# A POSTERIORI ERROR ANALYSIS OF MIXED FINITE ELEMENT APPROXIMATIONS TO QUASI-NEWTONIAN INCOMPRESSIBLE FLOWS*

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**Abstract.** We develop the *a posteriori* error analysis of mixed finite element approximations of a general family of steady, viscous, incompressible quasi-Newtonian fluids in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$; the family includes degenerate models such as the power-law model, as well as non-degenerate ones such as the Carreau model. The unified theoretical framework developed herein yields a residual-based *a posteriori* bound which measures the error in the approximation of the velocity in the $\mathrm{W}^{1,r}(\Omega)$ norm and that of the pressure in the $\mathrm{L}^{r'}(\Omega)$ norm, $1/r + 1/r' = 1$.

**Key words.** finite elements, a posteriori error estimates, non-Newtonian fluids

**AMS subject classifications.** 65N30, 65N15, 65G99, 76D07, 76D99

## 1. Introduction.

Suppose that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, scaled so that $|\Omega| = 1$, and, for $r \in (1, \infty)$, let $r' = r/(r-1)$. The fluid, whose motion in $\Omega$ is due to an external body force $\mathbf{f} \in [\mathrm{L}^{r'}(\Omega)]^d$, has velocity $\mathbf{u}$ and kinematic pressure $p$. For ease of exposition, $\mathbf{u}$ will be assumed to satisfy the homogeneous Dirichlet boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$. The functions $\mathbf{u} \in V$ and $p \in Q$ are to be found from the boundary value problem whose weak formulation is

$$a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in V, \tag{1.1}$$

$$b(q, \mathbf{u}) = 0 \qquad \forall q \in Q, \tag{1.2}$$

where $V = [\mathrm{W}_0^{1,r}(\Omega)]^d$, $Q = \mathrm{L}_0^{r'}(\Omega) = \mathrm{L}^{r'}(\Omega)/\mathbb{R}$,

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega k(x, |e(\mathbf{u})|) e(\mathbf{u}) : e(\mathbf{v}) \, \mathrm{d}\Omega, \qquad b(q, \mathbf{v}) = -\int_\Omega (\nabla \cdot \mathbf{v}) \, q \, \mathrm{d}\Omega.$$

The strain tensor $e(\mathbf{u}) \in \mathbb{R}^{d\times d}_{\mathrm{symm}}$ has components

$$e(\mathbf{u})_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, \ldots, d,$$

where $\mathbb{R}^{d\times d}_{\mathrm{symm}}$ denotes the set of all symmetric real-valued $d \times d$ matrices.

*Assumption (A)*: We assume that $k \in \mathrm{C}(\bar{\Omega} \times (0, \infty))$ and that, given $r \in (1, \infty)$ as above, there exist constants $\alpha \in [0, 1]$ and $\varepsilon$, $K_1$, $K_2 > 0$ such that, for all $x \in \bar{\Omega}$,

*(A1)* $\quad k(x,t)t - k(x,s)s \geq K_1(t-s)[(t+s)^\alpha(1+t+s)^{1-\alpha}]^{r-2} \qquad$ for all $t \geq s > 0$;

*(A2)* $\quad k(x,t) \leq K_2[t^\alpha(1+t)^{1-\alpha}]^{r-2} \qquad$ for all $t > 0$, and

$$|k(x,t)t - k(x,s)s| \leq K_2|t-s|[(t+s)^\alpha(1+t+s)^{1-\alpha}]^{r-2}$$

for all $s, t > 0$ satisfying $|(s/t) - 1| \leq \varepsilon$.

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For the sake of brevity and notational simplicity, we shall write $k(\cdot)$ instead of $k(x,\cdot)$. The parameter $\alpha$ in (A) measures the degree of degeneracy in $k(\cdot)$ for a given value of $r \in (1,\infty)$ in the sense that the closer $\alpha$ is to 1 the more degenerate $k(\cdot)$ is. For example:

(a) the power law model with $k(t) = 2\mu t^{r-2}$ corresponds to $\alpha = 1$; when $r = 2$, this reduces to $k(t) \equiv 2\mu$, yielding the Stokes equations which govern the stationary flow of a viscous incompressible Newtonian fluid;

(b) the Carreau law $k(t) = k_\infty + (k_0 - k_\infty)(1 + \lambda t^2)^{(\theta-2)/2}$ with $k_0 > k_\infty \geq 0$, $\lambda > 0$, $\theta \in (1,\infty)$ corresponds to $\alpha = 0$ with $r = \theta$ if $k_\infty = 0$, and $r = 2$ if $\theta \in (1,2]$ and $k_\infty > 0$.

Partial differential equations with nonlinearities of the kind considered here arise in a number of application areas, including geophysical models of the lithosphere, as well as chemical engineering, particularly in the modelling of the flow of pastes and dies.

We equip the spaces $V$ and $Q$ with the norms

$$\|\mathbf{v}\|_V = \|e(\mathbf{v})\|_{\mathrm{L}^r(\Omega)} \quad \text{and} \quad \|q\|_Q = \inf_{c\in\mathbb{R}} \|q + c\|_{\mathrm{L}^{r'}(\Omega)},$$

and recall from [1] that the bilinear form $b(\cdot,\cdot)$ satisfies the following inf-sup condition: there exists a positive constant $c_0$ such that

$$\inf_{q\in Q} \sup_{\mathbf{v}\in V} \frac{b(q,\mathbf{v})}{\|q\|_Q \|\mathbf{v}\|_V} \geq c_0 \qquad \forall q \in Q. \tag{1.3}$$

In Section 2, we shall *assume* that the finite element subspaces $V_h$ and $Q_h$ of the spaces $V$ and $Q$ satisfy an analogous inf-sup condition, with inf-sup constant $c_0' > 0$.

The fact that $\|\cdot\|_V$ is a norm on $V$ is a consequence of Korn's inequality (cf. [12], [14]) which asserts the existence of a constant $C = C(r,d,\Omega)$, $1 < r < \infty$, such that

$$\|\mathbf{v}\|_{\mathrm{W}^{1,r}(\Omega)} \leq C\|e(\mathbf{v})\|_{\mathrm{L}^r(\Omega)} \qquad \forall \mathbf{v} \in V. \tag{1.4}$$

In the sequel, for the sake of notational simplicity, we shall suppress the dependence of all constants on $d$ and $\Omega$; in particular, we shall write $C(r)$ instead of $C(r,d,\Omega)$.

The definition of the norm on $Q$ reflects the fact that in the case of Dirichlet boundary condition on $\partial\Omega$ the pressure in the model is determined only up to an additive constant. Let $V'$ denote the dual space of $V$ and let $Q'$ be the dual space of $Q$; the spaces $V'$ and $Q'$ have the norms

$$\|\mathbf{f}\|_{V'} = \sup_{\mathbf{v}\in V} \frac{\langle \mathbf{f},\mathbf{v}\rangle}{\|\mathbf{v}\|_V} \quad \text{and} \quad \|g\|_{Q'} = \sup_{q\in Q} \frac{\langle g,q\rangle}{\|q\|_Q}.$$

Here, in the definition of $\|\cdot\|_{V'}$, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $V'$ and $V$, and in the definition of $\|\cdot\|_{Q'}$ it signifies the duality pairing between $Q'$ and $Q$; as the choice of spaces over which the duality pairings act will always be clear from the context we have chosen not to indicate them explicitly in our notation $\langle\cdot,\cdot\rangle$.

Over the last decade, there has been considerable interest both in the mathematical analysis of quasi-Newtonian flow problems of this kind and in their finite element approximation. The existence and uniqueness of solutions to the boundary value problem (1.1), (1.2) was studied by Baranger and Najib [4] and Barrett and Liu [6]. In particular, it is known that (1.1), (1.2) has a unique solution $(\mathbf{u},p) \in V \times Q$. Concerning the *a priori* error analysis of finite element methods for quasi-Newtonian flow equations, we refer to Baranger and Najib [4], Du and Gunzburger [10], Sandri [16], Barrett and Liu [6, 7], Barrett and Bao [5], and Bao [2]. Baranger and El

Amri [3] were the first to pursue the *a posteriori* error analysis of conforming finite element approximations to a quasi-Newtonian problem in the case of the Carreau law. Subsequently, Simms [18] considered the *a posteriori* error analysis of Fortin's element for conforming mixed finite element approximations of quasi-Newtonian flow problems and, more recently, Sandri [17] studied the *a posteriori* error analysis of conforming mixed finite element approximations of the power-law model and derived a posteriori error bounds for the case of $1 < r < 2$. In fact, Sandri's bounds on the velocity and the pressure will emerge from our analysis for the special case of $k(t) = 2\mu t^{r-2}$, $1 < r < 2$; similarly, the error bounds of Baranger and El Amri [3] are arrived at by selecting $k(t) = k_\infty + (k_0 - k_\infty)(1 + \lambda t^2)^{(\theta-2)/2}$, $1 < \theta \leq 2$. For nonconforming finite element methods, Padra [15] derived *a posteriori* error bounds for Fortin–Soulie [11] piecewise quadratic approximations of quasi-Newtonian flows. In the case of Carreau-type nonlinearities in two space dimensions, Bao and Barrett [5] developed *a posteriori* error bounds based on the linear nonconforming element of Kouhia and Stenberg [13] which involves continuous piecewise linear approximation for one velocity component and a discontinuous linear Crouzeix–Raviart element for the other in tandem with piecewise constant approximation of the pressure. More recently, Carstensen and Funken [9] established *a posteriori* error bounds for quite a general class of conforming and nonconforming finite element methods for steady quasi-Newtonian flows with uniformly monotone and uniformly Lipschitz-continuous nonlinearities.

The purpose of the present paper is to develop the *a posteriori* error analysis of $(V, Q)$-conforming finite element approximations to (1.1), (1.2), for the entire range of $r \in (1, \infty)$. A distinctive feature of problem (1.1), (1.2) is that, in general, there is no value of $r > 1$ other than $r = 2$ such that the nonlinear differential operator is both uniformly monotone and uniformly Lipschitz-continuous in the Sobolev norm $\|\cdot\|_{W^{1,r}(\Omega)}$. Hence, following the work of Barrett and Liu [6], [7], we shall rely here on uniform monotonicity and local Lipschitz continuity properties in *Sobolev quasi-norms.*

The paper is structured as follows. In the next section we state the finite element discretisation of the boundary value problem and derive an *error representation formula* which expresses the error between the analytical solution of the boundary-value problem and its finite element approximation in terms of computable finite element residuals. In Section 3, we establish some preliminary results which will then be used in Section 4 to derive our *a posteriori* bounds on the error in the approximations $\mathbf{u}_h$ and $p_h$ to the velocity and the pressure in $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. The main result of the paper is the following *a posteriori* upper bound on the error.

THEOREM 1.1. *Let $(\mathbf{u}, p) \in V \times Q$ denote the solution to (1.1), (1.2), and let $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ denote its finite element approximation defined by (2.2), (2.3). Then, there exists a positive constant $C = C(K_1, K_2, c_0, c_0', r, \|\mathbf{f}\|_{V'})$ such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_V^{\mathrm{R}} + \|p - p_h\|_Q^{\text{я}} \leq C \left( \|\mathbf{S}_1\|_{V'}^{\mathrm{R}'} + \|\mathrm{S}_2\|_{Q'}^{\text{я}'} \right), \tag{1.5}$$

*where* $\mathrm{R} = \max\{r, 2\}$, $\text{я} = \max\{r', 2\}$, $1/\mathrm{R} + 1/\mathrm{R}' = 1$, $1/\text{я} + 1/\text{я}' = 1$, *and* $\mathbf{S}_1$ *and* $\mathrm{S}_2$ *residual functionals which are computably bounded according to (2.18) and (2.19).*

**2. Finite element approximation.** Henceforth, we shall suppose that $\Omega \subset \mathbb{R}^d$ is a bounded polyhedral domain and that $(\mathcal{T}_h)_{h>0}$ is a shape-regular family of subdivisions of $\Omega$ consisting of $d$-dimensional open simplexes $T \in \mathcal{T}_h$, each of which is an affine image of the open unit simplex

$$\hat{T} = \{\hat{x} = (\hat{x}_1, \ldots, \hat{x}_d) \in \mathbb{R}^d \ : \ 0 < \hat{x}_i < 1, \ i = 1, \ldots, d, \ 0 < \hat{x}_1 + \ldots + \hat{x}_d < 1\}.$$

Suppose that $V_h \subset V$ is a finite element space consisting of continuous piecewise polynomial $d$-component vector functions defined on the triangulation $\mathcal{T}_h$ of $\Omega$ and $Q_h \subset Q$ is a finite element space consisting of continuous or discontinuous piecewise polynomial functions defined on $\mathcal{T}_h$. We shall assume that the pair $(V_h, Q_h)$ satisfies the following inf-sup condition: there exists a positive constant $c_0'$, independent of the discretisation parameter $h > 0$, such that

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{b(q_h, \mathbf{v}_h)}{\|q_h\|_Q \|\mathbf{v}_h\|_V} \geq c_0'. \tag{2.1}$$

The finite element approximation of our model problem has the following form: find $\mathbf{u}_h \in V_h$ and $p_h \in Q_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in V_h, \tag{2.2}$$

$$b(q_h, \mathbf{u}_h) = 0 \qquad \forall q_h \in Q_h. \tag{2.3}$$

Under the stated hypotheses problem (2.2), (2.3) has a unique solution $(\mathbf{u}_h, p_h)$ in $V_h \times Q_h$ (c.f. [3], [6]).

In addition, we shall suppose that the finite element space $V_h$ has the following approximation property: there exists a (possibly nonlinear) mapping $I_h : V \to V_h$ and a positive constant $C_1$ such that, for all $\mathbf{w} \in V = [\mathrm{W}_0^{1,r}(\Omega)]^d$ and all $T \in \mathcal{T}_h$,

$$\|\mathbf{w} - I_h \mathbf{w}\|_{\mathrm{L}^r(T)} + h_T |\mathbf{w} - I_h \mathbf{w}|_{\mathrm{W}^{1,r}(T)} \leq C_1 h_T |\mathbf{w}|_{\mathrm{W}^{1,r}(S_T)}, \tag{2.4}$$

where $h_T$ is the diameter of the element $T$, and $S_T$ is the patch of elements surrounding $T$. Condition (2.4) can be fulfilled by selecting $I_h \mathbf{w}$ as the Scott–Zhang quasi-interpolant of $\mathbf{w} \in V$ (cf. [8] and [19]).

We deduce from (1.1) and (1.2) that, for all $\mathbf{w} \in V$ and all $q \in Q$,

$$a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p - p_h, \mathbf{w}) = (\mathbf{f}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) - b(p_h, \mathbf{w}), \tag{2.5}$$

$$b(q, \mathbf{u} - \mathbf{u}_h) = -b(q, \mathbf{u}_h). \tag{2.6}$$

Adding (2.5) to (2.6) and using (2.2) with $\mathbf{v}_h = I_h \mathbf{w}$ yields

$$\begin{aligned} &a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p - p_h, \mathbf{w}) + b(q, \mathbf{u} - \mathbf{u}_h) \\ &\quad = \{(\mathbf{f}, \mathbf{w} - I_h \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w} - I_h \mathbf{w}) - b(p_h, \mathbf{w} - I_h \mathbf{w})\} - b(q, \mathbf{u}_h), \end{aligned} \tag{2.7}$$

for all $\mathbf{w} \in V$ and all $q \in Q$.

We proceed by decomposing the inner product $(\cdot, \cdot)$, the semilinear form $a(\cdot, \cdot)$ and the bilinear form $b(\cdot, \cdot)$ as sums of integrals over elements $T \in \mathcal{T}_h$, and integrating by parts over each element $T \in \mathcal{T}_h$; thus,

$$\begin{aligned} a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p - p_h, \mathbf{w}) + b(q, \mathbf{u} - \mathbf{u}_h) &= \sum_{T \in \mathcal{T}_h} \int_T \mathbf{f} \cdot (\mathbf{w} - I_h \mathbf{w}) \, \mathrm{d}T \\ &\quad - \sum_{T \in \mathcal{T}_h} \int_T k(|e(\mathbf{u}_h)|) e(\mathbf{u}_h) : e(\mathbf{w} - I_h \mathbf{w}) \, \mathrm{d}T \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{T\in\mathcal{T}_h} \int_T p_h \nabla\cdot(\mathbf{w} - I_h\mathbf{w})\,\mathrm{d}T + \sum_{T\in\mathcal{T}_h} \int_T (\nabla\cdot\mathbf{u}_h) q\,\mathrm{d}T \\
&= \sum_{T\in\mathcal{T}_h} (\mathbf{f} + \nabla\cdot(k(|e(\mathbf{u}_h)|)e(\mathbf{u}_h)) - \nabla p_h)\cdot(\mathbf{w} - I_h\mathbf{w})\,\mathrm{d}T \\
&- \sum_{T\in\mathcal{T}_h} \int_{\partial T} [k(|e(\mathbf{u}_h)|)e(\mathbf{u}_h)\mathbf{n}_T - p_h\mathbf{n}_T]\cdot(\mathbf{w} - I_h\mathbf{w})\,\mathrm{d}s \\
&+ \sum_{T\in\mathcal{T}_h} \int_T (\nabla\cdot\mathbf{u}_h)\, q\,\mathrm{d}T \\
&\equiv \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3, \qquad (2.8)
\end{aligned}
$$

where $\mathbf{n}_T$ is the unit outward normal vector to the boundary $\partial T$ of the simplex $T \in \mathcal{T}_h$.

Let us define the residuals $\mathbf{R}_1$ and $\mathrm{R}_3$ on $\bigcup_{T\in\mathcal{T}_h} T$ by setting, on each $T \in \mathcal{T}_h$,

$$\mathbf{R}_1 = \mathbf{f} + \nabla\cdot(k(|e(\mathbf{u}_h)|)e(\mathbf{u}_h)) - \nabla p_h$$

and

$$\mathrm{R}_3 = \nabla\cdot\mathbf{u}_h.$$

Clearly, $\mathbf{R}_1$ is a vector and $\mathrm{R}_3$ is a scalar, – and this is reflected by our notation.

The term $\mathrm{T}_2$ in (2.8) can be rewritten as

$$\mathrm{T}_2 = \sum_{T\in\mathcal{T}_h} \int_{\partial T} (\sigma_T \mathbf{n}_T)\cdot(\mathbf{w} - I_h\mathbf{w})\,\mathrm{d}s,$$

where

$$\sigma = -\left(k(|e(\mathbf{u}_h)|)e(\mathbf{u}_h) - p_h I\right),$$

$I$ denotes the $d\times d$ identity matrix, and $\sigma_T = \sigma|_T$. As $\mathbf{w} - I_h\mathbf{w} = \mathbf{0}$ on $\partial\Omega$, parts of $\partial T$ which intersect $\partial\Omega$ can be omitted from the region of integration in term $\mathrm{T}_2$. Hence, only faces $e \subset \partial T$ internal to $\Omega$ need to be considered in detail.

Let $e \subset \Omega$ be a $(d-1)$-dimensional face shared by elements $T$ and $T'$; i.e., $e = \partial T \cap \partial T'$. Then, during the summation $\sum_{T\in\mathcal{T}_h}$ and surface integration $\int_{\partial T} \dots \mathrm{d}s$ involved in $\mathrm{T}_2$, the face $e$ will be traversed twice: once in the course of integration over $\partial T$ and then in the course of integration over $\partial T'$ (cf. Figure 2.1).

Since during the two passes through $e = \partial T \cap \partial T'$ the orientation of the unit outward normal changes, we deduce that

$$\mathrm{T}_2 = \sum_{T\in\mathcal{T}_h} \sum_{e\subset\partial T\cap\Omega} \int_e \tfrac{1}{2}[\![\sigma\mathbf{n}]\!]\cdot(\mathbf{w} - I_h\mathbf{w})\,\mathrm{d}s,$$

where, on the face $e \subset \partial T \cap \Omega$,

$$[\![\mathbf{v}]\!] = \mathbf{v}|_{\partial T'\cap\Omega} - \mathbf{v}|_{\partial T\cap\Omega}.$$

The presence of the factor $\frac{1}{2}$ is due to the fact that in the double summation over $T \in \mathcal{T}_h$ and $e \subset \partial T \cap \Omega$ each face $e$ has been counted twice. In the definition of $[\![\cdot]\!]$, for the sake of notational simplicity, we suppressed the reference to the element $T$

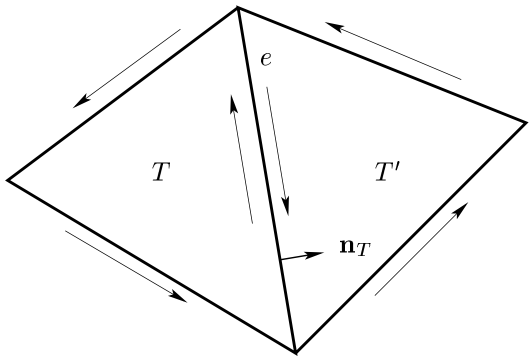


FIG. 2.1. *A face $e$ shared by elements $T$ and $T'$ in the triangulation and the unit outward normal vector $\mathbf{n}_T$ to the boundary $\partial T$ of $T$, in the case $d = 2$.*

considered. A more precise notation would have been to write $[\![\cdot]\!]_T$ instead of $[\![\cdot]\!]$ to highlight the fact that $[\![\cdot]\!]_{T'} = -[\![\cdot]\!]_T$.

Motivated by the form of $\mathrm{T}_2$, for each element $T \in \mathcal{T}_h$ and each face $e \subset \partial T \cap \Omega$ we define

$$\mathbf{R}_2 = \tfrac{1}{2}[\![\sigma \mathbf{n}]\!].$$

On expressing the right-hand side of (2.7) in terms of $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathrm{R}_3$, we obtain the following *error representation formula*:

$$\begin{aligned}
&a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p - p_h, \mathbf{w}) + b(q, \mathbf{u} - \mathbf{u}_h) \\
&\quad = \sum_{T \in \mathcal{T}_h} \int_T \mathbf{R}_1 \cdot (\mathbf{w} - I_h \mathbf{w})\, \mathrm{d}T \\
&\qquad + \sum_{T \in \mathcal{T}_h} \sum_{e \subset \partial T \cap \Omega} \int_e \mathbf{R}_2 \cdot (\mathbf{w} - I_h \mathbf{w})\, \mathrm{d}s \\
&\qquad + \sum_{T \in \mathcal{T}_h} \int_T \mathrm{R}_3 (q + c)\, \mathrm{d}T
\end{aligned} \tag{2.9}$$

for all $\mathbf{w} \in V$, all $q \in Q$, and all $c \in \mathbb{R}$; here we made use of the fact that

$$c \sum_{T \in \mathcal{T}_h} \int_T \mathrm{R}_3 \, \mathrm{d}T = c \int_\Omega \nabla \cdot \mathbf{u}_h \, \mathrm{d}\Omega = c \int_{\partial\Omega} \mathbf{u}_h \cdot \mathbf{n} \, \mathrm{d}s = 0$$

for all $c \in \mathbb{R}$ since $\mathbf{u}_h|_{\partial\Omega} = \mathbf{0}$.

Applying Hölder's inequality to each of the terms on the right-hand side of (2.9) and then taking the infimum over all $c \in \mathbb{R}$, we have that

$$\begin{aligned}
&a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p - p_h, \mathbf{w}) + b(q, \mathbf{u} - \mathbf{u}_h) \\
&\quad \le \sum_{T \in \mathcal{T}_h} \|\mathbf{R}_1\|_{\mathrm{L}^{r'}(T)} \|\mathbf{w} - I_h \mathbf{w}\|_{\mathrm{L}^r(T)} \\
&\qquad + \sum_{T \in \mathcal{T}_h} \sum_{e \subset \partial T \cap \Omega} \|\mathbf{R}_2\|_{\mathrm{L}^{r'}(e)} \|\mathbf{w} - I_h \mathbf{w}\|_{\mathrm{L}^r(e)} \\
&\qquad + \sum_{T \in \mathcal{T}_h} \|\mathrm{R}_3\|_{\mathrm{L}^r(T)} \inf_{c \in \mathbb{R}} \|q + c\|_{\mathrm{L}^{r'}(T)}
\end{aligned} \tag{2.10}$$

for all $\mathbf{w} \in V$ and all $q \in Q$.

According to the Trace Inequality, there exists a positive constant $C_0$ such that, for all $\mathbf{w} \in [\mathrm{W}^{1,r}(T)]^d$, $1 < r < \infty$, and any face $e \subset \partial T$,

$$\|\mathbf{w}\|_{\mathrm{L}^r(e)} \leq C_0 \left( h_T^{-1/r} \|\mathbf{w}\|_{\mathrm{L}^r(T)} + h_T^{1/r'} |\mathbf{w}|_{\mathrm{W}^{1,r}(T)} \right).$$

Hence, (2.4) implies that, for any $\mathbf{w} \in V$, any $T \in \mathcal{T}_h$ and any face $e \subset \partial T$,

$$\|\mathbf{w} - I_h \mathbf{w}\|_{\mathrm{L}^r(e)} \leq C_0 C_1 h_T^{1/r'} |\mathbf{w}|_{\mathrm{W}^{1,r}(S_T)}. \tag{2.11}$$

Applying (2.4) and (2.11) in (2.10) and using Hölder's inequality for finite sums, we have that

$$\begin{aligned} & a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p - p_h, \mathbf{w}) + b(q, \mathbf{u} - \mathbf{u}_h) \\ & \qquad \leq C \left( \sum_{T \in \mathcal{T}_h} h_T^{r'} \|\mathbf{R}_1\|_{\mathrm{L}^{r'}(T)}^{r'} \right)^{1/r'} |\mathbf{w}|_{\mathrm{W}^{1,r}(\Omega)} \\ & \qquad \quad + C \left( \sum_{T \in \mathcal{T}_h} \sum_{e \subset \partial T \cap \Omega} h_T \|\mathbf{R}_2\|_{\mathrm{L}^{r'}(e)}^{r'} \right)^{1/r'} |\mathbf{w}|_{\mathrm{W}^{1,r}(\Omega)} \\ & \qquad \quad + \left( \sum_{T \in \mathcal{T}_h} \|\mathrm{R}_3\|_{\mathrm{L}^r(T)}^r \right)^{1/r} \inf_{c \in \mathbb{R}} \|q + c\|_{\mathrm{L}^{r'}(\Omega)} \end{aligned} \tag{2.12}$$

for all $\mathbf{w} \in V$ and all $q \in Q$. Inequality (2.12) implies that

$$\begin{aligned} & a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p - p_h, \mathbf{w}) + b(q, \mathbf{u} - \mathbf{u}_h) \\ & \quad \leq C \left[ \left( \sum_{T \in \mathcal{T}_h} h_T^{r'} \|\mathbf{R}_1\|_{\mathrm{L}^{r'}(T)}^{r'} \right)^{1/r'} + \left( \sum_{T \in \mathcal{T}_h} \sum_{e \subset \partial T \cap \Omega} h_T \|\mathbf{R}_2\|_{\mathrm{L}^{r'}(e)}^{r'} \right)^{1/r'} \right] |\mathbf{w}|_{\mathrm{W}^{1,r}(\Omega)} \\ & \quad + \left( \sum_{T \in \mathcal{T}_h} \|\mathrm{R}_3\|_{\mathrm{L}^r(T)}^r \right)^{1/r} \inf_{c \in \mathbb{R}} \|q + c\|_{\mathrm{L}^{r'}(\Omega)} \end{aligned} \tag{2.13}$$

for all $\mathbf{w} \in V$ and all $q \in Q$. Taking $q = 0$ in (2.13) and then the supremum over all $\mathbf{w} \in V = [\mathrm{W}_0^{1,r}(\Omega)]^d$, using (1.4), and recalling that $\|e(\mathbf{w})\|_{\mathrm{L}^r(\Omega)} = \|\mathbf{w}\|_V$, we get

$$\begin{aligned} & \sup_{\mathbf{w} \in V} \frac{a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p - p_h, \mathbf{w})}{\|\mathbf{w}\|_V} \\ & \quad \leq C \left[ \left( \sum_{T \in \mathcal{T}_h} h_T^{r'} \|\mathbf{R}_1\|_{\mathrm{L}^{r'}(T)}^{r'} \right)^{1/r'} + \left( \sum_{T \in \mathcal{T}_h} \sum_{e \subset \partial T \cap \Omega} h_T \|\mathbf{R}_2\|_{\mathrm{L}^{r'}(e)}^{r'} \right)^{1/r'} \right]. \end{aligned} \tag{2.14}$$

On the other hand, taking $\mathbf{w} = \mathbf{0}$ in (2.13) and then the supremum over $q \in Q$ yields

$$\sup_{q \in Q} \frac{b(q, \mathbf{u} - \mathbf{u}_h)}{\|q\|_Q} \leq \left( \sum_{T \in \mathcal{T}_h} \|\mathrm{R}_3\|_{\mathrm{L}^r(T)}^r \right)^{1/r}. \tag{2.15}$$

Now, let us rewrite the left-hand side of (2.14) as the norm of a certain element $\mathbf{S}_1 \in V'$ and the left-hand side of (2.15) as the norm of a certain element $\mathrm{S}_2 \in Q'$;

$\mathbf{S}_1$ and $\mathrm{S}_2$ are referred to as *residual functionals*. We shall complete the *a posteriori* error analysis by showing that $\|\mathbf{u}-\mathbf{u}_h\|_V$ and $\|p-p_h\|_Q$ can be bounded in terms of $\|\mathbf{S}_1\|_{V'}$ and $\|\mathrm{S}_2\|_{Q'}$, and thereby, also, in terms of the right-hand sides of (2.14) and (2.15). We define $\mathbf{S}_1 \in V'$ by

$$\langle \mathbf{S}_1, \mathbf{w}\rangle = (\mathbf{f}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) - b(p_h, \mathbf{w}) \qquad \forall \mathbf{w} \in V. \tag{2.16}$$

Similarly, we define $\mathrm{S}_2 \in Q'$ by

$$\langle \mathrm{S}_2, q\rangle = -b(q, \mathbf{u}_h) \qquad \forall q \in Q. \tag{2.17}$$

The existence of the functionals $\mathbf{S}_1$ and $\mathrm{S}_2$ as elements of $V'$ and $Q'$, respectively, is the consequence of Remark 1 following Lemma 3.4 in the next section.

Let us compute the norms $\|\mathbf{S}_1\|_{V'}$ and $\|\mathrm{S}_2\|_{Q'}$. We begin by noting that (1.1) and (2.16) imply that

$$\langle \mathbf{S}_1, \mathbf{w}\rangle = a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p, \mathbf{w}) - b(p_h, \mathbf{w}_h).$$

Hence,

$$\|\mathbf{S}_1\|_{V'} = \sup_{\mathbf{w}\in V} \frac{\langle \mathbf{S}_1, \mathbf{w}\rangle}{\|\mathbf{w}\|_V} = \sup_{\mathbf{w}\in V} \frac{a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p-p_h, \mathbf{w})}{\|\mathbf{w}\|_V}.$$

Applying (2.14) to the right-most expression in this chain, we deduce that

$$\|\mathbf{S}_1\|_{V'} \leq C \left[ \left( \sum_{T\in\mathcal{T}_h} h_T^{r'} \|\mathbf{R}_1\|^{r'}_{\mathrm{L}^{r'}(T)} \right)^{1/r'} + \left( \sum_{T\in\mathcal{T}_h} \sum_{e\subset\partial T\cap\Omega} h_T \|\mathbf{R}_2\|^{r'}_{\mathrm{L}^{r'}(e)} \right)^{1/r'} \right]. \tag{2.18}$$

Analogously, by (2.17),

$$\|\mathrm{S}_2\|_{Q'} = \sup_{q\in Q} \frac{\langle \mathrm{S}_2, q\rangle}{\|q\|_Q} = \sup_{q\in Q} \frac{-b(q, \mathbf{u}_h)}{\|q\|_Q} = \sup_{q\in Q} \frac{b(q, \mathbf{u}-\mathbf{u}_h)}{\|q\|_Q},$$

and therefore, by (2.15),

$$\|\mathrm{S}_2\|_{Q'} \leq \left( \sum_{T\in\mathcal{T}_h} \|\mathrm{R}_3\|^r_{\mathrm{L}^r(T)} \right)^{1/r}. \tag{2.19}$$

For $1 < r < \infty$ the reflexive Banach space $V$ is continuously and densely embedded into the reflexive Banach space $[\mathrm{L}^r(\Omega)]^d$. Hence, $[\mathrm{L}^{r'}(\Omega)]^d$, the dual space of $[\mathrm{L}^r(\Omega)]^d$, is continuously and densely embedded into $V'$. In particular, $\mathbf{f} \in [\mathrm{L}^{r'}(\Omega)]^d$ can be identified with an element of $V'$ (also denoted $\mathbf{f}$ for the sake of notational simplicity) *via* $\langle \mathbf{f}, \mathbf{w}\rangle = (\mathbf{f}, \mathbf{w})$ for all $\mathbf{w} \in V$. Hence, the definitions (2.16) and (2.17) imply that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{w}) + b(p_h, \mathbf{w}) &= \langle \mathbf{f} - \mathbf{S}_1, \mathbf{w}\rangle \qquad && \forall \mathbf{w} \in V, \\ b(q, \mathbf{u}_h) &= \langle -\mathrm{S}_2, q\rangle && \forall q \in Q. \end{aligned}$$

On subtracting these from (1.1) and (1.2), respectively, we obtain

$$a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p-p_h, \mathbf{w}) = \langle \mathbf{S}_1, \mathbf{w}\rangle \qquad \forall \mathbf{w} \in V, \tag{2.20}$$

$$b(q, \mathbf{u}-\mathbf{u}_h) = \langle \mathrm{S}_2, q\rangle \qquad \forall q \in Q. \tag{2.21}$$

Since $\|\mathbf{S}_1\|_{V'}$ and $\|\mathrm{S}_2\|_{Q'}$ have been bounded in terms of the computable residuals $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathrm{R}_3$ through (2.18) and (2.19), the desired *a posteriori* error bound will be arrived at by deducing from (2.20) and (2.21) that $\|\mathbf{u}-\mathbf{u}_h\|_V$ and $\|p-p_h\|_Q$ can, in turn, be bounded in terms of $\|\mathbf{S}_1\|_{V'}$ and $\|\mathrm{S}_2\|_{Q'}$, as stated in the next proposition.

PROPOSITION 2.1. *Let $(\mathbf{u},p) \in V \times Q$ denote the solution to (1.1), (1.2), and let $(\mathbf{u}_h,p_h) \in V_h \times Q_h$ denote its finite element approximation defined by (2.2), (2.3); then, there exists a positive constant $C = C(K_1,K_2,c_0,r,\|\mathbf{f}\|_{V'})$ such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_V^{\mathrm{R}} \le C\left(\|\mathbf{S}_1\|_{V'}^{\mathrm{R}'} + \|\mathrm{S}_2\|_{Q'}^{\text{я}'} + \|\mathbf{S}_1\|_{V'}\|\mathrm{S}_2\|_{Q'}\right), \tag{2.22}$$

$$\|p-p_h\|_Q^{\text{я}} \le C\left(\|\mathbf{S}_1\|_{V'}^{\text{я}} + \|\mathbf{S}_1\|_{V'}^{\mathrm{R}'} + \|\mathrm{S}_2\|_{Q'}^{\text{я}'} + \|\mathbf{S}_1\|_{V'}\|\mathrm{S}_2\|_{Q'}\right), \tag{2.23}$$

*where* $\mathrm{R} = \max\{r,2\}$, $\text{я} = \max\{r',2\}$, $1/\mathrm{R} + 1/\mathrm{R}' = 1$, $1/\text{я} + 1/\text{я}' = 1$, *and* $\mathbf{S}_1$ *and* $\mathrm{S}_2$ *are bounded according to (2.18) and (2.19).*

In the special case of a power law-model, $k(t) = 2\mu t^{r-2}$ with $r \in (1,2)$, the bounds (2.22) and (2.23) collapse to those of Sandri [17]. To prove the theorem, we require some preliminary results.

**3. Preliminary results.** For $\alpha \in [0,1]$ and $t \in (0,\infty)$, we define

$$\Xi_\alpha(t) = t^\alpha(1+t)^{1-\alpha}.$$

Hence,

$$\Xi'_\alpha(t) = (\alpha+t)(1+t)^{-\alpha}t^{\alpha-1} \quad \text{and} \quad \Xi''_\alpha(t) = -\alpha(1-\alpha)t^{\alpha-2}(1+t)^{-\alpha-1}.$$

Therefore, for any $\alpha \in [0,1]$, $t \mapsto \Xi_\alpha(t)$ is a strictly monotonic increasing function of $t \in (0,\infty)$; in particular, $t \mapsto \Xi_0(t)$ and $t \mapsto \Xi_1(t)$ are affine functions of $t \in (0,\infty)$. Furthermore, for $\alpha \in (0,1)$, $t \mapsto \Xi_\alpha(t)$ is a strictly concave function of $t \in (0,\infty)$. The following Jensen-type inequality is easily proved by using Hölder's inequality and the triangle inequality in $\mathrm{L}^r(\Omega)$: for any $r \in [1,\infty)$, $\alpha \in [0,1]$, and all $w \in \mathrm{L}^r(\Omega)$,

$$\left(\frac{1}{|\Omega|}\int_\Omega [\Xi_\alpha(|w(x)|)]^r \,\mathrm{d}\Omega\right)^{\frac{1}{r}} \le \Xi_\alpha\!\left(\left(\frac{1}{|\Omega|}\int_\Omega |w(x)|^r \,\mathrm{d}\Omega\right)^{\frac{1}{r}}\right).$$

According to our simplifying assumption from the start of the paper, $|\Omega| = 1$; hence,

$$\|\Xi_\alpha(|w|)\|_{\mathrm{L}^r(\Omega)} \le \Xi_\alpha(\|w\|_{\mathrm{L}^r(\Omega)}) \tag{3.1}$$

for all $r \in [1,\infty)$, $\alpha \in [0,1]$, and all $w \in \mathrm{L}^r(\Omega)$.

We recall the following result from the paper of Barrett and Liu [6].

LEMMA 3.1. *Let $k$ satisfy assumption (A1) for $r \in (1,\infty)$ and $\alpha \in [0,1]$. Then, for all $M_1$, $M_2$ in $\mathbb{R}^{d\times d}_{\mathrm{symm}}$ and $\delta \ge 0$, we have that*

$$(k(|M_1|)M_1 - k(|M_2|)M_2) : (M_1 - M_2) \ge K_1[\Xi_\alpha(|M_1|+|M_2|)]^{r-2-\delta}|M_1-M_2|^{2+\delta}. \tag{3.2}$$

*Let $k$ satisfy assumption (A2) for $r \in (1,\infty)$ and $\alpha \in [0,1]$. Then, for all $M_1, M_2$ in $\mathbb{R}^{d\times d}_{\mathrm{symm}}$ and $\delta \ge 0$, we have that*

$$|k(|M_1|)M_1 - k(|M_2|)M_2| \le K_2[\Xi_\alpha(|M_1|+|M_2|)]^{r-2+\delta}|M_1-M_2|^{1-\delta}. \tag{3.3}$$

Next, we introduce the notation

$$|\mathbf{v}|^2_{(\mathbf{w},r,\alpha)} = \int_\Omega [\Xi_\alpha(|e(\mathbf{v})| + |e(\mathbf{w})|)]^{r-2} |e(\mathbf{v})|^2 \,\mathrm{d}\Omega, \qquad \mathbf{v},\mathbf{w} \in [\mathrm{W}^{1,r}(\Omega)]^d,\ 1 < r < \infty.$$

PROPOSITION 3.2. *Suppose that* $r \in (1,\infty)$, $\alpha \in [0,1]$ *and* $\mathbf{w} \in [\mathrm{W}^{1,r}(\Omega)]^d$*; then, the following hold:*

*(i)* $|\mathbf{v}|_{(\mathbf{w},r,\alpha)} \geq 0$ *for all* $\mathbf{v} \in [\mathrm{W}^{1,r}(\Omega)]^d$*. In particular, when* $\mathbf{v} \in V = [\mathrm{W}^{1,r}_0(\Omega)]^d$, $|\mathbf{v}|_{(\mathbf{w},r,\alpha)} = 0$ *if, and only if,* $\mathbf{v} = \mathbf{0}$*;*

*(ii) (Quasi-triangle-inequality): there exists a constant* $C = C(r)$ *such that*

$$|\mathbf{v}_1 + \mathbf{v}_2|_{(\mathbf{w},r,\alpha)} \leq C \left( |\mathbf{v}_1|_{(\mathbf{w},r,\alpha)} + |\mathbf{v}_2|_{(\mathbf{w},r,\alpha)} \right)$$

*for all* $\mathbf{v}_1, \mathbf{v}_2 \in [\mathrm{W}^{1,r}(\Omega)]^d$*;*

*(iii) For* $1 < r \leq 2$,

$$|\mathbf{v}|^{2/r}_{(\mathbf{w},r,\alpha)} \leq \|e(\mathbf{v})\|_{\mathrm{L}^r(\Omega)} \leq [\Xi_\alpha(\|e(\mathbf{v})\|_{\mathrm{L}^r(\Omega)} + \|e(\mathbf{w})\|_{\mathrm{L}^r(\Omega)})]^{(2-r)/2}\, |\mathbf{v}|_{(\mathbf{w},r,\alpha)}$$

*for all* $\mathbf{v} \in [\mathrm{W}^{1,r}(\Omega)]^d$*.*

*For* $2 \leq r < \infty$,

$$\|e(\mathbf{v})\|^{r/2}_{\mathrm{L}^r(\Omega)} \leq |\mathbf{v}|_{(\mathbf{w},r,\alpha)} \leq [\Xi_\alpha(\|e(\mathbf{v})\|_{\mathrm{L}^r(\Omega)} + \|e(\mathbf{w})\|_{\mathrm{L}^r(\Omega)})]^{(r-2)/2}\, \|e(\mathbf{v})\|_{\mathrm{L}^r(\Omega)}$$

*for all* $\mathbf{v} \in [\mathrm{W}^{1,r}(\Omega)]^d$*.*

Part (i) of this proposition follows from the definition of $|\cdot|_{(\mathbf{w},r,\alpha)}$, and Korn's inequality. Part (ii) has been proved in the paper of Barrett and Liu [7]. The proof of (iii) is based on a straightforward application of Hölder's inequality and Jensen's inequality (3.1). Properties (i) and (ii) in Proposition 3.2 are the axioms of *quasi-norm*. Thus, for $\mathbf{w} \in V = [\mathrm{W}^{1,r}(\Omega)]^d$, $|\cdot|_{(\mathbf{w},r,\alpha)}$ is a quasi-norm on $V$. Property (iii) relates the Sobolev norm $\|\cdot\|_V = |\cdot|_{\mathrm{W}^{1,r}(\Omega)}$ to the quasi-norm $|\cdot|_{(\mathbf{w},r,\alpha)}$.

Now, we show the uniform monotonicity and local Lipschitz continuity of the semilinear form $a(\cdot,\cdot)$ with respect to the quasi-norm.

LEMMA 3.3. *Suppose that* $r \in (1,\infty)$ *and define the constants* $C_2 = 2^{-|r-2|}K_1$ *and* $C_3 = 2^{|r-2|/\max\{2,r'\}}K_2$*; then, for* $i = 1,2$, *and all* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$ *in* $V$,

$$a(\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2) - a(\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \geq C_2 |\mathbf{v}_1 - \mathbf{v}_2|^2_{(\mathbf{v}_i,r,\alpha)}; \tag{3.4}$$

$$|a(\mathbf{v}_1,\mathbf{w}) - a(\mathbf{v}_2,\mathbf{w})| \leq C_3 |\mathbf{v}_1 - \mathbf{v}_2|^{\min\{1,\frac{2}{r'}\}}_{(\mathbf{v}_i,r,\alpha)} [\Xi_\alpha(\|\mathbf{v}_1\|_V + \|\mathbf{v}_2\|_V)]^{\max\{0,\frac{r-2}{2}\}} \|\mathbf{w}\|_V. \tag{3.5}$$

*Proof.* To prove (3.4), we use (3.2) with $\delta = 0$, and $M_i = e(\mathbf{v}_i)$, $i = 1,2$. Hence, we deduce that, for any $\mathbf{v}_1, \mathbf{v}_2 \in V$,

$$\begin{aligned} &a(\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2) - a(\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \\ &\quad \geq K_1 \int_\Omega |e(\mathbf{v}_1) - e(\mathbf{v}_2)|^2 [\Xi_\alpha(|e(\mathbf{v}_1)| + |e(\mathbf{v}_2)|)]^{r-2} \,\mathrm{d}\Omega. \end{aligned} \tag{3.6}$$

We note that, for $i = 1,2$,

$$\frac{1}{2}\left(|e(\mathbf{v}_1 - \mathbf{v}_2)| + |e(\mathbf{v}_i)|\right) \leq |e(\mathbf{v}_1)| + |e(\mathbf{v}_2)| \leq 2\left(|e(\mathbf{v}_1 - \mathbf{v}_2)| + |e(\mathbf{v}_i)|\right). \tag{3.7}$$

Suppose that $1 < r \leq 2$; then, (3.6), the second inequality in (3.7) and the definition of the quasi-norm $|\cdot|_{(\mathbf{v}_i,r,\alpha)}$ imply that

$$a(\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2) - a(\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \geq 2^{r-2} K_1 |\mathbf{v}_1 - \mathbf{v}_2|^2_{(\mathbf{v}_i,r,\alpha)}, \quad i = 1, 2,$$

and hence (3.4) with $C_2 = 2^{r-2} K_1$ for $1 < r \leq 2$.

Similarly, when $2 \leq r < \infty$, the first inequality in (3.7) and the definition of the quasi-norm $|\cdot|_{(\mathbf{v}_i,r,\alpha)}$ imply that

$$a(\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2) - a(\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \geq 2^{2-r} K_1 |\mathbf{v}_1 - \mathbf{v}_2|^2_{(\mathbf{v}_i,r,\alpha)}, \quad i = 1, 2,$$

and hence (3.4) with $C_2 = 2^{2-r} K_1$ for $2 \leq r < \infty$.

To show (3.5), we apply Hölder's inequality, the fact that $\|e(\mathbf{w})\|_{\mathrm{L}^r(\Omega)} = \|\mathbf{w}\|_V$, and the inequality (3.2) with $M_i = e(\mathbf{v}_i)$, $i = 1, 2$; hence, we deduce that

$$\begin{aligned}
|a(\mathbf{v}_1, \mathbf{w}) - a(\mathbf{v}_2, \mathbf{w})| &= \int_\Omega \big(k(|e(\mathbf{v}_1)|)e(\mathbf{v}_1) - k(|e(\mathbf{v}_2)|)e(\mathbf{v}_2)\big) : e(\mathbf{w})\, \mathrm{d}\Omega \\
&\leq \left(\int_\Omega |k(|e(\mathbf{v}_1)|)e(\mathbf{v}_1) - k(|e(\mathbf{v}_2)|)e(\mathbf{v}_2)|^{r'}\, \mathrm{d}\Omega\right)^{1/r'} \|\mathbf{w}\|_V \\
&\leq K_2 \left(\int_\Omega |e(\mathbf{v}_1) - e(\mathbf{v}_2)|^{(1-\delta)r'} [\Xi_\alpha(|e(\mathbf{v}_1)| + |e(\mathbf{v}_2)|)]^{(r-2+\delta)r'}\, \mathrm{d}\Omega\right)^{1/r'} \|\mathbf{w}\|_V. \qquad (3.8)
\end{aligned}$$

Let $1 < r \leq 2$ and define $\delta = 1 - (2/r')$; then $(1-\delta)r' = 2$, $(r - 2 + \delta)r' = r - 2$. Therefore, using the first inequality in (3.7),

$$|a(\mathbf{v}_1, \mathbf{w}) - a(\mathbf{v}_2, \mathbf{w})| \leq 2^{(2-r)/r'} K_2 |\mathbf{v}_1 - \mathbf{v}_2|^{2/r'}_{(\mathbf{v}_i,r,\alpha)} \|\mathbf{w}\|_V,$$

which is (3.5) with $C_3 = 2^{(2-r)/r'} K_2$ for $1 < r \leq 2$.

Now, let $2 < r \leq \infty$ and hence $r' = r/(r-1) \in (1, 2)$; we shall use Hölder's inequality in the integral on the right-hand side of (3.8). Thus, we take $\delta = 0$, split

$$[\Xi_\alpha(|e(\mathbf{v}_1)|+|e(\mathbf{v}_2)|)]^{(r-2)r'} = [\Xi_\alpha(|e(\mathbf{v}_1)|+|e(\mathbf{v}_2)|)]^{\frac{(r-2)r'}{2}} [\Xi_\alpha(|e(\mathbf{v}_1)|+|e(\mathbf{v}_2)|)]^{\frac{(r-2)r'}{2}}$$

and group the first factor on the right with $|e(\mathbf{v}_1) - e(\mathbf{v}_2)|^{r'}$. The application of Hölder's inequality with exponents $\alpha = 2/r'$ and $\alpha' = 2/(2 - r')$, $1/\alpha + 1/\alpha' = 1$, corresponding to the factors

$$|e(\mathbf{v}_1) - e(\mathbf{v}_2)|^{r'} [\Xi_\alpha(|e(\mathbf{v}_1)| + |e(\mathbf{v}_2)|)]^{(r-2)r'/2} \quad \text{and} \quad [\Xi_\alpha(|e(\mathbf{v}_1)| + |e(\mathbf{v}_2)|)]^{(r-2)r'/2},$$

respectively, yields

$$\begin{aligned}
|a(\mathbf{v}_1, \mathbf{w}) - a(\mathbf{v}_2, \mathbf{w})| &\leq K_2 \left(\int_\Omega |e(\mathbf{v}_1) - e(\mathbf{v}_2)|^2 [\Xi_\alpha(|e(\mathbf{v}_1)| + |e(\mathbf{v}_2)|)]^{r-2}\, \mathrm{d}\Omega\right)^{1/2} \\
&\quad \times \left(\int_\Omega [\Xi_\alpha(|e(\mathbf{v}_1)| + |e(\mathbf{v}_2)|)]^{\frac{(r-2)r'}{2}\cdot\frac{2}{2-r'}}\, \mathrm{d}\Omega\right)^{(2-r')/(2r')} \|\mathbf{w}\|_V.
\end{aligned}$$

As $\frac{(r-2)r'}{2} \cdot \frac{2}{2-r'} = r$, $\frac{2-r'}{2r'} = \frac{r-2}{2r}$, using the second inequality in (3.7) we get that

$$|a(\mathbf{v}_1, \mathbf{w}) - a(\mathbf{v}_2, \mathbf{w})| \leq C_3 |\mathbf{v}_1 - \mathbf{v}_2|_{(\mathbf{v}_i,r,\alpha)} \|[\Xi_\alpha(|e(\mathbf{v}_1)| + |e(\mathbf{v}_2)|)]\|^{(r-2)/2}_{\mathrm{L}^r(\Omega)} \|\mathbf{w}\|_V$$

for all $\mathbf{v}_1, \mathbf{v}_2$ in $V$, with $C_3 = 2^{(r-2)/2} K_2$. On noting (3.1), the triangle inequality for the $\|\cdot\|_{\mathrm{L}^r(\Omega)}$ norm, and that $t \mapsto \Xi_\alpha(t)$ is monotonic increasing, we have (3.5) for $2 < r < \infty$. □

Next, we show that the solutions to problems (1.1), (1.2) and (2.2), (2.3) can be bounded in terms of $\|\mathbf{f}\|_{V'}$.

LEMMA 3.4. *Let* $(\mathbf{u}, p) \in V \times Q$ *and* $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ *denote the solutions to problems (1.1), (1.2) and (2.2), (2.3), respectively, and let* $r \in (1, \infty)$*; then*

$$\|\mathbf{u}\|_V \le G^{-1}(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'}), \quad \|p\|_Q \le \tfrac{1}{c_0}\Big(\|\mathbf{f}\|_{V'} + K_2 (H \circ G^{-1})(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'})\Big), \tag{3.9}$$

$$\|\mathbf{u}_h\|_V \le G^{-1}(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'}), \quad \|p_h\|_Q \le \tfrac{1}{c_0'}\Big(\|\mathbf{f}\|_{V'} + K_2 (H \circ G^{-1})(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'})\Big), \tag{3.10}$$

*where* $c_0$ *and* $c_0'$ *are the inf-sup constants from (1.3) and (2.1), respectively, and* $G$ *and* $H$ *are continuous strictly monotonic increasing functions defined on* $[0, \infty)$.

*Proof.* Taking $\mathbf{v} = \mathbf{u}$ in (1.1) and using (1.2), we have that

$$a(\mathbf{u}, \mathbf{u}) = (\mathbf{f}, \mathbf{u}) = \langle \mathbf{f}, \mathbf{u} \rangle \le \|\mathbf{f}\|_{V'} \|\mathbf{u}\|_V.$$

Now, using (3.2) with $M_1 = e(\mathbf{u})$, $M_2 = 0$, $\delta = 0$, and (iii) of Proposition 3.2 with $\mathbf{w} = \mathbf{0}$, we obtain

$$a(\mathbf{u}, \mathbf{u}) \ge K_1 |\mathbf{u}|^2_{(\mathbf{0}, r, \alpha)} \ge K_1 \|\mathbf{u}\|_V \, G(\|\mathbf{u}\|_V); \quad G(t) = \begin{cases} t \cdot [\Xi_\alpha(t)]^{r-2}, & 1 < r \le 2, \\ t^{r-1}, & r \ge 2. \end{cases}$$

Since $G : t \mapsto G(t)$ is continuous and strictly monotonic increasing on $[0, \infty)$, its inverse function $G^{-1}$ is continuous and strictly monotonic increasing on $[0, \infty)$. Hence,

$$\|\mathbf{u}\|_V \le G^{-1}(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'}).$$

To bound $\|p\|_Q$, note that, by the inf-sup condition (1.3),

$$c_0 \|p\|_Q \le \sup_{\mathbf{v} \in V} \frac{b(p, \mathbf{v})}{\|\mathbf{v}\|_V}.$$

On the other hand, from (1.1), and using using (3.3) with $M_1 = e(\mathbf{u})$, $M_2 = 0$, $\delta = 0$, and (iii) of Proposition 3.2 with $\mathbf{w} = \mathbf{0}$, we obtain

$$b(p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle - a(\mathbf{u}, \mathbf{v}) \le \|\mathbf{f}\|_{V'} \|\mathbf{v}\|_V + K_2 H(\|\mathbf{u}\|_V) \|\mathbf{v}\|_V,$$

where

$$H(t) = \begin{cases} t^{r-1}, & 1 < r \le 2, \\ t \cdot [\Xi_\alpha(t)]^{r-2}, & 2 \le r, \end{cases}$$

and therefore,

$$c_0 \|p\|_Q \le \|\mathbf{f}\|_{V'} + K_2 H(\|\mathbf{u}\|_V).$$

Clearly, $H : t \mapsto H(t)$ is continuous and strictly monotonic increasing on $[0, \infty)$. Together with our earlier bound $\|\mathbf{u}\|_V \le G^{-1}(\frac{1}{K_1}\|\mathbf{f}\|_{V'})$, this proves (3.9); the proof of (3.10) is identical, except that (2.1) is used instead of (1.3). □

REMARK 1. *As* $|(\mathbf{f},\mathbf{w})| = |\langle \mathbf{f},\mathbf{w}\rangle| \leq \|\mathbf{f}\|_{V'}\|\mathbf{w}\|_V$ *and, by Hölder's inequality,* $|a(\mathbf{v},\mathbf{w})| \leq K_2 H(\|\mathbf{v}\|_V)\|\mathbf{w}\|_V$ *and* $|b(q,\mathbf{w})| \leq \|q\|_Q\|\mathbf{w}\|_V$*, it follows, using (3.10), that*

$$|(\mathbf{f},\mathbf{w}) - a(\mathbf{u}_h,\mathbf{w}) - b(p_h,\mathbf{w})| \leq \left(1+\frac{1}{c_0'}\right)\Big(\|\mathbf{f}\|_{V'} + K_2(H\circ G^{-1})(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'})\Big)\,\|\mathbf{w}\|_V.$$

*Hence, for* $\mathbf{f}\in[\mathrm{L}^{r'}(\Omega)]^d$ *fixed, and the corresponding unique solution* $(\mathbf{u}_h,p_h)$ *of (2.2), (2.3) in* $V_h\times Q_h\subset V\times Q$ *thereby also fixed,* $\mathbf{w}\mapsto(\mathbf{f},\mathbf{w}) - a(\mathbf{u}_h,\mathbf{w}) - b(p_h,\mathbf{w})$ *is a bounded (and therefore continuous) linear functional on* $V$*; as such, it belongs to* $V'$*. It is this element of* $V'$ *that was denoted earlier by* $\mathbf{S}_1$*. Similarly, for* $\mathbf{u}_h\in V_h\subset V$ *fixed,* $q\mapsto b(q,\mathbf{u}_h)$ *is a bounded (and therefore continuous) linear functional on* $Q$*, and as such, it belongs to* $Q'$*; it is this element of* $Q'$ *that was denoted above by* $\mathrm{S}_2$*.*

Using $\mathbf{v}_1=\mathbf{u}$ and $\mathbf{v}_2=\mathbf{u}_h$ in (3.4) and (3.5), together with the bounds on $\|\mathbf{u}\|_V$ and $\|\mathbf{u}_h\|_V$ from Lemma 3.4, we obtain the following result.

LEMMA 3.5. *Let* $(\mathbf{u},p)\in V\times Q$ *and* $(\mathbf{u}_h,p_h)\in V_h\times Q_h$ *denote the solutions to problems (1.1), (1.2) and (2.2), (2.3), respectively, and suppose that* $r\in(1,\infty)$*; then,*

$$a(\mathbf{u},\mathbf{u}-\mathbf{u}_h) - a(\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h) \geq C_2|\mathbf{u}-\mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)}; \tag{3.11}$$

$$|a(\mathbf{u},\mathbf{w}) - a(\mathbf{u}_h,\mathbf{w})| \leq C_4|\mathbf{u}-\mathbf{u}_h|^{\min\{1,\frac{2}{r'}\}}_{(\mathbf{u},r,\alpha)}\|\mathbf{w}\|_V, \tag{3.12}$$

*where* $C_4 = C_3 2^{\max\{0,\frac{r-2}{2}\}}[(\Xi_\alpha\circ G^{-1})(\frac{1}{K_1}\|\mathbf{f}\|_{V'})]^{\max\{0,\frac{r-2}{2}\}}$*, and* $C_2=C_2(K_1,r)$*,* $C_3=C_3(K_2,r)$ *are as in Lemma 3.3.*

**4. Proof of the *a posteriori* error bound.** Equipped with the results of the previous section, we now return to the proof of Proposition 2.1, whereupon we shall prove Theorem 1.1. Our starting point is the following result.

PROPOSITION 4.1. *Let* $(\mathbf{u},p)\in V\times Q$ *denote the solution to (1.1), (1.2), and let* $(\mathbf{u}_h,p_h)\in V_h\times Q_h$ *denote its finite element approximation defined by (2.2), (2.3); then, there exists a positive constant* $C=C(K_1,K_2,c_0,r,\|\mathbf{f}\|_{V'})$ *such that*

$$|\mathbf{u}-\mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)} \leq C\left(\|\mathbf{S}_1\|^{\mathrm{R}'}_{V'} + \|\mathrm{S}_2\|^{\text{я}'}_{Q'} + \|\mathbf{S}_1\|_{V'}\|\mathrm{S}_2\|_{Q'}\right), \tag{4.1}$$

$$\|p-p_h\|^{\text{я}}_Q \leq C\left(\|\mathbf{S}_1\|^{\text{я}}_{V'} + |\mathbf{u}-\mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)}\right), \tag{4.2}$$

*where* $\mathrm{R}=\max\{r,2\}$*,* $\text{я}=\max\{r',2\}$*,* $1/\mathrm{R}+1/\mathrm{R}'=1$*,* $1/\text{я}+1/\text{я}'=1$*, and* $\mathbf{S}_1$ *and* $\mathrm{S}_2$ *are bounded according to (2.18) and (2.19).*

*Proof. (Proposition 4.1.)* According to the inf-sup condition (1.3), identity (2.20), the definition of the norm $\|\cdot\|_{V'}$ and (3.12), we have that

$$c_0\|p-p_h\|_Q \leq \|\mathbf{S}_1\|_{V'} + C_4|\mathbf{u}-\mathbf{u}_h|^{\min\{1,\frac{2}{r'}\}}_{(\mathbf{u},r,\alpha)}, \tag{4.3}$$

and hence (4.2) on noting that $\min\{1,\frac{2}{r'}\} = 2\min\{\frac{1}{2},\frac{1}{r'}\} = 2/\text{я}$.

On the other hand, taking $\mathbf{w}=\mathbf{u}-\mathbf{u}_h$ in (2.20), then using (2.21) with $q=p-p_h$, (3.11), and the definitions of the norms $\|\cdot\|_{V'}$ and $\|\cdot\|_{Q'}$, we get that

$$C_2|\mathbf{u}-\mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)} \leq \|\mathbf{S}_1\|_{V'}\|\mathbf{u}-\mathbf{u}_h\|_V + \|\mathrm{S}_2\|_{Q'}\|p-p_h\|_Q. \tag{4.4}$$

Multiplying (4.4) by $c_0$ and then eliminating $c_0\|p-p_h\|_Q$ using (4.3) gives

$$\begin{aligned} c_0 C_2 |\mathbf{u}-\mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)} &\leq c_0\|\mathbf{S}_1\|_{V'}\|\mathbf{u}-\mathbf{u}_h\|_V + \|\mathbf{S}_1\|_{V'}\|\mathrm{S}_2\|_{Q'} \\ &\quad + C_4\|\mathrm{S}_2\|_{Q'}|\mathbf{u}-\mathbf{u}_h|^{\min\{1,\frac{2}{r'}\}}_{(\mathbf{u},r,\alpha)}. \end{aligned} \tag{4.5}$$

Part (iii) of Proposition 3.2, with $\mathbf{v}=\mathbf{u}-\mathbf{u}_h$ and $\mathbf{w}=\mathbf{u}$ implies that

$$\|\mathbf{u}-\mathbf{u}_h\|_V \leq [\Xi_\alpha(\|\mathbf{u}-\mathbf{u}_h\|_V + \|\mathbf{u}\|_V)]^{\max\{0,\frac{2-r}{2}\}}\,|\mathbf{u}-\mathbf{u}_h|^{\min\{1,\frac{2}{r}\}}_{(\mathbf{u},r,\alpha)}. \tag{4.6}$$

Also, recalling from Lemma 3.4 that

$$\|\mathbf{u}\|_V \leq G^{-1}(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'}) \quad \text{and} \quad \|\mathbf{u}_h\|_V \leq G^{-1}(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'})$$

gives

$$\|\mathbf{u}-\mathbf{u}_h\|_V + \|\mathbf{u}\|_V \leq 2\,(\|\mathbf{u}\|_V + \|\mathbf{u}_h\|_V) \leq 2G^{-1}(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'}). \tag{4.7}$$

Hence, (4.6) and the fact that $t \mapsto \Xi_\alpha(t)$ is monotonic increasing on $[0,\infty)$ imply that

$$\|\mathbf{u}-\mathbf{u}_h\|_V \leq 2^{\max\{0,\frac{2-r}{2}\}}[(\Xi_\alpha \circ G^{-1})(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'})]^{\max\{0,\frac{2-r}{2}\}}|\mathbf{u}-\mathbf{u}_h|^{\min\{1,\frac{2}{r}\}}_{(\mathbf{u},r,\alpha)}.$$

We substitute this into the right-hand side of (4.5) to eliminate $\|\mathbf{u}-\mathbf{u}_h\|_V$; thus,

$$\begin{aligned} |\mathbf{u}-\mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)} &\leq C\Big(\|\mathbf{S}_1\|_{V'}|\mathbf{u}-\mathbf{u}_h|^{\min\{1,\frac{2}{r}\}}_{(\mathbf{u},r,\alpha)} + \|\mathbf{S}_1\|_{V'}\|\mathrm{S}_2\|_{Q'} \\ &\quad + \|\mathrm{S}_2\|_{Q'}|\mathbf{u}-\mathbf{u}_h|^{\min\{1,\frac{2}{r'}\}}_{(\mathbf{u},r,\alpha)}\Big), \end{aligned} \tag{4.8}$$

where $C=C(K_1,K_2,c_0,r,\|\mathbf{f}\|_{V'})$ is a positive constant. We apply Young's inequality

$$ab \leq \varepsilon^{1-s}\frac{a^s}{s} + \varepsilon\frac{b^{s'}}{s'}, \qquad 1/s+1/s'=1, \quad 1<s<\infty, \quad a,b\geq 0, \quad \varepsilon>0,$$

to the first and the third terms on the right-hand side of (4.8), with $a = C\|\mathbf{S}_1\|_{V'}$ and $s' = 2/\min\{1,2/r\} = \max\{r,2\}$ in the case of the first term and $a = C\|\mathrm{S}_2\|_{Q'}$ and $s' = 2/\min\{1,2/r'\} = \max\{r',2\}$ in the case of the third term, and take $\varepsilon>0$ sufficiently small so as to hide the term $\varepsilon(1/\max\{r,2\}+1/\max\{r',2\})|\mathbf{u}-\mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)}$ thus resulting from the right-hand side of (4.8) into $|\mathbf{u}-\mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)}$ appearing on the left-hand side of (4.8); any $\varepsilon\in(0,1)$ will suffice. Hence we deduce that

$$|\mathbf{u}-\mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)} \leq C\left(\|\mathbf{S}_1\|^{\mathrm{R}'}_{V'} + \|\mathbf{S}_1\|_{V'}\|\mathrm{S}_2\|_{Q'} + \|\mathrm{S}_2\|^{\text{я}'}_{Q'}\right), \tag{4.9}$$

where $\mathrm{R}=\max\{r,2\}$, $\text{я}=\max\{r',2\}$, $1/\mathrm{R}+1/\mathrm{R}'=1$, $1/\text{я}+1/\text{я}'=1$, and $C$ is at most $1/(1-\varepsilon)$ times what it was in (4.8). ◻

*Proof. (Proposition 2.1.)* Part (iii) of Proposition 3.2 with $\mathbf{v}=\mathbf{u}-\mathbf{u}_h$ and $\mathbf{w}=\mathbf{u}$, (4.7) and (4.9) imply that

$$\|\mathbf{u}-\mathbf{u}_h\|^{\mathrm{R}}_V \leq C\left(\|\mathbf{S}_1\|^{\mathrm{R}'}_{V'} + \|\mathbf{S}_1\|_{V'}\|\mathrm{S}_2\|_{Q'} + \|\mathrm{S}_2\|^{\text{я}'}_{Q'}\right), \tag{4.10}$$

and hence (2.22).

For the error in the pressure, substitution of (4.9) into the right-hand side of (4.2) yields (2.23). That completes the proof of Proposition 2.1. ◻

The *a posteriori* error bounds stated in Proposition 2.1 can be simplified, thus leading to Theorem 1.1 which we now prove.

*Proof. (Theorem 1.1.)* By virtue of (2.16) and Remark 1,

$$\|\mathbf{S}_1\|_{V'} \leq \left(1 + \frac{1}{c_0'}\right) \left(\|\mathbf{f}\|_{V'} + K_2 (H \circ G^{-1})(\tfrac{1}{K_1}\|\mathbf{f}\|_{V'})\right).$$

Thus, on noting that $\mathrm{R}' = \min\{r', 2\} \leq \max\{r', 2\} = \text{я}$, we have that

$$\|\mathbf{S}_1\|_{V'}^{\mathrm{R}'} + \|\mathbf{S}_1\|_{V'}^{\text{я}} = \|\mathbf{S}_1\|_{V'}^{\mathrm{R}'} \left(1 + \|\mathbf{S}_1\|_{V'}^{\text{я}-\mathrm{R}'}\right) \leq C \|\mathbf{S}_1\|_{V'}^{\mathrm{R}'}, \tag{4.11}$$

where $C = C(K_1, K_2, c_0', r, \|\mathbf{f}\|_{V'})$. Also, by Young's inequality and (4.11),

$$\|\mathbf{S}_1\|_{V'} \|\mathrm{S}_2\|_{Q'} \leq \frac{1}{\text{я}} \|\mathbf{S}_1\|_{V'}^{\text{я}} + \frac{1}{\text{я}'} \|\mathrm{S}_2\|_{Q'}^{\text{я}'} \leq C \left(\|\mathbf{S}_1\|_{V'}^{\mathrm{R}'} + \|\mathrm{S}_2\|_{Q'}^{\text{я}'}\right), \tag{4.12}$$

where $C = C(K_1, K_2, c_0', r, \|\mathbf{f}\|_{V'})$. Hence, (4.11), (4.12), together with (2.22) and (2.23) of Proposition 2.1, yield the *a posteriori* error bound (1.5) of Theorem 1.1 stated in the Introduction. □

**5. Concluding remarks.** We presented a general framework for energy-norm-based *a posteriori* error analysis of conforming mixed finite element approximations to quasi-Newtonian flow models. As has been noted in the Introduction, Proposition 2.1 and Theorem 1.1 recover a number of known *a posteriori* bounds from the literature; they also provide new bounds for a very general class of quasi-Newtonian flow models.

When $r = 2$, we have $\mathrm{R} = \mathrm{R}' = \text{я} = \text{я}' = 2$; then, (1.5) of Theorem 1.1 collapses to the *a posteriori* error bound of Barrett and Bao [5] for inf-sup-stable mixed finite element approximations of Carreau-type quasi-Newtonian flows, the linear Stokes problem being a special case [20]. For $r \neq 2$, the bound (1.5) represents an improvement over several earlier results (cf. [3] and [15], in particular) in that the powers of $\|\mathbf{S}_1\|_{V'}$ and $\|\mathrm{S}_2\|_{Q'}$ in our *a posteriori* error bound are larger than the ones in those papers.

The validity of Propositions 2.1 and 4.1 is independent of whether or not the pair of spaces $(V_h, Q_h)$ is inf-sup stable in the sense of (2.1): it is only in the transition from Proposition 2.1 to Theorem 1.1 that we made use of the bound on $\|p_h\|_Q$ from (3.10) which relied on (1.3). Indeed, suppose that problem (1.1), (1.2) has been approximated by the finite element method: find $\mathbf{u}_h \in V_h$ and $p_h \in Q_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \qquad && \forall \mathbf{v}_h \in V_h, \\ b(q_h, \mathbf{u}_h) &= c_h(q_h, p_h) \qquad && \forall q_h \in Q_h, \end{aligned}$$

where $c_h(\cdot,\cdot)$ is a bilinear form on $Q_h \times Q_h$ satisfying $c_h(q_h, q_h) \geq 0$, $q_h \in Q_h$ (the discretisation (2.2), (2.3) being a special case with $c_h(q_h, p_h) = 0$ for all $q_h \in Q_h$). A number of pressure-stabilised finite element discretisations of (1.1), (1.2) are of this form. It is then easy to see that the bound on $\|\mathbf{u}_h\|_V$ from (3.10) still holds irrespective of (2.1), and, if instead of assuming (2.1) we suppose that the sequence $(\|p_h\|_Q)_{h>0}$ is bounded, independent of $h$, then, once again, Theorem 1.1 follows from Proposition 2.1 in exactly the same way as before.

The computational assessment of the sharpness of the *a posteriori* upper bound stated in Theorem 1.1 and the derivation of *a posteriori* lower bounds on the error will be considered in forthcoming papers.

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