = \ell^n_{\alpha,\varepsilon}(\varphi) \qquad \forall \varphi \in X_q, \tag{3.40b}$$

and from (3.9b) we have that

$$\underset{\sim}{J}^x_\alpha\, \widehat{\underset{\sim}{u}}^{(i)} \to \underset{\sim}{J}^x_\alpha\, \widehat{\underset{\sim}{u}} \quad \text{strongly in } \underset{\sim}{W}^{1,\infty}(\Omega) \quad \text{as } i \to \infty. \tag{3.40c}$$

Choosing $\underset{\sim}{w} \equiv \underset{\sim}{v}^{(i)}$ in (3.40a), and noting (3.29), (3.9a) and (3.13), yields that, for all $i \geq 0$, $\widehat{\underset{\sim}{v}}^{(i)} \in \underset{\sim}{V}$ satisfies

$$\int_\Omega \left[|\widehat{\underset{\sim}{v}}^{(i)}|^2 + |\widehat{\underset{\sim}{v}}^{(i)} - \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}|^2 - |\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}|^2\right] \mathrm{d}\underset{\sim}{x} + \Delta t\, \nu \int_\Omega |\underset{\approx}{\nabla}_x\, \widehat{\underset{\sim}{v}}^{(i)}|^2\, \mathrm{d}\underset{\sim}{x}$$
$$\leq C\, \Delta t \int_{\Omega \times D} \frac{|\widehat{\psi}^{(i)}|^2}{M}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\,; \tag{3.41}$$

where we have noted the simple identity

$$2\,(s_1 - s_2)\,s_1 = s_1^2 + (s_1 - s_2)^2 - s_2^2 \qquad \forall s_1, s_2 \in \mathbb{R}. \tag{3.42}$$

Choosing $\varphi \equiv \widehat{\psi}^{(i)}$ in (3.40b), and noting (3.34) and (3.32b,c), yields, for all $i \geq 0$, that

$$\int_{\Omega\times D} \left[W_c\,|\widehat{\psi}^{(i)}|^2 + 2\,\Delta t\,\varepsilon\,M \left| \underset{\sim}{\nabla}_x \left(\frac{\widehat{\psi}^{(i)}}{M}\right)\right|^2 + \frac{\Delta t\,M}{2\,\lambda} \left| \underset{\sim}{\nabla}_q \left(\frac{\widehat{\psi}^{(i)}}{M}\right)\right|^2 \right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$\leq (1 + \lambda\,A_{\alpha,\varepsilon}^{n-1})^2 \int_{\Omega\times D} W_c\,|\psi_{\alpha,\varepsilon}^{n-1}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \leq C(\alpha). \tag{3.43}$$

On combining (3.41) and (3.43), and noting a well-known embedding result, and a Poincaré inequality, we have for all $i \geq 0$ that

$$\|\widehat{\underset{\sim}{v}}^{(i)}\|_{L^r(\Omega)} \leq C\,\|\underset{\approx}{\nabla}_x\,\widehat{\underset{\sim}{v}}^{(i)}\|_{L^2(\Omega)} \leq C(\alpha)\,. \tag{3.44}$$

It follows from (3.43), (3.44), (3.13) and (2.4a), on noting the compactness of the embedding $\underset{\sim}{H}^1(\Omega) \hookrightarrow \underset{\sim}{L}^r(\Omega)$, $r \in (d,6)$, that there exists a subsequence $\{\widehat{\psi}^{(i_k)}, \widehat{\underset{\sim}{v}}^{(i_k)}\}_{i_k\geq 0}$ and functions $\widehat{\psi} \in X_q$ and $\widehat{\underset{\sim}{v}} \in \underset{\sim}{V}$ such that

$$W_c^{\frac{1}{2}}\,\widehat{\psi}^{(i_k)} \to W_c^{\frac{1}{2}}\,\widehat{\psi} \qquad \text{weakly in } L^2(\Omega\times D) \qquad \text{as } i_k \to \infty, \tag{3.45a}$$

$$M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_x \left(\frac{\widehat{\psi}^{(i_k)}}{M}\right) \to M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_x \left(\frac{\widehat{\psi}}{M}\right) \qquad \text{weakly in } \underset{\sim}{L}^2(\Omega\times D) \qquad \text{as } i_k \to \infty, \tag{3.45b}$$

$$M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q \left(\frac{\widehat{\psi}^{(i_k)}}{M}\right) \to M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q \left(\frac{\widehat{\psi}}{M}\right) \qquad \text{weakly in } \underset{\sim}{L}^2(\Omega\times D) \qquad \text{as } i_k \to \infty, \tag{3.45c}$$

$$\underset{\approx}{C}(\widehat{\psi}^{(i_k)}) \to \underset{\approx}{C}(\widehat{\psi}) \qquad \text{weakly in } \underset{\approx}{L}^2(\Omega) \qquad \text{as } i_k \to \infty, \tag{3.45d}$$

$$\widehat{\underset{\sim}{v}}^{(i_k)} \to \widehat{\underset{\sim}{v}} \qquad \text{weakly in } \underset{\sim}{H}^1(\Omega) \qquad \text{as } i_k \to \infty, \tag{3.45e}$$

$$\widehat{\underset{\sim}{v}}^{(i_k)} \to \widehat{\underset{\sim}{v}} \qquad \text{strongly in } \underset{\sim}{L}^r(\Omega) \qquad \text{as } i_k \to \infty. \tag{3.45f}$$

It follows from (3.40a), (3.28) and (3.45d,e), that $\widehat{\underset{\sim}{v}} \in \underset{\sim}{V}$ and $\widehat{\psi} \in X_q$ satisfy

$$b(\underset{\sim}{u}_{\alpha,\varepsilon}^{n-1})(\widehat{\underset{\sim}{v}}, \underset{\sim}{w}) = \int_\Omega \left[\underset{\sim}{u}_{\alpha,\varepsilon}^{n-1} \cdot \underset{\sim}{w} - \Delta t\,k\,\mu\,\underset{\approx}{C}(\widehat{\psi}) : \underset{\approx}{\nabla}_x\,(\underset{\sim}{I}_\alpha \underset{\sim}{w})\right] \mathrm{d}\underset{\sim}{x} \qquad \forall \underset{\sim}{w} \in \underset{\sim}{V}. \tag{3.46}$$

It follows from (3.40b,c), (3.38), (3.32a–c), (3.25) and (3.45a–c), that $\widehat{\underset{\sim}{u}} \in \underset{\sim}{Y}^r$, $\underset{\sim}{J}_\alpha^x \widehat{\underset{\sim}{u}} \in \underset{\sim}{W}^{1,\infty}(\Omega)$ and $\widehat{\psi} \in X_q$ satisfy

$$a_{\alpha,\varepsilon}(\widehat{\underset{\sim}{u}})(\widehat{\psi}, \varphi) = \ell_{\alpha,\varepsilon}^n(\varphi) \qquad \forall \varphi \in X_q. \tag{3.47}$$

Combining (3.47) and (3.46), we have that $\widehat{\underset{\sim}{v}} = \underset{\sim}{G}(\widehat{\underset{\sim}{u}}) \in \underset{\sim}{V}$. Therefore the whole sequence $\widehat{\underset{\sim}{v}}^{(i)} \equiv \underset{\sim}{G}(\widehat{\underset{\sim}{u}}^{(i)}) \to \underset{\sim}{G}(\widehat{\underset{\sim}{u}})$ strongly in $\underset{\sim}{L}^r(\Omega)$ as $i \to \infty$, and so (i) holds.

As the embedding $\underset{\sim}{V} \hookrightarrow \underset{\sim}{L}^r(\Omega)$, $r \in (d,6)$, is compact; it follows that (ii) holds.

As regards (iii), $\widehat{\underset{\sim}{u}} = \beta\,\underset{\sim}{G}(\widehat{\underset{\sim}{u}})$ implies that $\{\widehat{\psi}, \widehat{\underset{\sim}{u}}\} \in X_q \times \underset{\sim}{V}$ satisfies

$$a_{\alpha,\varepsilon}(\widehat{\underset{\sim}{u}})(\widehat{\psi}, \varphi) = \ell_{\alpha,\varepsilon}^n(\varphi) \qquad \forall \varphi \in X_q, \tag{3.48a}$$

$$b(\underset{\sim}{u}_{\alpha,\varepsilon}^{n-1})(\widehat{\underset{\sim}{u}}, \underset{\sim}{w}) = \beta \int_\Omega \left[\underset{\sim}{u}_{\alpha,\varepsilon}^{n-1} \cdot \underset{\sim}{w} - \Delta t\,k\,\mu\,\underset{\approx}{C}(\widehat{\psi}) : \underset{\approx}{\nabla}_x\,(\underset{\sim}{I}_\alpha \underset{\sim}{w})\right] \mathrm{d}\underset{\sim}{x} \quad \forall \underset{\sim}{w} \in \underset{\sim}{V}. \tag{3.48b}$$

On choosing $\underset{\sim}{w} \equiv \widehat{\underset{\sim}{u}}$ in (3.48b) yields, similarly to (3.41), that

$$\tfrac{1}{2} \int_{\Omega} \left[|\widehat{\underset{\sim}{u}}|^2 + |\widehat{\underset{\sim}{u}} - \beta\, \underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}|^2 - \beta^2\, |\underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}|^2 \right] \,\mathrm{d}\underset{\sim}{x} + \Delta t\, \nu \int_{\Omega} |\underset{\approx}{\nabla}_x\, \widehat{\underset{\sim}{u}}|^2 \,\mathrm{d}\underset{\sim}{x}$$
$$= -\Delta t\, \beta\, k\, \mu \int_{\Omega} \underset{\approx}{C}(\widehat{\psi}) : \underset{\approx}{\nabla}_x\, (\underset{\sim}{I}_{\alpha} \widehat{\underset{\sim}{u}}) \,\mathrm{d}\underset{\sim}{x} \leq C\, \Delta t \int_{\Omega \times D} \frac{|\widehat{\psi}|^2}{M} \,\mathrm{d}\underset{\sim}{q} \,\mathrm{d}\underset{\sim}{x}. \quad (3.49)$$

On choosing $\varphi = \widehat{\psi}$ in (3.48a) yields, similarly to (3.43) that

$$\int_{\Omega \times D} \left[W_c\, |\widehat{\psi}|^2 + 2\, \Delta t\, \varepsilon\, M \left| \underset{\sim}{\nabla}_x \left(\frac{\widehat{\psi}}{M} \right) \right|^2 + \frac{\Delta t\, M}{2\, \lambda} \left| \underset{\sim}{\nabla}_q \left(\frac{\widehat{\psi}}{M} \right) \right|^2 \right] \,\mathrm{d}\underset{\sim}{q} \,\mathrm{d}\underset{\sim}{x}$$
$$\leq (1 + \lambda\, A_{\alpha,\varepsilon}^{n-1})^2 \int_{\Omega \times D} W_c\, |\psi_{\alpha,\varepsilon}^{n-1}|^2 \,\mathrm{d}\underset{\sim}{q} \,\mathrm{d}\underset{\sim}{x} \leq C(\alpha). \quad (3.50)$$

Combining (3.49) and (3.50), and noting the embedding $\underset{\sim}{V} \hookrightarrow \underset{\sim}{L}^r(\Omega)$ gives rise to the desired bound (3.37) with C dependent on Δt and α. Hence (iii) holds and so $\underset{\sim}{G}$ has a fixed point. Finally, as $\psi_{\alpha,\varepsilon}^{n-1} \in X_q^+$ and $\psi_{\alpha,\varepsilon}^{n} \in X_q \implies [\psi_{\alpha,\varepsilon}^{n}]_- \in X_q$, recall (3.17), it follows from (3.31) and (3.32a,b) that

$$a_{\alpha,\varepsilon}(\underset{\sim}{u}_{\alpha,\varepsilon}^{n})([\psi_{\alpha,\varepsilon}^{n}]_-, [\psi_{\alpha,\varepsilon}^{n}]_-) = a_{\alpha,\varepsilon}(\underset{\sim}{u}_{\alpha,\varepsilon}^{n})(\psi_{\alpha,\varepsilon}^{n}, [\psi_{\alpha,\varepsilon}^{n}]_-) = \ell_{\alpha,\varepsilon}^{n}([\psi_{\alpha,\varepsilon}]_-) \leq 0. \quad (3.51)$$

Therefore (3.34) yields that $[\psi_{\alpha,\varepsilon}^{n}]_- = 0$; that is, $\psi_{\alpha,\varepsilon}^{n} \in X_q^+$. Thus we have proved existence of a solution to (3.26a,b). ■

On choosing $\underset{\sim}{w} \equiv \underset{\sim}{u}_{\alpha,\varepsilon}^{n}$ in (3.27) yields, similarly to (3.49), that

$$\tfrac{1}{2} \int_{\Omega} \left[|\underset{\sim}{u}_{\alpha,\varepsilon}^{n}|^2 + |\underset{\sim}{u}_{\alpha,\varepsilon}^{n} - \underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}|^2 - |\underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}|^2 \right] \,\mathrm{d}\underset{\sim}{x} + \Delta t\, \nu \int_{\Omega} |\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\alpha,\varepsilon}^{n}|^2 \,\mathrm{d}\underset{\sim}{x}$$
$$= -\Delta t\, k\, \mu \int_{\Omega} \underset{\approx}{C}(\psi_{\alpha,\varepsilon}^{n}) : \underset{\approx}{\nabla}_x\, (\underset{\sim}{I}_{\alpha} \underset{\sim}{u}_{\alpha,\varepsilon}^{n}) \,\mathrm{d}\underset{\sim}{x} \leq C\, \Delta t \int_{\Omega \times D} \frac{|\psi_{\alpha,\varepsilon}^{n}|^2}{M} \,\mathrm{d}\underset{\sim}{q} \,\mathrm{d}\underset{\sim}{x}. \quad (3.52)$$

On choosing $\underset{\sim}{w} \equiv \underset{\sim}{S}_{\gamma} \left(\frac{\underset{\sim}{u}_{\alpha,\varepsilon}^{n} - \underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}}{\Delta t} \right) \in \underset{\sim}{V}$ in (3.27) yields, on noting (3.3), (3.20), (3.9a) and (3.29), that

$$\int_{\Omega} \left[\gamma \left| \underset{\approx}{\nabla}_x \left[\underset{\sim}{S}_{\gamma} \left(\frac{\underset{\sim}{u}_{\alpha,\varepsilon}^{n} - \underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}}{\Delta t} \right) \right] \right|^2 + \left| \underset{\sim}{S}_{\gamma} \left(\frac{\underset{\sim}{u}_{\alpha,\varepsilon}^{n} - \underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}}{\Delta t} \right) \right|^2 \right] \,\mathrm{d}\underset{\sim}{x}$$
$$= -\nu \int_{\Omega} \underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\alpha,\varepsilon}^{n} : \underset{\approx}{\nabla}_x \left[\underset{\sim}{S}_{\gamma} \left(\frac{\underset{\sim}{u}_{\alpha,\varepsilon}^{n} - \underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}}{\Delta t} \right) \right] \,\mathrm{d}\underset{\sim}{x}$$
$$+ \int_{\Omega} \underset{\approx}{C}(\psi_{\alpha,\varepsilon}^{n}) : \underset{\approx}{\nabla}_x \left(\underset{\sim}{I}_{\alpha} \left[\underset{\sim}{S}_{\gamma} \left(\frac{\underset{\sim}{u}_{\alpha,\varepsilon}^{n} - \underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}}{\Delta t} \right) \right] \right) \,\mathrm{d}\underset{\sim}{x}$$
$$+ \int_{\Omega} \underset{\sim}{u}_{\alpha,\varepsilon}^{n} \cdot \left[(\underset{\sim}{u}_{\alpha,\varepsilon}^{n-1} \cdot \underset{\sim}{\nabla}_x) \left[\underset{\sim}{S}_{\gamma} \left(\frac{\underset{\sim}{u}_{\alpha,\varepsilon}^{n} - \underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}}{\Delta t} \right) \right] \right] \,\mathrm{d}\underset{\sim}{x}$$
$$\leq C \int_{\Omega} \left[|\underset{\approx}{C}(\psi_{\alpha,\varepsilon}^{n})|^2 + |\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\alpha,\varepsilon}^{n}|^2 + |\underset{\sim}{u}_{\alpha,\varepsilon}^{n-1}|^2\, |\underset{\sim}{u}_{\alpha,\varepsilon}^{n}|^2 \right] \,\mathrm{d}\underset{\sim}{x}. \quad (3.53)$$

Applying the Cauchy–Schwarz and the algebraic-geometric mean inequalities, and a Gagliardo–Nirenberg inequality yields that

$$\begin{aligned}\int_\Omega |\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}|^2\,|\underset{\sim}{u}^{n}_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x} &\leq \left(\int_\Omega |\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}|^4\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}} \left(\int_\Omega |\underset{\sim}{u}^{n}_{\alpha,\varepsilon}|^4\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}} \\ &\leq \tfrac{1}{2} \sum_{m=n-1}^{n} \int_\Omega |\underset{\sim}{u}^{m}_{\alpha,\varepsilon}|^4\,\mathrm{d}\underset{\sim}{x} \\ &\leq C \sum_{m=n-1}^{n} \left[\left(\int_\Omega |\underset{\sim}{u}^{m}_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x}\right)^{2-\frac{d}{2}} \left(\int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{u}^{m}_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{d}{2}}\right]. \quad (3.54)\end{aligned}$$

First we consider the corotational case for (3.31), where we can exploit (3.19a). On choosing $\varphi = \psi^n_{\alpha,\varepsilon}$ in (3.31) and noting (3.42), (3.25), (3.33) and (3.19a), yields that

$$\begin{aligned}&\int_{\Omega\times D} W_c\,\left[|\psi^n_{\alpha,\varepsilon}|^2 + |\psi^n_{\alpha,\varepsilon} - \psi^{n-1}_{\alpha,\varepsilon}|^2\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + 2\,\Delta t\,\varepsilon \int_{\Omega\times D} M \left|\underset{\sim}{\nabla}_x \left(\frac{\psi^n_{\alpha,\varepsilon}}{M}\right)\right|^2 \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ &\quad + \frac{\Delta t}{\lambda} \int_{\Omega\times D} M \left|\underset{\sim}{\nabla}_q \left(\frac{\psi^n_{\alpha,\varepsilon}}{M}\right)\right|^2 \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \int_{\Omega\times D} W_c\,|\psi^{n-1}_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}. \quad (3.55)\end{aligned}$$

Summing (3.52) and (3.55) from $n = 1 \to m$, with $1 \leq m \leq N$, and noting (3.24) yields that

$$\begin{aligned}&\max_{n=0\to N}\left[\int_{\Omega\times D} W_c\,|\psi^n_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \frac{1}{\lambda} \sum_{n=1}^{N} \Delta t \int_{\Omega\times D} M \left|\underset{\sim}{\nabla}_q \left(\frac{\psi^n_{\alpha,\varepsilon}}{M}\right)\right|^2 \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ &\quad + \varepsilon \sum_{n=1}^{N} \Delta t \int_{\Omega\times D} M \left|\underset{\sim}{\nabla}_x \left(\frac{\psi^n_{\alpha,\varepsilon}}{M}\right)\right|^2 \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + \sum_{n=1}^{N} \int_{\Omega\times D} W_c\,|\psi^n_{\alpha,\varepsilon} - \psi^{n-1}_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ &\qquad + \max_{n=0\to N}\left[\int_\Omega |\underset{\approx}{C}(\psi^n_{\alpha,\varepsilon})|^2\,\mathrm{d}\underset{\sim}{x}\right] \\ &\qquad\qquad \leq C \int_{\Omega\times D} W_c\,|\psi^0|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \leq C\,; \quad (3.56a)\end{aligned}$$

$$\begin{aligned}&\max_{n=0\to N}\left[\int_\Omega |\underset{\sim}{u}^n_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x}\right] + \sum_{n=1}^{N} \int_\Omega |\underset{\sim}{u}^n_{\alpha,\varepsilon} - \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x} + \nu \sum_{n=1}^{N} \Delta t \int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{u}^n_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x} \\ &\qquad\qquad \leq C \int_\Omega |\underset{\sim}{u}^0|^2\,\mathrm{d}\underset{\sim}{x} + C\,T \int_{\Omega\times D} W_c\,|\psi^0|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \leq C(T). \quad (3.56b)\end{aligned}$$

In addition, taking the $\frac{2}{d}$ power of both sides of (3.53), summing from $n = 1 \to N$

and noting (3.54), (3.56a,b) and (3.24) yields that

$$\sum_{n=1}^{N} \Delta t \left(\int_\Omega \left[\gamma \left| \underset{\approx}{\nabla}_x \left[\underset{\sim}{S}_\gamma \left(\frac{\underset{\sim}{u}^n_{\alpha,\varepsilon} - \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}}{\Delta t} \right) \right] \right|^2 + \left| \underset{\sim}{S}_\gamma \left(\frac{\underset{\sim}{u}^n_{\alpha,\varepsilon} - \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}}{\Delta t} \right) \right|^2 \right] \mathrm{d}\underset{\sim}{x} \right)^{\frac{2}{d}}$$

$$\leq C \left[\sum_{n=1}^{N} \Delta t \left(\int_\Omega |\underset{\approx}{C}(\psi^n_{\alpha,\varepsilon})|^2 \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{2}{d}} \right] + C(T) \left[\sum_{n=1}^{N} \Delta t \int_\Omega |\underset{\approx}{\nabla}_x \, \underset{\sim}{u}^n_{\alpha,\varepsilon}|^2 \, \mathrm{d}\underset{\sim}{x} \right]^{\frac{2}{d}}$$

$$+ C(T) \left[\max_{n=0 \to N} \left(\int_\Omega |\underset{\sim}{u}^n_{\alpha,\varepsilon}|^2 \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{4}{d}-1} \right] \left[\sum_{n=0}^{N} \Delta t \int_\Omega |\underset{\approx}{\nabla}_x \, \underset{\sim}{u}^n_{\alpha,\varepsilon}|^2 \, \mathrm{d}\underset{\sim}{x} \right]$$

$$\leq C(T). \tag{3.57}$$

We now consider the noncorotational case for (3.31), where at first we have to use a different testing procedure to $\varphi = \psi^n_{\alpha,\varepsilon}$ as used in the corotational case. Choosing $\varphi = M$ in (3.31) yields that

$$\int_{\Omega \times D} [\, 1 + \Delta t \, |\underset{\sim}{q}|^2 \, (1 + \lambda \, A(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon}) \,) \,] \, \psi^n_{\alpha,\varepsilon} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$

$$= \int_{\Omega \times D} [\, 1 + \Delta t \, |\underset{\sim}{q}|^2 \, (1 + \lambda \, A^{n-1}_{\alpha,\varepsilon}) \,] \, \psi^{n-1}_{\alpha,\varepsilon} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}. \tag{3.58}$$

Choosing $\varphi = U \, M$ in (3.31), and noting (2.10), (2.4a), (3.16a,b), (2.9) and (2.19) yields that

$$\int_{\Omega \times D} (\psi^n_{\alpha,\varepsilon} - \psi^{n-1}_{\alpha,\varepsilon}) \, U \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} - \Delta t \int_\Omega \underset{\approx}{C}(\psi^n_{\alpha,\varepsilon}) : \underset{\approx}{\nabla}_x \, (\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon}) \, \mathrm{d}\underset{\sim}{x}$$

$$+\Delta t \int_{\Omega \times D} |\underset{\sim}{q}|^2 \, \left[\, (1 + \lambda \, A(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon}) \,) \, \psi^n_{\alpha,\varepsilon} - (\, 1 + \lambda \, A^{n-1}_{\alpha,\varepsilon}) \, \psi^{n-1}_{\alpha,\varepsilon} \, \right] \, U \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$

$$= -\frac{\Delta t}{2\,\lambda} \int_{\Omega \times D} M \, \underset{\sim}{\nabla}_q \left(\frac{\psi^n_{\alpha,\varepsilon}}{M} \right) \cdot U' \, \underset{\sim}{q} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$

$$= \frac{\Delta t}{2\,\lambda} \int_{\Omega \times D} \left[\left(U'' - (U')^2 \right) \, |\underset{\sim}{q}|^2 + d \, U' \right] \, \psi^n_{\alpha,\varepsilon} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$

$$\leq \frac{-\Delta t \, c_2}{2\,\lambda} \int_{\Omega \times \{|\underset{\sim}{q}|^2 \geq \frac{d}{c_2}\}} |\underset{\sim}{q}|^2 \, U' \, \psi^n_{\alpha,\varepsilon} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} + \frac{\Delta t \, d}{2\,\lambda} \int_{\Omega \times \{|\underset{\sim}{q}|^2 \leq \frac{d}{c_2}\}} U' \, \psi^n_{\alpha,\varepsilon} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$

$$\leq \frac{-\Delta t \, c_2}{2\,\lambda} \int_{\Omega \times \{|\underset{\sim}{q}|^2 \geq \frac{d}{c_2}\}} |\underset{\sim}{q}|^2 \, U' \, \psi^n_{\alpha,\varepsilon} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$

$$+ C \, \Delta t \int_{\Omega \times D} \left[1 + \Delta t \, |\underset{\sim}{q}|^2 \, (1 + \lambda \, A(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon}) \,) \right] \, \psi^n_{\alpha,\varepsilon} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}. \tag{3.59}$$

Combining (3.52) in the noncorotational case and (3.59) multiplied by $k \, \mu$, and noting (3.58) yields that

$$\begin{aligned}
&\tfrac{1}{2} \int_\Omega \left[\, |\underset{\sim}{u}^n_{\alpha,\varepsilon}|^2 + |\underset{\sim}{u}^n_{\alpha,\varepsilon} - \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}|^2 \,\right] \mathrm{d}\underset{\sim}{x} + \Delta t\, \nu \int_\Omega |\underset{\approx}{\nabla}_x\, \underset{\sim}{u}^n_{\alpha,\varepsilon}|^2 \,\mathrm{d}\underset{\sim}{x} \\
&\qquad + k\,\mu \int_{\Omega\times D} \left[1 + \Delta t\, |\underset{\sim}{q}|^2 \,(1 + \lambda\, A(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon})\,) \right] U\, \psi^n_{\alpha,\varepsilon} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&\qquad\qquad + \frac{\Delta t\, k\,\mu\, c_2}{2\,\lambda} \int_{\Omega\times\{|\underset{\sim}{q}|^2 \geq \frac{d}{c_2}\}} |\underset{\sim}{q}|^2\, U'\, \psi^n_{\alpha,\varepsilon} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&\leq \tfrac{1}{2} \int_\Omega |\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}|^2 \,\mathrm{d}\underset{\sim}{x} + k\,\mu \int_{\Omega\times D} \left[1 + \Delta t\, |\underset{\sim}{q}|^2 \,(1 + \lambda\, A^{n-1}_{\alpha,\varepsilon}\,) \right] U\, \psi^{n-1}_{\alpha,\varepsilon} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&\quad + C\, k\,\mu\,\Delta t \int_{\Omega\times D} \left[1 + \Delta t\, |\underset{\sim}{q}|^2 \,(1 + \lambda\, A^{n-1}_{\alpha,\varepsilon}\,) \right] \psi^{n-1}_{\alpha,\varepsilon} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}. \qquad (3.60)
\end{aligned}$$

Summing this from $n = 1 \to m$, $m = 1 \to N$, and noting by induction on (3.58) that

$$\begin{aligned}
&\int_{\Omega\times D} \left[1 + \Delta t\, |\underset{\sim}{q}|^2 \,(1 + \lambda\, A^{n-1}_{\alpha,\varepsilon}) \right] \psi^{n-1}_{\alpha,\varepsilon} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&\qquad\qquad = \int_{\Omega\times D} \left[1 + \Delta t\, |\underset{\sim}{q}|^2 \,(1 + \lambda\, A^0_{\alpha,\varepsilon}) \right] \psi^0 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}, \qquad (3.61)
\end{aligned}$$

and (3.24) yields that

$$\begin{aligned}
&\max_{n=0\to N} \left[\int_\Omega |\underset{\sim}{u}^n_{\alpha,\varepsilon}|^2 \,\mathrm{d}\underset{\sim}{x} \right] + \sum_{n=1}^N \int_\Omega |\underset{\sim}{u}^n_{\alpha,\varepsilon} - \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}|^2 \,\mathrm{d}\underset{\sim}{x} \\
&\quad + \nu \sum_{n=1}^N \Delta t \int_\Omega |\underset{\approx}{\nabla}_x\, \underset{\sim}{u}^n_{\alpha,\varepsilon}|^2 \,\mathrm{d}\underset{\sim}{x} + k\,\mu \max_{n=1\to N} \left[\int_{\Omega\times D} U\, \psi^n_{\alpha,\varepsilon} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right] \\
&\quad + \frac{k\,\mu\, c_2}{2\,\lambda} \sum_{n=1}^N \Delta t \int_{\Omega\times\{|\underset{\sim}{q}|^2 \geq \frac{d}{c_2}\}} |\underset{\sim}{q}|^2\, U'\, \psi^n_{\alpha,\varepsilon} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&\quad + k\,\mu\,\Delta t \max_{n=1\to N} \left[\int_{\Omega\times D} |\underset{\sim}{q}|^2 \,(1 + \lambda\, A(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^n_{\alpha,\varepsilon})\,)\, U\, \psi^n_{\alpha,\varepsilon} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right] \\
&\leq C \int_\Omega |\underset{\sim}{u}^0|^2 \,\mathrm{d}\underset{\sim}{x} + C(T) \int_{\Omega\times D} \left[1 + \Delta t\, |\underset{\sim}{q}|^2 \,(1 + \lambda\, A^0_{\alpha,\varepsilon}) \right] (1 + U)\, \psi^0 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&\leq C(T). \qquad (3.62)
\end{aligned}$$

The bounds on $\psi^n_{\alpha,\varepsilon}$ in (3.62) for the noncorotational case do not suffice in order to pass to the limit $\Delta t \to 0$ in the summation over n of (3.26b). One needs to establish additional bounds on $\psi^n_{\alpha,\varepsilon}$. We confine ourselves to the physically more realistic case of FENE-type models, i.e. D bounded, for the general noncorotational case.

It follows from (3.25), (3.9b) and (3.62) that

$$\sum_{n=1}^{N} \Delta t\, A^{n-1}_{\alpha,\varepsilon} = \sum_{n=1}^{N} \Delta t\, \|\underset{\approx}{\nabla}_x (\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon})\|^2_{L^\infty(\Omega)} \leq C(\alpha) \sum_{n=1}^{N} \Delta t\, \|\underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}\|^2_{L^2(\Omega)} \leq C_1(\alpha, T)\,. \tag{3.63}$$

Choosing $\varphi = \psi^n_{\alpha,\varepsilon}$ in (3.31) and noting (3.32c), (3.35) and (3.34) yields that

$$\begin{aligned}\int_{\Omega\times D} W_c\, \psi^n_{\alpha,\varepsilon}\, (\psi^n_{\alpha,\varepsilon} - \psi^{n-1}_{\alpha,\varepsilon})\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ + \Delta t\, \varepsilon \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_x \left(\frac{\psi^n_{\alpha,\varepsilon}}{M} \right) \right|^2 \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} + \frac{\Delta t}{4\,\lambda} \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_q \left(\frac{\psi^n_{\alpha,\varepsilon}}{M} \right) \right|^2 \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ \leq \Delta t\, \lambda\, A^{n-1}_{\alpha,\varepsilon} \int_{\Omega\times D} \frac{|\underset{\sim}{q}|^2}{M}\, \psi^n_{\alpha,\varepsilon}\, \psi^{n-1}_{\alpha,\varepsilon}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}.\end{aligned} \tag{3.64}$$

Applying the identity (3.42) and a Young's inequality to (3.64), and noting that D is bounded and (3.63), with $C_1 \equiv C_1(\alpha, T)$, yields that

$$\begin{aligned}(1 - \tfrac{1}{2}\, C_1^{-1}\, \Delta t\, A^{n-1}_{\alpha,\varepsilon}) \int_{\Omega\times D} W_c\, |\psi^n_{\alpha,\varepsilon}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} + \int_{\Omega\times D} W_c\, |\psi^n_{\alpha,\varepsilon} - \psi^{n-1}_{\alpha,\varepsilon}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ + 2\, \Delta t\, \varepsilon \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_x \left(\frac{\psi^n_{\alpha,\varepsilon}}{M} \right) \right|^2 \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} + \frac{\Delta t}{2\,\lambda} \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_q \left(\frac{\psi^n_{\alpha,\varepsilon}}{M} \right) \right|^2 \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ \leq (1 + C_2\, \Delta t\, A^{n-1}_{\alpha,\varepsilon}) \int_{\Omega\times D} W_c\, |\psi^{n-1}_{\alpha,\varepsilon}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x},\end{aligned} \tag{3.65}$$

where $C_2(\alpha, T)$. It follows from (3.65) and (3.63) that

$$\begin{aligned}\int_{\Omega\times D} W_c\, |\psi^n_{\alpha,\varepsilon}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} &\leq \frac{1 + C_2\, \Delta t\, A^{n-1}_{\alpha,\varepsilon}}{1 - \frac{1}{2}\, C_1^{-1}\, \Delta t\, A^{n-1}_{\alpha,\varepsilon}} \int_{\Omega\times D} W_c\, |\psi^{n-1}_{\alpha,\varepsilon}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ &\leq \mathrm{e}^{C(\alpha,T)\, \Delta t\, A^{n-1}_{\alpha,\varepsilon}} \int_{\Omega\times D} W_c\, |\psi^{n-1}_{\alpha,\varepsilon}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}.\end{aligned} \tag{3.66}$$

Hence combining (3.66) and (3.63), summing (3.65) from $n = 1 \rightarrow N$, and noting (3.13) and (3.24) yields the bounds (3.56a) for the general noncorotational FENE model; in particular:

$$\begin{aligned}\max_{n=0\rightarrow N} \left[\int_{\Omega\times D} W_c\, |\psi^n_{\alpha,\varepsilon}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \right] + \varepsilon \sum_{n=1}^{N} \Delta t \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_x \left(\frac{\psi^n_{\alpha,\varepsilon}}{M} \right) \right|^2 \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ + \frac{1}{\lambda} \sum_{n=1}^{N} \Delta t \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_q \left(\frac{\psi^n_{\alpha,\varepsilon}}{M} \right) \right|^2 \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} + \sum_{n=1}^{N} \int_{\Omega\times D} W_c\, |\psi^n_{\alpha,\varepsilon} - \psi^{n-1}_{\alpha,\varepsilon}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ + \max_{n=0\rightarrow N} \left[\int_{\Omega} |\underset{\approx}{C}(\psi^n_{\alpha,\varepsilon})|^2\, \mathrm{d}\underset{\sim}{x} \right] \leq C(\alpha, T).\end{aligned} \tag{3.67}$$

Finally, taking the $\frac{2}{d}$ power of both sides of (3.53), summing from $n = 1 \rightarrow N$ and noting (3.54), (3.62), (3.67) and (3.24) yields, similarly to (3.57), that

$$\sum_{n=1}^{N} \Delta t \left(\int_{\Omega} \left[\gamma \left| \underset{\approx}{\nabla}_x \left[\underset{\sim}{S}_\gamma \left(\frac{\underset{\sim}{u}^n_{\alpha,\varepsilon} - \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}}{\Delta t} \right) \right] \right|^2 + \left| \underset{\sim}{S}_\gamma \left(\frac{\underset{\sim}{u}^n_{\alpha,\varepsilon} - \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}}{\Delta t} \right) \right|^2 \right] \mathrm{d}\underset{\sim}{x} \right)^{\frac{2}{d}} \\ \leq C(\alpha, T). \tag{3.68}$$

We have now established all of the analogues of the bounds (3.56a,b) and (3.57) in the corotational case for the general noncorotational FENE-type potentials, see (3.62), (3.67) and (3.68) above. The key difference is that the corotational bounds are independent of $\alpha > 0$, whereas the noncorotational bounds (3.67) and (3.68) are α dependent.

Finally, we note that in the corotational case one can derive an upper bound, in addition to the zero lower bound, on $\psi^n_{\alpha,\varepsilon}$. To do so, we proceed inductively. Assuming that for some $L^{n-1} \in \mathbb{R}^+$, $\psi^{n-1}_{\alpha,\varepsilon} \leq L^{n-1}\, M$ a.e. in $\Omega \times D$, we then determine $L^n \in \mathbb{R}^+$ in terms of L^{n-1} such that $\psi^n_{\alpha,\varepsilon} \leq L^n\, M$ a.e. in $\Omega \times D$. Now, from (3.17), (3.31), (3.32a,b), (3.19b) and (3.30), we have, for any $L^n \in \mathbb{R}^+$, that $[\psi^n_{\alpha,\varepsilon} - L^n\, M]_+ \in X_q$ and

$$\begin{aligned} & a_{\alpha,\varepsilon}(\underset{\sim}{u}^n_{\alpha,\varepsilon})([\psi^n_{\alpha,\varepsilon} - L^n\, M]_+, [\psi^n_{\alpha,\varepsilon} - L^n\, M]_+) \\ &\qquad = a_{\alpha,\varepsilon}(\underset{\sim}{u}^n_{\alpha,\varepsilon})(\psi^n_{\alpha,\varepsilon}, [\psi^n_{\alpha,\varepsilon} - L^n\, M]_+) - L^n\, a_{\alpha,\varepsilon}(\underset{\sim}{u}^n_{\alpha,\varepsilon})(M, [\psi^n_{\alpha,\varepsilon} - L^n\, M]_+) \\ &\qquad = \ell^n_{\alpha,\varepsilon}([\psi^n_{\alpha,\varepsilon} - L^n\, M]_+) - L^n\, a^n_c(M, [\psi^n_{\alpha,\varepsilon} - L^n\, M]_+) \\ &\qquad = \int_{\Omega \times D} W_c\, (\psi^{n-1}_{\alpha,\varepsilon} - L^n\, M\,)\, [\psi^n_{\alpha,\varepsilon} - L^n\, M]_+ \,\mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ &\qquad \leq \int_{\Omega \times D} \left[W_c\, (L^{n-1} - L^n\,)\, M \right]\, [\psi^n_{\alpha,\varepsilon} - L^n\, M]_+ \,\mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}. \end{aligned} \tag{3.69}$$

On choosing $L^n = L^{n-1}$ yields that the right-hand side of (3.69) is zero and hence from (3.34) that $[\psi^n_{\alpha,\varepsilon} - L^n\, M]_+ \equiv 0$. Thus, by induction, we have for $n = 1 \rightarrow N$ that

$$0 \leq \psi^n_{\alpha,\varepsilon} \leq L^n\, M = L^0\, M \quad \text{a.e. in } \Omega \times D, \quad \text{where} \quad L^0 := \sup_{(\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega \times D} \frac{\psi^0(\underset{\sim}{x}, \underset{\sim}{q})}{M(\underset{\sim}{q})}. \tag{3.70}$$

Now we introduce some definitions prior to passing to the limit $\Delta t \rightarrow 0_+$. Let

$$\underset{\sim}{u}^{\Delta t}_{\alpha,\varepsilon}(\cdot, t) := \frac{t - t_{n-1}}{\Delta t}\, \underset{\sim}{u}^n_{\alpha,\varepsilon}(\cdot) + \frac{t_n - t}{\Delta t}\, \underset{\sim}{u}^{n-1}_{\alpha,\varepsilon}(\cdot), \qquad t \in [t_{n-1}, t_n], \quad n \geq 1, \tag{3.71a}$$

and

$$\underset{\sim}{u}^{\Delta t,+}_{\alpha,\varepsilon}(\cdot, t) := \underset{\sim}{u}^n(\cdot), \qquad \underset{\sim}{u}^{\Delta t,-}_{\alpha,\varepsilon}(\cdot, t) := \underset{\sim}{u}^{n-1}(\cdot), \qquad t \in (t_{n-1}, t_n], \quad n \geq 1. \tag{3.71b}$$

We note for future reference that

$$\underset{\sim}{u}^{\Delta t}_{\alpha,\varepsilon} - \underset{\sim}{u}^{\Delta t,\pm}_{\alpha,\varepsilon} = (t - t^{\pm}_n)\, \frac{\partial \underset{\sim}{u}^{\Delta t}_{\alpha,\varepsilon}}{\partial t}, \qquad t \in (t_{n-1}, t_n), \quad n \geq 1, \tag{3.72}$$

where $t_n^+ := t_n$ and $t_n^- := t_{n-1}$. Using the above notation, and introducing analogous notation for and $\{\psi_{\alpha,\varepsilon}^n\}_{n=0}^N$, (3.27) summed for $n = 1 \to N$ can be restated as:

$$\int_0^T \left\langle \frac{\partial \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t}}{\partial t}, \underset{\sim}{w} \right\rangle \mathrm{d}t + \int_0^T \int_\Omega \left[\left[(\underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,-} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+} \right] \cdot \underset{\sim}{w} + \nu \underset{\approx}{\nabla}_x \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+} : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$

$$= -k\,\mu \int_0^T \int_\Omega \underset{\approx}{C}(\psi_{\alpha,\varepsilon}^{\Delta t,+}) : \underset{\approx}{\nabla}_x (\underset{\sim}{I}_\alpha \underset{\sim}{w})\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \qquad \forall \underset{\sim}{w} \in L^{\frac{4}{4-d}}(0,T;\underset{\sim}{V}). \qquad (3.73)$$

Similarly, (3.31) summed for $n = 1 \to N$ can be restated as:

$$\int_0^T \int_{\Omega\times D} W_c \frac{\psi_{\alpha,\varepsilon}^{\Delta t,+} - \psi_{\alpha,\varepsilon}^{\Delta t,-}}{\Delta t} \varphi \,\mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$+ \int_0^T \int_{\Omega\times D} \left[\frac{M}{2\,\lambda} \underset{\sim}{\nabla}_q \left(\frac{\psi_{\alpha,\varepsilon}^{\Delta t,+}}{M} \right) - [\underset{\approx}{\sigma}(\underset{\sim}{J}_\alpha^x \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+})\, \underset{\sim}{q}]\, \psi_{\alpha,\varepsilon}^{\Delta t,+} \right] \cdot \underset{\sim}{\nabla}_q \left(\frac{\varphi}{M} \right) \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$+ \int_0^T \int_{\Omega\times D} \left[\varepsilon\, M \underset{\sim}{\nabla}_x \left(\frac{\psi_{\alpha,\varepsilon}^{\Delta t,+}}{M} \right) - \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+}\, \psi_{\alpha,\varepsilon}^{\Delta t,+} \right] \cdot \underset{\sim}{\nabla}_x \left(\frac{\varphi}{M} \right) \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$+ \int_0^T \int_{\Omega\times D} \frac{\lambda\,|\underset{\sim}{q}|^2}{M} \left[A(\underset{\sim}{J}_\alpha^x \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+})\, \psi_{\alpha,\varepsilon}^{\Delta t,+} - A(\underset{\sim}{J}_\alpha^x \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,-})\, \psi_{\alpha,\varepsilon}^{\Delta t,-} \right] \varphi \,\mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t = 0$$
$$\forall \varphi \in L^2(0,T;X_q). \qquad (3.74)$$

We have from (3.56b) and (3.62) that

$$\sup_{t\in(0,T)} \left[\int_\Omega |\underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t(,\pm)}|^2 \,\mathrm{d}\underset{\sim}{x} \right] + \int_0^T \int_\Omega \frac{|\underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+} - \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,-}|^2}{\Delta t} \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$+ \nu \int_0^T \int_\Omega |\underset{\approx}{\nabla}_x \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t(,\pm)}|^2 \,\mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \le C(T). \qquad (3.75)$$

In the above, the notation $\underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t(,\pm)}$ means $\underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t}$ with or without the superscripts $\pm$. Similarly, we have from (3.56a), (3.57), (3.67) and (3.68) that

$$\sup_{t\in(0,T)} \left[\int_{\Omega\times D} \frac{|\psi_{\alpha,\varepsilon}^{\Delta t(,\pm)}|^2}{M} \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \right] + \frac{1}{\lambda} \int_0^T \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_q \left(\frac{\psi_{\alpha,\varepsilon}^{\Delta t,+}}{M} \right) \right|^2 \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$+ \varepsilon \int_0^T \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_x \left(\frac{\psi_{\alpha,\varepsilon}^{\Delta t,+}}{M} \right) \right|^2 \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t + \sup_{t\in(0,T)} \left[\int_\Omega |\underset{\approx}{C}(\psi_{\alpha,\varepsilon}^{\Delta t(,\pm)})|^2 \,\mathrm{d}\underset{\sim}{x} \right]$$
$$+ \int_0^T \left[\int_{\Omega\times D} W_c \frac{|\psi_{\alpha,\varepsilon}^{\Delta t,+} - \psi_{\alpha,\varepsilon}^{\Delta t,-}|^2}{\Delta t} \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} + \left\| \underset{\approx}{S}_\gamma \frac{\partial \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t}}{\partial t} \right\|_{H^1(\Omega)}^{\frac{4}{d}} \right] \mathrm{d}t$$
$$\le \begin{cases} C(T) & \text{if} \quad \underset{\approx}{\sigma}(\cdot) = \underset{\approx}{\omega}(\cdot), \\ C(\alpha,T) & \text{if} \quad \underset{\approx}{\sigma}(\cdot) = \underset{\approx}{\nabla}_x \,\cdot\, \text{ and } D \text{ is bounded.} \end{cases} \qquad (3.76)$$

We are now in a position to prove the following convergence result.

Lemma 5 *There exists a subsequence of* $\{\underset{\sim}{u}^{\Delta t}_{\alpha,\varepsilon}, \psi^{\Delta t}_{\alpha,\varepsilon}\}_{\Delta t}$, *and functions* $\underset{\sim}{u}_{\alpha,\varepsilon} \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}')$ *and* $\psi_{\alpha,\varepsilon} \in L^2(0,T;X^+)$ *with* $M^{-\frac{1}{2}}\psi_{\alpha,\varepsilon} \in L^\infty(0,T;L^2(\Omega\times D))$, *such that, as* $\Delta t \to 0$,

$$
\begin{aligned}
\frac{\psi^{\Delta t(,\pm)}_{\alpha,\varepsilon}}{M^{\frac{1}{2}}} &\to \frac{\psi_{\alpha,\varepsilon}}{M^{\frac{1}{2}}} && \textit{weak* in } L^\infty(0,T;L^2(\Omega\times D)), && (3.77a)\\
M^{\frac{1}{2}}\underset{\sim}{\nabla}_q\left(\frac{\psi^{\Delta t,+}_{\alpha,\varepsilon}}{M}\right) &\to M^{\frac{1}{2}}\underset{\sim}{\nabla}_q\left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) && \textit{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), && (3.77b)\\
M^{\frac{1}{2}}\underset{\sim}{\nabla}_x\left(\frac{\psi^{\Delta t,+}_{\alpha,\varepsilon}}{M}\right) &\to M^{\frac{1}{2}}\underset{\sim}{\nabla}_x\left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) && \textit{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), && (3.77c)\\
\underset{\approx}{C}(\psi^{\Delta t(,\pm)}_{\alpha,\varepsilon}) &\to \underset{\approx}{C}(\psi_{\alpha,\varepsilon}) && \textit{weak* in } L^\infty(0,T;\underset{\approx}{L}^2(\Omega)); && (3.77d)
\end{aligned}
$$

and

$$
\begin{aligned}
\underset{\sim}{u}^{\Delta t(,\pm)}_{\alpha,\varepsilon} &\to \underset{\sim}{u}_{\alpha,\varepsilon} && \textit{weak* in } L^\infty(0,T;\underset{\sim}{L}^2(\Omega)), && (3.78a)\\
\underset{\sim}{u}^{\Delta t(,\pm)}_{\alpha,\varepsilon} &\to \underset{\sim}{u}_{\alpha,\varepsilon} && \textit{weakly in } L^2(0,T;\underset{\sim}{V}), && (3.78b)\\
\underset{\sim}{S}_\gamma\frac{\partial \underset{\sim}{u}^{\Delta t}_{\alpha,\varepsilon}}{\partial t} &\to \underset{\sim}{S}_\gamma\frac{\partial \underset{\sim}{u}_{\alpha,\varepsilon}}{\partial t} && \textit{weakly in } L^{\frac{4}{d}}(0,T;\underset{\sim}{V}), && (3.78c)\\
\underset{\sim}{u}^{\Delta t(,\pm)}_{\alpha,\varepsilon} &\to \underset{\sim}{u}_{\alpha,\varepsilon} && \textit{strongly in } L^2(0,T;\underset{\sim}{L}^r(\Omega)), && (3.78d)\\
\underset{\sim}{J}^x_\alpha \underset{\sim}{u}^{\Delta t(,\pm)}_{\alpha,\varepsilon} &\to \underset{\sim}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon} && \textit{strongly in } L^\infty(0,T;\underset{\sim}{W}^{1,\infty}(\Omega)); && (3.78e)
\end{aligned}
$$

where $r \in [1,\infty)$ *if* $d = 2$ *and* $r \in [1,6)$ *if* $d = 3$.

Proof The result (3.77a) follows immediately from the bounds on the first and third terms on the left-hand side of (3.76), on noting (3.35) and the notation (3.71a,b).

It follows immediately from the bound on the second term on the left-hand side of (3.76) that (3.77b) holds for some limit $\underset{\sim}{g} \in L^2(0,T;\underset{\sim}{L}^2(\Omega\times D))$, which we need to identify. However for any $\underset{\sim}{\eta} \in L^2(0,T;\underset{\sim}{C}^\infty_0(\Omega\times D))$, it follows from (2.9) and the compact support of $\underset{\sim}{\eta}$ on D that $[\underset{\sim}{\nabla}_q\cdot(M^{\frac{1}{2}}\underset{\sim}{\eta})]/M^{\frac{1}{2}} \in L^2(0,T;L^2(\Omega\times D))$ and hence the above convergence implies, on noting (3.77a), that

$$
\begin{aligned}
\int_0^T\int_{\Omega\times D}\underset{\sim}{g}\cdot\underset{\sim}{\eta}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t &\leftarrow -\int_0^T\int_{\Omega\times D}\frac{\psi^{\Delta t,+}_{\alpha,\varepsilon}}{M^{\frac{1}{2}}}\,\frac{\underset{\sim}{\nabla}_q\cdot(M^{\frac{1}{2}}\underset{\sim}{\eta})}{M^{\frac{1}{2}}}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&\to -\int_0^T\int_{\Omega\times D}\frac{\psi_{\alpha,\varepsilon}}{M^{\frac{1}{2}}}\,\frac{\underset{\sim}{\nabla}_q\cdot(M^{\frac{1}{2}}\underset{\sim}{\eta})}{M^{\frac{1}{2}}}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \qquad (3.79)
\end{aligned}
$$

as $\Delta t \to 0$. Hence the desired result (3.77b) follows from (3.79) on noting the denseness of $C^\infty_0(\Omega\times D)$ in $L^2(\Omega\times D)$. A similar argument also proves (3.77c). The desired result (3.77d) follows immediately from (3.77a), and (2.4a). Finally, the non-negativity of $\psi_{\alpha,\varepsilon}$ follows from that of $\psi^{\Delta t(,\pm)}_{\alpha,\varepsilon}$, recall Lemma 4.

The results (3.78a–c) follow immediately from the bounds (3.75) and the bound on $\underset{\sim}{u}^{\Delta t}_{\alpha,\varepsilon}$ in (3.76). The strong convergence result (3.78d) for $\underset{\sim}{u}^{\Delta t}_{\alpha,\varepsilon}$ follows immediately from (3.78a–c), (3.3) and a standard compactness result, on noting that $\underset{\sim}{V} \subset \underset{\sim}{H}^1_0(\Omega)$ is compactly embedded in $\underset{\sim}{L}^r(\Omega)$ for the stated values of r. We now prove (3.78d) for $\underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,\pm}$. First we obtain from the bound on the second term on the left-hand side of (3.75) and (3.72) that

$$\|\underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t} - \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,\pm}\|^2_{L^2(0,T,L^2(\Omega))} \leq C\,\Delta t\,. \tag{3.80}$$

Second, we note from Sobolev embedding that, for all $\eta \in L^2(0,T;H^1(\Omega))$,

$$\|\eta\|_{L^2(0,T;L^r(\Omega))} \leq \|\eta\|^{\beta}_{L^2(0,T;L^2(\Omega))}\,\|\eta\|^{1-\beta}_{L^2(0,T;L^s(\Omega))} \leq C\,\|\eta\|^{\beta}_{L^2(0,T;L^2(\Omega))}\,\|\eta\|^{1-\beta}_{L^2(0,T;H^1(\Omega))} \tag{3.81}$$

for all $r \in [2,s)$, with any $s \in (2,\infty)$ if $d = 2$ or any $s \in (2,6]$ if $d = 3$, and

$$\beta = \frac{2\,(s-r)}{r\,(s-2)} \in (0,1].$$

Hence, combining (3.80), (3.81) and (3.78d) for $\underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t}$ yields (3.78d) for $\underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,\pm}$. Finally, the desired result (3.78e) follows immediately from (3.9b) and (3.78d). ∎

It follows from (3.78a–d), (3.77d) and (3.2) that we may pass to the limit, $\Delta t \to 0$, in (3.73) to obtain that $\underset{\sim}{u}_{\alpha,\varepsilon} \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}')$ and $\underset{\approx}{C}(\psi_{\alpha,\varepsilon}) \in L^\infty(0,T;\underset{\approx}{L}^2(\Omega))$ satisfy (3.21a). It also follows from (3.23) that $\underset{\sim}{u}_{\alpha,\varepsilon}(\cdot,0) = \underset{\sim}{u}_0(\cdot)$ in the required sense.

As we have no control of the time derivative $\psi_{\alpha,\varepsilon}^{\Delta t}$, in order to pass to the $\Delta t \to 0$ limit in (3.74) this derivative has to be transferred to the test function. We have for any fixed $\varphi \in C_0^\infty((-T,T);\mathcal{K})$ that

$$\begin{aligned}
&\int_0^T \int_{\Omega\times D} W_c\, \frac{\psi_{\alpha,\varepsilon}^{\Delta t,+}(\underset{\sim}{x},\underset{\sim}{q},t) - \psi_{\alpha,\varepsilon}^{\Delta t,-}(\underset{\sim}{x},\underset{\sim}{q},t)}{\Delta t}\, \varphi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\
&\qquad = -\int_0^T \int_{\Omega\times D} W_c\, \psi_{\alpha,\varepsilon}^{\Delta t,-}(\underset{\sim}{x},\underset{\sim}{q},t) \frac{\varphi(\underset{\sim}{x},\underset{\sim}{q},t) - \varphi(\underset{\sim}{x},\underset{\sim}{q},t-\Delta t)}{\Delta t}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\
&\qquad\quad - \int_{\Omega\times D} W_c\, \psi^0(\underset{\sim}{x},\underset{\sim}{q}) \left(\frac{1}{\Delta t}\int_0^{t_1} \varphi(\underset{\sim}{x},\underset{\sim}{q},t-\Delta t)\,\mathrm{d}t\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,.
\end{aligned} \tag{3.82}$$

It follows for all $\varphi \in C_0^\infty((-T,T);\mathcal{K})$ and for all $(\underset{\sim}{x},\underset{\sim}{q},t) \in \Omega \times D \times (0,T)$ that

$$\frac{\varphi(\underset{\sim}{x},\underset{\sim}{q},t) - \varphi(\underset{\sim}{x},\underset{\sim}{q},t-\Delta t)}{\Delta t} = \frac{\partial \varphi}{\partial t}(\underset{\sim}{x},\underset{\sim}{q},t) + R_{\Delta t}(\varphi)(\underset{\sim}{x},\underset{\sim}{q},t)\,,$$

where

$$|R_{\Delta t}(\varphi)(\underset{\sim}{x},\underset{\sim}{q},t)| \leq \frac{\Delta t}{2} \max_{(\underset{\sim}{x},\underset{\sim}{q},t)\in\overline{\Omega}\times D\times[-T,T]} \left|\frac{\partial^2\varphi}{\partial t^2}(\underset{\sim}{x},\underset{\sim}{q},t)\right|. \tag{3.83}$$

Hence, on combining (3.74), (3.82) and (3.83), we have for any fixed function φ

contained in $C_0^\infty((-T,T);\mathcal{K})$ that

$$
\begin{aligned}
&-\int_0^T \int_{\Omega\times D} W_c\, \psi_{\alpha,\varepsilon}^{\Delta t,-} \left[\frac{\partial\varphi}{\partial t} + R_{\Delta t}(\varphi)\right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\
&-\int_{\Omega\times D} W_c\, \psi^0(\underset{\sim}{x},\underset{\sim}{q}) \left(\frac{1}{\Delta t}\int_0^{t_1} \varphi(\underset{\sim}{x},\underset{\sim}{q},t-\Delta t)\,\mathrm{d}t\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&+\int_0^T \int_{\Omega\times D} \left[\frac{M}{2\,\lambda}\underset{\sim}{\nabla}_q \left(\frac{\psi_{\alpha,\varepsilon}^{\Delta t,+}}{M}\right) - [\underset{\approx}{\sigma}(\underset{\sim}{J}_\alpha^x \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+})\,\underset{\sim}{q}]\,\psi_{\alpha,\varepsilon}^{\Delta t,+}\right] \cdot \underset{\sim}{\nabla}_q \left(\frac{\varphi}{M}\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\
&+\int_0^T \int_{\Omega\times D} \left[\varepsilon\, M\, \underset{\sim}{\nabla}_x \left(\frac{\psi_{\alpha,\varepsilon}^{\Delta t,+}}{M}\right) - \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+}\,\psi_{\alpha,\varepsilon}^{\Delta t,+}\right] \cdot \underset{\sim}{\nabla}_x \left(\frac{\varphi}{M}\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\
&+\int_0^T \int_{\Omega\times D} \frac{\lambda\,|\underset{\sim}{q}|^2}{M} \left[A(\underset{\sim}{J}_\alpha^x\, \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+})\,\psi_{\alpha,\varepsilon}^{\Delta t,+} - A(\underset{\sim}{J}_\alpha^x\, \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,-})\,\psi_{\alpha,\varepsilon}^{\Delta t,-}\right] \varphi\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0\,. \qquad (3.84)
\end{aligned}
$$

Now, similarly to (3.82), we have from (3.83), (3.76), (3.25), (3.9b) and (3.75) for any $\varphi \in C_0^\infty((-T,T);\mathcal{K})$ that

$$
\begin{aligned}
&\left|\int_0^T \int_{\Omega\times D} \frac{|\underset{\sim}{q}|^2}{M} \left[A(\underset{\sim}{J}_\alpha^x\, \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,+})\,\psi_{\alpha,\varepsilon}^{\Delta t,+} - A(\underset{\sim}{J}_\alpha^x\, \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,-})\,\psi_{\alpha,\varepsilon}^{\Delta t,-}\right] \varphi\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}t\right| \\
&\qquad = \Delta t \left|\int_0^T \int_{\Omega\times D} \frac{|\underset{\sim}{q}|^2}{M} A(\underset{\sim}{J}_\alpha^x\, \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,-})\,\psi_{\alpha,\varepsilon}^{\Delta t,-} \left[\frac{\partial\varphi}{\partial t} + R_{\Delta t}(\varphi)\right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right. \\
&\qquad\quad \left. + \int_{\Omega\times D} \frac{|\underset{\sim}{q}|^2}{M} A(\underset{\sim}{J}_\alpha^x\, \underset{\sim}{u}^0)\,\psi^0(\underset{\sim}{x},\underset{\sim}{q}) \left(\frac{1}{\Delta t}\int_0^{t_1} \varphi(\underset{\sim}{x},\underset{\sim}{q},t-\Delta t)\,\mathrm{d}t\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right| \\
&\qquad \leq C(\varphi)\,\Delta t \int_0^T A(\underset{\sim}{J}_\alpha^x\, \underset{\sim}{u}_{\alpha,\varepsilon}^{\Delta t,-})\,\mathrm{d}t \leq C(\varphi)\,\Delta t\,. \qquad (3.85)
\end{aligned}
$$

It follows from (3.77a–c), (3.78d,e), (3.83), (3.85), (3.35) and (3.23) that we may pass to the limit $\Delta t \to 0$ in (3.84) to obtain that $\psi_{\alpha,\varepsilon} \in L^2(0,T;X)$ with $M^{-\frac{1}{2}}\,\psi_{\alpha,\varepsilon} \in L^\infty(0,T;L^2(\Omega\times D))$ and $\underset{\sim}{u}_{\alpha,\varepsilon} \in L^2(0,T;\underset{\sim}{V})$ satisfy

$$
\begin{aligned}
&-\int_0^T \int_{\Omega\times D} \frac{\psi_{\alpha,\varepsilon}}{M}\frac{\partial\varphi}{\partial t}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t - \int_{\Omega\times D} \frac{\psi_0(\cdot,\cdot)}{M}\,\varphi(\cdot,\cdot,0)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&\quad+\int_0^T \int_{\Omega\times D} \left[\frac{M}{2\,\lambda}\underset{\sim}{\nabla}_q \left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) - [\underset{\approx}{\omega}(\underset{\sim}{J}_\alpha^x \underset{\sim}{u}_{\alpha,\varepsilon})\,\underset{\sim}{q}]\,\psi_{\alpha,\varepsilon}\right] \cdot \underset{\sim}{\nabla}_q \left(\frac{\varphi}{M}\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\
&\quad+\int_0^T \int_{\Omega\times D} \left[\varepsilon\, M\, \underset{\sim}{\nabla}_x \left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) - \underset{\sim}{u}_{\alpha,\varepsilon}\,\psi_{\alpha,\varepsilon}\right] \cdot \underset{\sim}{\nabla}_x \left(\frac{\varphi}{M}\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0 \\
&\qquad\qquad\qquad\qquad \forall \varphi \in C_0^\infty((-T,T);\mathcal{K})\,. \qquad (3.86)
\end{aligned}
$$

Noting that $C_0^\infty((-T,T);\mathcal{K})$ is a dense subset of $\mathcal{X}$, recall (3.22), it follows that (3.86) remains true for all $\varphi \in \mathcal{X}$. Hence we have proved existence of a global weak solution of $(\mathrm{P}_{\alpha,\varepsilon})$, (3.21a,b). Moreover, it follows from (3.75), (3.76), (3.77a–d) and (3.78a–c) that

$$\sup_{t\in(0,T)}\left[\int_\Omega |\underset{\sim}{u}_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x}\right]+\nu\int_0^T\int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\alpha,\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\le C(T); \tag{3.87a}$$

$$\begin{aligned}
&\sup_{t\in(0,T)}\left[\int_{\Omega\times D}\frac{|\psi_{\alpha,\varepsilon}|^2}{M}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right]+\frac{1}{\lambda}\int_0^T\int_{\Omega\times D} M\left|\underset{\sim}{\nabla}_q\left(\frac{\psi_{\alpha,\varepsilon}}{M}\right)\right|^2\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&\quad+\varepsilon\int_0^T\int_{\Omega\times D} M\left|\underset{\sim}{\nabla}_x\left(\frac{\psi_{\alpha,\varepsilon}}{M}\right)\right|^2\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t+\sup_{t\in(0,T)}\left[\int_\Omega|\underset{\approx}{C}(\psi_{\alpha,\varepsilon})|^2\,\mathrm{d}\underset{\sim}{x}\right]\\
&\quad+\int_0^T\left\|\underset{\sim}{S}_\gamma\frac{\partial\underset{\sim}{u}_{\alpha,\varepsilon}}{\partial t}\right\|_{H^1(\Omega)}^{\frac{4}{d}}\mathrm{d}t\\
&\quad\le\begin{cases} C(T) & \text{if}\quad \underset{\approx}{\sigma}(\cdot)=\underset{\approx}{\omega}(\cdot),\\ C(\alpha,T) & \text{if}\quad \underset{\approx}{\sigma}(\cdot)=\underset{\sim}{\nabla}_x\,\cdot\ \text{ and } D \text{ is bounded.}\end{cases}
\end{aligned}\tag{3.87b}$$

Remark 6 *In the corotational case if L^0 is finite in (3.70), then we have a uniform $L^\infty(0,T;\,L^\infty(\Omega\times D))$ bound on $M^{-1}\,\psi_{\alpha,\varepsilon}^{\Delta(,\pm)}$. Moreover, it is then easily established that the limit $M^{-1}\,\psi_{\alpha,\varepsilon}\in L^\infty(0,T;L^\infty(\Omega\times D))$ with $\psi_{\alpha,\varepsilon}\ge 0$ a.e. on $\Omega\times D\times(0,T)$.*

Remark 7 *The argument presented above for noncorotational FENE-type models breaks down for noncorotational Hookean models, since in the transition from bound (3.64) to (3.65) we exploit the fact that D is bounded. The difficulty could be overcome if one could obtain a maximum principle on ψ^n along the lines of (3.69). Unfortunately, in the case of $D=\mathbb{R}^d$ this does not appear to be readily achievable. Having said this, our main focus of interest in the present article is FENE-type microscopic-macroscopic models for diluted polymers where D is a bounded open ball in $\mathbb{R}^d$: for, the fact that in Hookean-type models the domain D is equal to the whole of $\mathbb{R}^d$ stems from the physically unrealistic modelling assumption that the length $|\underset{\sim}{q}|$ of the elongation-vector $\underset{\sim}{q}\in D$ of a polymer chain may be arbitrarily large.*

Remark 8 *Since the test functions in $\underset{\sim}{V}$ are divergence-free, the pressure has been eliminated in (3.21a,b); it can be recovered in a very weak sense following the same procedure as for the incompressible Navier–Stokes equations discussed on p.208 in Temam [28]; i.e., one obtains that $\int_0^t p_{\alpha,\varepsilon}(\cdot,s)\,\mathrm{d}s\in C([0,T];L^2(\Omega))$.*

Remark 9 *It is a simple matter to adapt the proofs above to show that the main results above remain true for $(P_{\alpha,\varepsilon})$, if we replace the smoothing procedure $\underset{\sim}{J}^x_\alpha$ by $\underset{\sim}{S}_\alpha$, (including the definition of $\underset{\sim}{I}_\alpha$ in (3.20)). The key results, (3.8b) and (3.9a,b), that we exploit for $\underset{\sim}{J}^x_\alpha$ in the above, are now replaced by (3.4b) and (3.6b) for $\underset{\sim}{S}_\alpha$. Unfortunately, we require $\partial\Omega\in C^2$ for (3.6b), as opposed to $\partial\Omega\in C^{0,1}$ for (3.9b). Hence our preference, in general, for $\underset{\sim}{J}^x_\alpha$ over $\underset{\sim}{S}_\alpha$. However, $\underset{\sim}{S}_\alpha$ does have one key advantage over $\underset{\sim}{J}^x_\alpha$ in that $\underset{\sim}{S}_\alpha\underset{\sim}{v}\in\underset{\sim}{H}^1_0(\Omega)$ if $\underset{\sim}{v}\in\underset{\sim}{H}^1_0(\Omega)$.*

3.2 Existence for (P_α)

As the bounds (3.87a,b) are independent of the parameter ε, it follows immediately, similarly to (3.77a–d), (3.78a–d), and (3.87a,b), that the following lemma holds.

Lemma 10 *There exists a subsequence of $\{\underset{\sim}{u}_{\alpha,\varepsilon}, \psi_{\alpha,\varepsilon}\}_\varepsilon$, and functions $\underset{\sim}{u}_\alpha$ contained in $L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}')$ and ψ_α contained in $L^2(0,T;X^{0,+})$ with $M^{-\frac{1}{2}}\,\psi_\alpha \in L^\infty(0,T;L^2(\Omega\times D))$, such that, as $\varepsilon \to 0$,*

$$\begin{aligned}
\frac{\psi_{\alpha,\varepsilon}}{M^{\frac{1}{2}}} &\to \frac{\psi_\alpha}{M^{\frac{1}{2}}} && \textit{weak* in } L^\infty(0,T;L^2(\Omega\times D)), && (3.88a)\\
M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q\left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) &\to M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q\left(\frac{\psi_\alpha}{M}\right) && \textit{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), && (3.88b)\\
\varepsilon\, M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_x\left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) &\to 0 && \textit{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), && (3.88c)\\
\underset{\approx}{C}(\psi_{\alpha,\varepsilon}) &\to \underset{\approx}{C}(\psi_\alpha) && \textit{weak* in } L^\infty(0,T;\underset{\approx}{L}^2(\Omega)); && (3.88d)
\end{aligned}$$

and

$$\begin{aligned}
\underset{\sim}{u}_{\alpha,\varepsilon} &\to \underset{\sim}{u}_\alpha && \textit{weak* in } L^\infty(0,T;\underset{\sim}{L}^2(\Omega)), && (3.89a)\\
\underset{\sim}{u}_{\alpha,\varepsilon} &\to \underset{\sim}{u}_\alpha && \textit{weakly in } L^2(0,T;\underset{\sim}{V}), && (3.89b)\\
\underset{\approx}{S}_\gamma \frac{\partial \underset{\sim}{u}_{\alpha,\varepsilon}}{\partial t} &\to \underset{\approx}{S}_\gamma \frac{\partial \underset{\sim}{u}_\alpha}{\partial t} && \textit{weakly in } L^{\frac{4}{d}}(0,T;\underset{\sim}{V}), && (3.89c)\\
\underset{\sim}{u}_{\alpha,\varepsilon} &\to \underset{\sim}{u}_\alpha && \textit{strongly in } L^2(0,T;\underset{\sim}{L}^r(\Omega)), && (3.89d)\\
\underset{\approx}{J}^x_\alpha \underset{\sim}{u}_{\alpha,\varepsilon} &\to \underset{\approx}{J}^x_\alpha \underset{\sim}{u}_\alpha && \textit{strongly in } L^\infty(0,T;\underset{\sim}{W}^{1,\infty}(\Omega)); && (3.89e)
\end{aligned}$$

where $r \in [1,\infty)$ if $d=2$ and $r \in [1,6)$ if $d=3$.

In addition, we have that

$$\sup_{t\in(0,T)}\left[\int_\Omega |\underset{\sim}{u}_\alpha|^2\,\mathrm{d}\underset{\sim}{x}\right] + \nu\int_0^T\int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_\alpha|^2\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \le C(T); \qquad (3.90a)$$

$$\begin{aligned}
&\sup_{t\in(0,T)}\left[\int_{\Omega\times D}\frac{|\psi_\alpha|^2}{M}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \frac{1}{\lambda}\int_0^T\int_{\Omega\times D} M\left|\underset{\sim}{\nabla}_q\left(\frac{\psi_\alpha}{M}\right)\right|^2 \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&\qquad + \sup_{t\in(0,T)}\left[\int_\Omega |\underset{\approx}{C}(\psi_\alpha)|^2\,\mathrm{d}\underset{\sim}{x}\right] + \int_0^T\left\|\underset{\approx}{S}_\gamma\frac{\partial\underset{\sim}{u}_\alpha}{\partial t}\right\|_{H^1(\Omega)}^{\frac{4}{d}}\mathrm{d}t\\
&\le \begin{cases} C(T) & \textit{if} \quad \underset{\approx}{g}(\cdot) = \underset{\approx}{\omega}(\cdot),\\ C(\alpha,T) & \textit{if} \quad \underset{\approx}{g}(\cdot) = \underset{\approx}{\nabla}_x\,\cdot \ \ \textit{and } D \ \ \textit{is bounded.}\end{cases} \qquad (3.90b)
\end{aligned}$$

Therefore in both the corotational and the noncorotational cases we can then pass to limit $\varepsilon \to 0$ in $(\mathrm{P}_{\alpha,\varepsilon})$ to obtain existence of a weak solution to the following problem for a given $\alpha \in (0,1]$:

(P$_\alpha$) Find functions

$$\underset{\sim}{u}_\alpha \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}') \quad \text{and} \quad \psi_\alpha \in L^2(0,T;X^0),$$

with

$$\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_\alpha \in L^\infty(0,T;\underset{\sim}{W}^{1,\infty}(\Omega)), \quad M^{-\frac{1}{2}}\,\psi_\alpha \in L^\infty(0,T;L^2(\Omega \times D))$$

and

$$\underset{\approx}{C}(\psi_\alpha) \in L^\infty(0,T;\underset{\approx}{L}^2(\Omega)),$$

such that $\underset{\sim}{u}_\alpha(\cdot,0) = \underset{\sim}{u}_0(\cdot)$ and

$$\int_0^T \left\langle \frac{\partial \underset{\sim}{u}_\alpha}{\partial t}, \underset{\sim}{w} \right\rangle \mathrm{d}t + \int_0^T \int_\Omega \left[[(\underset{\sim}{u}_\alpha \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u}_\alpha] \cdot \underset{\sim}{w} + \nu\, \underset{\approx}{\nabla}_x \underset{\sim}{u}_\alpha : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$= -k\,\mu \int_0^T \int_\Omega \underset{\approx}{C}(\psi_\alpha) : \underset{\approx}{\nabla}_x (\underset{\sim}{I}_\alpha \underset{\sim}{w})\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \qquad \forall \underset{\sim}{w} \in L^{\frac{4}{4-d}}(0,T;\underset{\sim}{V}); \tag{3.91a}$$

$$- \int_0^T \int_{\Omega\times D} \frac{\psi_\alpha}{M} \frac{\partial \varphi}{\partial t}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t - \int_{\Omega\times D} \frac{\psi_0(\cdot,\cdot)}{M}\,\varphi(\cdot,\cdot,0)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$+ \int_0^T \int_{\Omega\times D} \left[\frac{M}{2\,\lambda} \underset{\sim}{\nabla}_q \left(\frac{\psi_\alpha}{M}\right) - [\,\underset{\approx}{\sigma}(\underset{\sim}{J}^x_\alpha \underset{\sim}{u}_\alpha)\,\underset{\sim}{q}\,]\,\psi_\alpha \right] \cdot \underset{\sim}{\nabla}_q \left(\frac{\varphi}{M}\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$- \int_0^T \int_{\Omega\times D} \underset{\sim}{u}_\alpha\,\psi_\alpha \cdot \underset{\sim}{\nabla}_x \left(\frac{\varphi}{M}\right) \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0 \qquad \forall \varphi \in \mathcal{X}^0; \tag{3.91b}$$

where $\mathcal{X}^0$ is the completion of $C_0^\infty((-T,T);\mathcal{K}^0)$ in the norm $\|\cdot\|_{\mathcal{X}^0}$ defined by

$$\|\varphi\|_{\mathcal{X}^0} := \|\varphi\|_{L^2(0,T;X^0_q)} + \|M^{-\frac{1}{2}} \underset{\sim}{\nabla}_x \varphi\|_{L^2(0,T;L^2(\Omega\times D))} + \left\| M^{-\frac{1}{2}} \frac{\partial \varphi}{\partial t} \right\|_{L^1(0;T;L^2(\Omega\times D))}. \tag{3.92}$$

This, in particular, implies that each $\varphi \in \mathcal{X}^0$ satisfies $\varphi(\cdot,\cdot,T) = 0$.

Remark 11 *In view of Remark 6, in the corotational case if L^0 is finite in (3.70), then one can show that $M^{-1}\,\psi_\alpha \in L^\infty(0,T;L^\infty(\Omega \times D))$ with $\psi_\alpha \geq 0$ a.e. on $\Omega \times D \times (0,T)$. Hence the norm $\|\cdot\|_{\mathcal{X}^0}$ in (3.92) can be relaxed to the weaker norm $\|\varphi\|_{L^2(0,T;X^0_q)} + \|\underset{\sim}{\nabla}_x \varphi\|_{L^1(0,T;L^{r'}(\Omega;L^1(D)))} + \|\frac{\partial\varphi}{\partial t}\|_{L^1(0;T;L^1(\Omega\times D))}$, where $r' > 1$ if $d = 2$ and $r' > \frac{6}{5}$ if $d = 3$.*

Remark 12 *Although we have introduced smoothing through the parameter $\alpha > 0$ into the model (P$_\alpha$) compared to the standard polymer model, (P$_0$); we wish to stress that in both the corotational and the noncorotational case the bounds on $\underset{\sim}{u}_\alpha$, the variable of real physical interest, in (3.90a) are independent of this smoothing parameter α.*

3.3 Existence for (P_ε) in the Corotational Case

Finally, in the corotational case we can pass to the limit $\alpha \to 0$ in ($\mathrm{P}_{\alpha,\varepsilon}$) with $\underset{\approx}{J}{}^x_\alpha$ replaced by $\underset{\approx}{S}_\alpha$ to obtain a weak formulation of (P_ε). Hence, on recalling Remark 9, we require $\partial\Omega \in C^2$ to obtain the existence result, and the bounds (3.87a,b), for ($\mathrm{P}_{\alpha,\varepsilon}$) with $\underset{\approx}{J}{}^x_\alpha$ replaced by $\underset{\approx}{S}_\alpha$. As the bounds (3.87a,b) are independent of α in the corotational case, we obtain immediately the following lemma.

Lemma 13 *Let $\partial\Omega \in C^2$ and $\underset{\approx}{\sigma}(\cdot) = \underset{\approx}{\omega}(\cdot)$. Then there exists a subsequence of $\{\underset{\sim}{u}_{\alpha,\varepsilon}, \psi_{\alpha,\varepsilon}\}_\alpha$, and functions $\underset{\sim}{u}_\varepsilon \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}')$ and $\psi_\varepsilon \in L^2(0,T;X^+)$ with $M^{-\frac{1}{2}}\,\psi_\varepsilon \in L^\infty(0,T;L^2(\Omega\times D))$, such that, as $\alpha \to 0$,*

$$\frac{\psi_{\alpha,\varepsilon}}{M^{\frac{1}{2}}} \to \frac{\psi_\varepsilon}{M^{\frac{1}{2}}} \qquad \textit{weak* in } L^\infty(0,T;L^2(\Omega\times D)), \tag{3.93a}$$

$$M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q \left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) \to M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q \left(\frac{\psi_\varepsilon}{M}\right) \qquad \textit{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), \tag{3.93b}$$

$$M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_x \left(\frac{\psi_{\alpha,\varepsilon}}{M}\right) \to M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_x \left(\frac{\psi_\varepsilon}{M}\right) \qquad \textit{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), \tag{3.93c}$$

$$\underset{\approx}{C}(\psi_{\alpha,\varepsilon}) \to \underset{\approx}{C}(\psi_\varepsilon) \qquad \textit{weak* in } L^\infty(0,T;\underset{\approx}{L}^2(\Omega)); \tag{3.93d}$$

and

$$\underset{\sim}{u}_{\alpha,\varepsilon} \to \underset{\sim}{u}_\varepsilon \qquad \textit{weak* in } L^\infty(0,T;\underset{\sim}{L}^2(\Omega)), \tag{3.94a}$$

$$\underset{\sim}{u}_{\alpha,\varepsilon} \to \underset{\sim}{u}_\varepsilon \qquad \textit{weakly in } L^2(0,T;\underset{\sim}{V}), \tag{3.94b}$$

$$\underset{\approx}{S}_\gamma \frac{\partial \underset{\sim}{u}_{\alpha,\varepsilon}}{\partial t} \to \underset{\approx}{S}_\gamma \frac{\partial \underset{\sim}{u}_\varepsilon}{\partial t} \qquad \textit{weakly in } L^{\frac{4}{d}}(0,T;\underset{\sim}{V}), \tag{3.94c}$$

$$\underset{\sim}{u}_{\alpha,\varepsilon} \to \underset{\sim}{u}_\varepsilon \qquad \textit{strongly in } L^2(0,T;\underset{\sim}{L}^r(\Omega)); \tag{3.94d}$$

where $r \in [1,\infty)$ if $d = 2$ and $r \in [1,6)$ if $d = 3$.

In addition, we have that

$$\sup_{t\in(0,T)} \left[\int_\Omega |\underset{\sim}{u}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{x}\right] + \nu \int_0^T \int_\Omega |\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \leq C(T); \tag{3.95a}$$

$$\sup_{t\in(0,T)} \left[\int_{\Omega\times D} \frac{|\psi_\varepsilon|^2}{M} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \int_0^T \int_{\Omega\times D} \left[\varepsilon\, M \left|\underset{\sim}{\nabla}_x \left(\frac{\psi_\varepsilon}{M}\right)\right|^2 + \frac{1}{\lambda}\left|\underset{\sim}{\nabla}_q \left(\frac{\psi_\varepsilon}{M}\right)\right|^2\right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+ \sup_{t\in(0,T)} \left[\int_\Omega |\underset{\approx}{C}(\psi_\varepsilon)|^2 \,\mathrm{d}\underset{\sim}{x}\right] + \int_0^T \left\|\underset{\approx}{S}_\gamma \frac{\partial \underset{\sim}{u}_\varepsilon}{\partial t}\right\|_{H^1(\Omega)}^{\frac{4}{d}} \mathrm{d}t \leq C(T). \tag{3.95b}$$

It follows immediately from (3.94d), (3.4a,b) and (3.95a) that

$$\underset{\approx}{S}_\alpha \underset{\sim}{u}_{\alpha,\varepsilon} \to \underset{\sim}{u}_\varepsilon \qquad \text{strongly in } L^2(0,T;\underset{\sim}{L}^2(\Omega)) \quad \text{as } \alpha \to 0. \tag{3.96}$$

Next, we note that for all $\underset{\sim}{v} \in \underset{\sim}{H}^1_0(\Omega)$ and $\underset{\sim}{\eta} \in H^1(\Omega; L^2(D;M))$

$$\int_{\Omega\times D} M\,[\underset{\approx}{\omega}(\underset{\sim}{v})\,\underset{\sim}{q}]\cdot\underset{\sim}{\eta}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \tfrac{1}{2}\int_{\Omega\times D} M\left[(\underset{\sim}{v}\cdot\underset{\sim}{q})\,(\underset{\sim}{\nabla}_x\cdot\underset{\sim}{\eta}) - [\underset{\approx}{\nabla}_x(\underset{\sim}{\eta})\,\underset{\sim}{q}]\cdot\underset{\sim}{v}\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,. \quad (3.97)$$

It is now a simple matter to prove existence of a solution to the following problem:

(P$_\varepsilon$) Find

$$\underset{\sim}{u}_\varepsilon \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}') \quad\text{and}\quad \psi_\varepsilon \in L^2(0,T;X),$$

with

$$M^{-\frac{1}{2}}\,\psi_\varepsilon \in L^\infty(0,T;L^2(\Omega\times D)) \quad\text{and}\quad \underset{\approx}{C}(\psi_\varepsilon) \in L^\infty(0,T;\underset{\approx}{L}^2(\Omega)),$$

such that $\underset{\sim}{u}_\varepsilon(\cdot,0) = \underset{\sim}{u}_0(\cdot)$ and

$$\int_0^T \left\langle \frac{\partial \underset{\sim}{u}_\varepsilon}{\partial t}, \underset{\sim}{w}\right\rangle \mathrm{d}t + \int_0^T\int_\Omega \left[[(\underset{\sim}{u}_\varepsilon\cdot\underset{\sim}{\nabla}_x)\underset{\sim}{u}_\varepsilon]\cdot\underset{\sim}{w} + \nu\,\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_\varepsilon : \underset{\approx}{\nabla}_x\,\underset{\sim}{w}\right]\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t.$$
$$= -\int_0^T\int_\Omega \underset{\approx}{C}(\psi_\varepsilon) : \underset{\approx}{\nabla}_x\,\underset{\sim}{w}\;\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \qquad \forall \underset{\sim}{w} \in L^{\frac{4}{4-d}}(0,T;\underset{\sim}{V}); \quad (3.98\text{a})$$

$$-\int_0^T\int_{\Omega\times D}\frac{\psi_\varepsilon}{M}\frac{\partial\varphi}{\partial t}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t - \int_{\Omega\times D}\frac{\psi_0(\cdot,\cdot)}{M}\,\varphi(\cdot,\cdot,0)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$+\int_0^T\int_{\Omega\times D}\left[\varepsilon\,M\,\underset{\sim}{\nabla}_x\left(\frac{\psi_\varepsilon}{M}\right) - \underset{\sim}{u}_\varepsilon\,\psi_\varepsilon\right]\cdot\underset{\sim}{\nabla}_x\left(\frac{\varphi}{M}\right)\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+\int_0^T\int_{\Omega\times D}\frac{M}{2\,\lambda}\underset{\sim}{\nabla}_q\left(\frac{\psi_\varepsilon}{M}\right)\cdot\underset{\sim}{\nabla}_q\left(\frac{\varphi}{M}\right)\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+\tfrac{1}{2}\int_0^T\int_{\Omega\times D}\left[\underset{\approx}{\nabla}_x\left(\psi_\varepsilon\,\underset{\sim}{\nabla}_q\left(\frac{\varphi}{M}\right)\right)\underset{\sim}{q}\right]\cdot\underset{\sim}{u}_\varepsilon\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$-\tfrac{1}{2}\int_0^T\int_{\Omega\times D}(\underset{\sim}{u}_\varepsilon\cdot\underset{\sim}{q})\left[\underset{\sim}{\nabla}_x\cdot\left(\psi_\varepsilon\,\underset{\sim}{\nabla}_q\left(\frac{\varphi}{M}\right)\right)\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = 0 \qquad \forall\varphi\in\mathcal{Y}; \quad (3.98\text{b})$$

where $\mathcal{Y}$ is defined as the completion of $C_0^\infty((-T,T);\mathcal{K})$ in the norm $\|\cdot\|_{\mathcal{Y}}$ defined by

$$\|\varphi\|_{\mathcal{Y}} := \|\varphi\|_{L^2(0,T;X_q)} + \left\|M^{\frac{1}{2}}\,|\underset{\sim}{q}|\,\underset{\sim}{\nabla}_q\left(\frac{\varphi}{M}\right)\right\|_{L^2(0;T;H^1(\Omega;L^2(D)))}$$
$$+\left\|M^{-\frac{1}{2}}\frac{\partial\varphi}{\partial t}\right\|_{L^1(0;T;L^2(\Omega\times D))}. \quad (3.99)$$

This, in particular, implies that each $\varphi\in\mathcal{Y}$ satisfies $\varphi(\cdot,\cdot,T)=0$.

Noting (3.94a–d) and (3.93d), we can pass to the limit $\alpha\to 0$ in (3.21a) in the corotational case to obtain (3.98a). As $\underset{\sim}{S}_\alpha\underset{\sim}{u}_{\alpha,\varepsilon}\in L^2(0,T;\underset{\sim}{V})$, we can apply (3.97) to (3.86) with $\underset{\sim}{v} = \underset{\sim}{S}_\alpha\underset{\sim}{u}_{\alpha,\varepsilon}$ and $\underset{\sim}{\eta} = \frac{\psi_{\alpha,\varepsilon}}{M}\,\underset{\sim}{\nabla}_q\,(\frac{\varphi}{M})$, then use (3.93a–d), (3.94d) and (3.96) to pass to the limit $\alpha\to 0$ to obtain (3.98b).

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