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A FINITE ELEMENT METHOD VIA NOISE REGULARIZATION FOR THE STOCHASTIC ALLEN-CAHN PROBLEM

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Abstract. We study finite element approximations of stochastic partial differential equations of Ginzburg-Landau type and the main paradigm considered in this paper is the stochastic Allen-Cahn model. We first demonstrate that the constructed stochastic finite element approximations are within an arbitrary level of tolerance from the corresponding one-dimensional stochastic partial differential equation; secondly we show that the finite element approximation is close to the most probable deterministic trajectory of the stochastic Allen-Cahn equation, even in large time intervals where interfaces form and evolve according to macroscopic mean curvature-dependent evolutions.

1. Introduction

Stochastic partial differential equation (SPDE) models arise in numerous applications ranging from materials science, surface processes and macromolecular dynamics [Coo70, Spo89, Str97], to atmosphere/ocean modeling [LN03] and epidemiology [Dur99]. These models are typically derived through mostly formal arguments from finer, more detailed, models were unresolved degrees of freedom are represented by suitable stochastic forcing terms. There are also some notable rigorous derivations from microscopic scales in special asymptotic regimes [BPRS93, MT95, e.g.].

An important class of these nonlinear SPDE are the stochastic Ginzburg–Landau models which are typically obtained from microscopic lattice models for a suitable order parameter (e.g., spin), by statistical mechanics renormalization arguments combined with detailed balance laws. They formally result to Langevin type dynamics in infinite dimensions and can be categorized as type A models, where the order parameter is not conserved, and type B, conservative, models [HH77].

A common feature of the aforementioned SPDE's is that they are nonlinear and typically have transition regimes associated with nucleation, phase transitions, pattern formation, etc. From a computational view-point such SPDE's are usually handled by using finite difference methods [KM99, KK01]. This approach, which is computationally robust as long as the underlying SPDE has a solution which is not oscillating at the finest resolution scales, has a drawback in that it does not allocate computational resources efficiently, the reason being that the computational space-time grids are uniform. This is a major computational issue in SPDE simulation, even more so than in deterministic PDE ones since many realizations must be carried out in order to reproduce statistical quantities with a satisfactory approximation.

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For this purpose Finite Element Methods (FEM) for SPDE could provide a more flexible framework than finite differences. In fact, it is a well known and a widely employed fact that FEM allow for space-time adaptivity. Heuristically, in an adaptive scheme the computational mesh is very fine in high activity regions (e.g., nucleation regimes, transition layers, etc.) of the computational domain, while the mesh stays coarse elsewhere. This process, should lead in principle to an efficient allocation of resources, in the sense that the best possible approximation is obtained within a given computational time and space. On the other hand, FEM have seen an enormous development of a posteriori error control, which permits one step further in adaptive methods, well beyond the solely heuristic ideas, by providing a mathematical foundation for automatic adaptivity. In this case, the mesh refinement and coarsening are driven by error indicators, that can be effectively computed from the numerical solution and form the building blocks of a bound on the numerical error. The ideal adaptive method, should therefore rely only on such automatically computed quantities, excluding human intervention, and be both efficient and reliable at the same time. Much on the subject of error control and adaptivity for deterministic PDE's can be found in some recent monographs dedicated to space-dependent problems [AO00, Bra01, BS94, Ver96] and, for space-time-dependent problems, in the extensive research paper literature some of which will be cited further on. Notice that recently there has also been a rapid development of a priori error analysis of FEM for linear SPDE's [ANZ98, BTZ04, ST03, e.g.] and various adaptive methods have been pursued for stochastic (ordinary) differential equations [STZ01] as well as for Monte-Carlo algorithms for space dependent problems [CKV05, CVK04]. However, there seems to be yet no studies regarding error control and adaptivity for linear, let alone nonlinear, evolution SPDE's in the literature.

Motivated by this need for robust FEM for nonlinear SPDE's, in this paper we begin a systematic study of a numerical approximation of stochastic Ginzburg–Landau type equations. We focus on the simplest example exhibiting the phenomena of interface formation and nucleation, namely the stochastic Allen–Cahn problem

$$\partial_t u(x,t) - \partial_{xx} u(x,t) + f_\epsilon(u(x,t)) = \epsilon^\gamma \partial_{xt} W(x,t), \text{ for } x \in D, t \in [0,\infty) \tag{1}$$

where $D = (-1,1)$, $\epsilon > 0$ and f_ϵ is an odd nonlinearity scaled by $1/\epsilon^2$ and $\partial_{xt} W$ is the space-time white noise; see §2.1 further for the details on f_ϵ. This is a stochastic version of the well-known deterministic Allen–Cahn problem describing, among the others, the evolution in time of a polycrystalline material [AC79]. We take boundary conditions of Neumann type and the initial condition to be a *resolved profile*; we refer again to §2 for the details.

Equation (1) is a type A model in Halperin's classification [HH77], i.e., it is nonconservative in the order parameter u and exhibits both nucleation and interface formation, while it retains a relatively simple structure without multiplicative or conservative noise terms as the ones arising in type B models, such as the Cahn–Hilliard–Cook equation [KM99, KV03].

While a thorough discussion of (1) is given in §2, it is worth mentioning that this SPDE, with the white noise term, is well-posed only in spatial dimension 1. Two important pieces of work concerned with the analytic and probabilistic aspects of (1) are those of Funaki [Fun95] and Brassesco, De Masi & Presutti [BDMP95]. In both papers, the authors study the asymptotic behavior of the solution processes as $\epsilon \to 0$. In particular, it turns out that, under suitable time-space rescaling, the solution with initial value taken to be (roughly speaking) a step function, converges (in an appropriate probabilistic sense) to the step function with its jump point performing a Brownian Motion.

Our first task, carried out in §3, is to construct a regularization, denoted $\partial_{xt}\bar{W}(x,t)$, of $\partial_{xt}W(x,t)$ (appearing in (1)) with respect to an underlying uniform partition, $\mathscr{D}_\sigma \times \mathscr{I}_\rho$, of the space-time domain $D \times I$. In the spirit of FEM, this regularization process consists of a projection of the white noise onto an appropriate space of piecewise constant space-time functions, which may be viewed as the mixed derivatives of hat functions. This idea, which has been successfully used in the context of the linear heat equation [ANZ98], leads to the *regularized problem*

$$\partial_t u(x,t) - \partial_{xx}u(x,t) + f_\epsilon(u(x,t)) = \epsilon^\gamma \partial_{xt}\bar{W}(x,t), \text{ for } x \in D, t \in [0,\infty). \tag{2}$$

Notice that $\partial_{xt}\bar{W}$ is still a stochastic process in space-time, but it is much smoother than the white noise which allows equation (2) to be interpreted in the usual PDE sense pathwise. In §4 we cover some basic properties of problem (2) and its solution. The main result of this section is Theorem 4.5, which states that the solution of the regularized problem converges—in an appropriate sense—to the solution of the original SPDE (1) as the space-time partition becomes infinitely fine.

In §5, we devote some more to the analysis of (2), taking it as a model for its own sake. We show, in Theorem 5.3, that if the noise is weak enough (namely $\gamma > 4$) then the solution of the regularized problem converges, in an appropriate sense, to the solution of the deterministic Allen–Cahn problem

$$\begin{gathered}\partial_t q - \partial_{xx}q + f_\epsilon(q) = 0, \text{ in } D \times I,\\ q(0) = u_0, \text{ on } D, \text{ and } \partial_x q(t,0) = \partial_x q(t,1),\ t \in I.\end{gathered} \tag{3}$$

Our proof makes use of the *spectrum estimates* of the linearized elliptic differential operator $-\partial_{xx}+f'_\epsilon(q)$, derived independently by Xinfu Chen [Che94] and de Mottoni & Schatzman [dMS95].

The derivation of (2) is the first step, out of two, towards the derivation of an implementable numerical scheme for the original SPDE. Indeed equation (2), having a more regular solution than (1), can be discretized using a standard finite element (or finite difference) scheme for evolution equations. Our second step consists therefore in introducing such a scheme in §6, using a FEM for the space variable and a Backward Euler discretization for the time variable. Related schemes have been thoroughly analyzed and successfully applied in the context of the *deterministic Allen-Cahn problem* [FP03, KNS04] and for the *stochastic linear heat diffusion problem* [ANZ98]. It is for the first time, up to our knowledge, that this scheme is employed in a *stochastic and nonlinear* setting. The issues of regularity of the regularized solution and the convergence of the FEM are left as a topic for a subsequent article [KKL05]. Let us mention just that while an adaptation of the ideas of Feng & Prohl [FP03] and Kessler, Nochetto & Schmidt [KNS04] may prove helpful, this is not straightforward and may be short from sufficient to conduct the analysis for any $\gamma \geq 0$ and $\epsilon > 0$. There are two main reasons for this. First, the spectral estimates derived by Xinfu Chen [Che94] and De Mottoni & Schatzman [dMS95] which allow a reasonable dependence of the computational parameters with respect to ϵ may cease to be valid as γ closes to 0. Secondly, in our case, the regularity assumptions on the "exact" solution $\bar{u}$ have to be weakened with respect to those assumed by Feng & Prohl in their a priori error analysis [FP03]; on the other hand, it is not clear how to conduct an adaptive method with the stochastic term using a posteriori error estimators similar to those of Kessler et al. [KNS04].

In §7, we test our scheme in a practical situation, geared towards reconciling the computational results with the main aspects of the theoretical results outlined by Funaki [Fun95] and Brassesco, De Masi & Presutti [BDMP95] independently. We focus on tracking the so-called *center* of a resolved profile of the Allen-Cahn equation as time evolves. For the deterministic Allen–Cahn it is well-known [CP89] that so-called *metastable states*, consisting of two separated phases, are non-equilibrium points that take times of the order $\exp(C_0/\epsilon)$ to reach a stable state, for which one phase prevails on the other. One of the main difference between the deterministic and the stochastic Allen–Cahn is that the speed that metastable states take to evolve into stable ones is much higher, indeed this takes times polynomial in ϵ. Furthermore, the *center of a resolved profile* follows a rescaled Brownian motion with diffusion coefficient given, in our scaling, by $\sqrt{3 \times 2^{-3/2}\epsilon^{1+2\gamma}}$. We use this fact as a theoretical benchmark to test the quality of our FEM solution of (2) by showing that, as ϵ decreases, the average of the numerical solution gets closer to the error function distribution with 0 mean and standard deviation $\sqrt{3 \times 2^{-3/2}\epsilon^{1+2\gamma}t}$. Although, this benchmarking procedure is not sufficient to validate the convergence of the FEM, it gives an as accurate as possible quantification of our method. It is worth mentioning that a similar procedure, albeit in the context of stochastic ODE's has been employed by Shardlow as a test case [Sha00]. The fact that the procedure succeeds, in that the behavior of the numerical solution reflects that of the exact solution, encourages us down the road of further studies of this method, its extension to higher dimensions and the derivation of adaptive methods.

Note also that the most probable path of (1) is given by its deterministic analogue as obtained also by large deviations results for stochastic reaction diffusion equations in [FW98] (cf [Fun95], [BDMP95]). The analysis in §5 can be seen as a quantification in terms of detailed error estimates. Furthermore, in view of well-known asymptotic limits in dimension 2 or higher of the deterministic Allen-Cahn equation to motion by mean curvature [BSS93], this latter result demonstrates that the FEM approximation presented here is a stochastic approximation to motion by mean curvature evolutions. The influence of stochastic corrections to motion by mean curvature evolutions and in particular to its instabilities such as interface fattening was recently demonstrated in [KK01, DLN01, SY04]. Our results give thus a first rigorous indication that this approximation framework could provide an accurate and potentially efficient algorithm (via FEM adaptivity) for simulating the stochastic motion by mean curvature and more complicated geometric motions that arise in stochastic PDE modeling.

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2. Set up

2.1. **Noisy Allen-Cahn problem.** We will study an initial-boundary value problem associated with the semilinear parabolic partial differential equation with additive white noise, known as the *stochastic* (or *noisy*) *Allen-Cahn equation* given by (1). The nonlinearity f_ϵ must be the derivative of an even coercive function F_ϵ with exactly two minimum points. Such a F_ϵ is known as a *double-well potential* and, for sake of conciseness, we focus on the model potential explicitly defined by

$$F_\epsilon(\xi) = \frac{1}{4\epsilon^2}(\xi^2 - 1)^2, \text{ for } \xi \in \mathbb{R}. \tag{4}$$

Here $\epsilon \in \mathbb{R}^+$ is a scaling parameter. The term $\partial_{xt}W$ is the space-time *Gaussian white noise*, which can be defined as the mixed distributional derivative of a *Brownian sheet* W [Wal86, KX95]. The parameter $\gamma \in \mathbb{R}$ models the intensity of the white noise and plays a delicate role in the analysis, as $\epsilon \to 0$.

The presence of the right-hand side makes (1) a randomly perturbed version of the Allen-Cahn equation which is a *stochastic PDE* (SPDE). A solution of such an equation has to be interpreted in the stochastic sense. That is, for each t, the solution $u(\cdot, t)$ is understood as a random process on an underlying probability measure space $(\Omega, \mathscr{F}, P)$ with values in a suitable function space defined on D. Equation (1), supplemented with the initial condition

$$u(x,0) = u_0(x),\ \forall x \in D, \tag{5}$$

and with the Neumann boundary conditions

$$\partial_x u(-1,t) = \partial_x u(1,t) = 0 \quad \forall t \in \mathbb{R}^+, \tag{6}$$

defines the *stochastic Allen-Cahn problem.* We will assume throughout the paper that u_0 is a *resolved profile solution*, that is, for all $x \in D$, $u_0(x)$ is a linear perturbation of $q_0((x-\alpha)/\epsilon)$, where $\alpha \in D$ and q_0 is the unique solution to

$$-q_0'' + f_1(q_0) = 0 \text{ in } \mathbb{R}, \quad q_0(\pm\infty) = \pm 1 \text{ and } q_0(0) = 0 \tag{7}$$

[Che94, p. 1374] [dMS95, Thm. 5.1].

2.2. **Space-time stochastic integral.** One can give a mathematically rigorous definition of a solution of the stochastic Allen-Cahn problem (1),(5)–(6) as a distribution-valued process [Wal86, KX95]. However, we find it more convenient, as in the case of the white noise generated from a Brownian motion, to work with the *stochastic integral* with respect to the Brownian sheet W denoted by "$\int \cdot \, \mathrm{d}W$" [Wal86, §II] [KX95, Ch. 3]. In our doing so, we bear in mind the formal relationship

$$\int_0^\infty \int_D f(x,t)\partial_{xt} W(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_D f(x,t)\,\mathrm{d}W(x,t) \tag{8}$$

that will inspire the weak formulation (14) and the definitions in §3. In the particular case where f is the characteristic function of a Borel-measurable set $A \in \mathscr{B}(\mathbb{R}^+ \times D)$ of Lebesgue measure $|A| < \infty$ the following basic property of the stochastic integral is satisfied

$$\int_0^\infty \int_A \mathrm{d}W(x,t) = W(A) \in \mathrm{N}\left(0, |A|\right), \tag{9}$$

i.e., $W(A)$ is a Gaussian random variable with mean zero and variance $|A|$.[1]

Since we are interested in numerical solutions, we consider the time domain to be a bounded interval $I = [0,T]$, for some fixed $T \in \mathbb{R}^+$. A fundamental property of the stochastic integral is the following well-known L_2-*isometry*, which holds for the Itô integral,

$$\mathbb{E}\left[\left(\int_I \int_D f(x,t)\,\mathrm{d}W(x,t)\right)^2\right] = \mathbb{E}\left[\int_I \int_D f(x,t)^2\,\mathrm{d}x\,\mathrm{d}t\right], \tag{10}$$

for any $\mathscr{F}_t^W$-measurable $f \in \mathrm{L}_2(I \times D \times \Omega)$, where

$$\mathscr{F}_t^W = \sigma\left\{W(A) : \ A \in \mathscr{B}(I \times D)\right\}, \tag{11}$$

is the sigma-field (or sigma-algebra) generated by W up to time t, and $\mathbb{E}$ denotes the *expectation* with respect to $(\Omega, \mathscr{F}, P)$.

A useful consequence of (10) is that

$$\mathbb{E}\left[\int_I \int_D f(x,t)\,\mathrm{d}W(x,t) \int_I \int_D g(y,s)\,\mathrm{d}W(y,s)\right] = \mathbb{E}\left[\int_I \int_D f(x,t)g(x,t)\,\mathrm{d}x\,\mathrm{d}t\right], \tag{12}$$

for any $\mathscr{F}_t^W$-measurable $f, g \in \mathrm{L}_2(I \times D \times \Omega)$. In the special case where f and g are, respectively, the characteristic functions of two Borel sets A and $B \in \mathscr{B}(I \times D)$, with $|A|, |B| < \infty$, (12) implies

$$\mathrm{Cov}(W(A), W(B)) = |A \cap B|. \tag{13}$$

[1]For $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ we denote by $\mathrm{N}\left(\mu, \sigma^2\right)$ the class of normally distributed (or Gaussian) random variables of mean μ and variance σ^2 on the space Ω.

2.3. **Integral solutions.** By multiplying (1) with a test function $\phi \in \mathrm{C}^2_{\mathrm{c}}(D \times (0,\infty))$ and using the formal relation (8), we can write the problem in the usual weak form[2]

$$\int_0^\infty \int_D u\partial_t\phi + \partial_x u\partial_x\phi + f_\epsilon(u)\phi + \epsilon^\gamma \int_0^\infty \int_D \phi \,\mathrm{d}W = 0. \tag{14}$$

Despite the above formulation being quite useful, especially for studying a numerical scheme, it is not very convenient to nail down the concept of solution. A rather more convenient way to give rigorous meaning to (1) is to look for an integral solution of an *equivalent integral equation* [DPZ92, Doe87, FJL82, Wal86], as we briefly illustrate next.

Introduce first the corresponding boundary value problem for the stochastic linear heat equation [DPZ92, Wal86]

$$\begin{gathered}\partial_t Z - \partial_{xx} Z = \partial_{xt} W, \text{ in } D \times \mathbb{R}^+_0\\ Z(x,0) = 0, \text{ on } D\\ \partial_x Z(1,t) = \partial_x Z(-1,t) = 0, \ \forall t \in [0,\infty).\end{gathered} \tag{15}$$

The solution to this problem can be defined as the Gaussian process in space-time produced by the stochastic integral

$$Z_t(x) = Z(x,t) := \int_0^t \int_D G_{t-s}(x,y)\,\mathrm{d}W(y,s), \tag{16}$$

where G is the heat kernel for the corresponding homogeneous Neumann problem. In our one-dimensional particular case, G can be explicitly written as

$$G_t(x,y) = 4\sum_{k=0}^\infty (2-\delta_0^k)\cos\frac{\pi k(x+1)}{2}\cos\frac{\pi k(y+1)}{2}\exp\frac{-\pi^2 k^2 t}{4}, \tag{17}$$

where δ_0^k is the Kronecker symbol.

The *integral solution* of (1) can then be defined as a solution of the equivalent integral equation

$$u(x,t) = -\int_0^t \int_D G_{t-s}(x,y) f_\epsilon(u(y,s))\,\mathrm{d}y\,\mathrm{d}s + \int_D G_t(x,y)u_0(y)\,\mathrm{d}y + \epsilon^\gamma Z_t(x). \tag{18}$$

It is known that such a solution exists uniquely as a $\mathrm{C}^0(D)$-valued continuous process, $t \mapsto u(.,t)$, adapted to Z_t, provided the initial condition u_0 satisfies the Neumann boundary conditions [BDMP95, Wal86, FJL82]. In this article we use this concept of solution which we refer to simply as the *solution of Problem* (1),(5)–(6) and we will denote it by u. Notice that u is also referred to by some authors as the *Ginzburg-Landau process* [BDMP95].

For the aims set in this paper, in order to study the error of convergence of an approximation of the solution of (1), we first need a uniform bound for u. In the deterministic case such a bound is direct consequence of the maximum principle. In our case we do not expect to have a uniform bound in the whole probability space. However, a bound on a set with large probability controlled by ϵ will suffice for our needs.

[2]Whenever the meaning is clear from the context, for sake of conciseness, we often drop the variables "x, t" and, in non-stochastic integrals, also the corresponding elementary terms "d".

2.4. **Theorem** (Probabilistic maximum principle) *Given T, there exist $c_1,\ c_2, \delta_0 > 0$ such that if $\|u_0\|_{\mathrm{L}_\infty(D)} \leq 1+\delta_0$ then*

$$P\left\{\sup_{t\in[0,T]} \|u(t)\|_{\mathrm{L}_\infty(D)} > 3\right\} \leq c_1 \exp(-c_2/\epsilon^{1+2\gamma}). \tag{19}$$

Proof

We conduct ourselves to Proposition 5.2 of Brassesco et al. [BDMP95] by introducing the time-space rescaling: $t \mapsto t/\epsilon^2$ and $x \mapsto x/(\sqrt{2}\epsilon)$ and extending the solution periodically to the whole space as to obtain the proper barrier function. Since we are dealing with the more general case $\gamma \geq 0$, while they deal with the case $\gamma = 0$ only, we retrace the salient points of their proof. The barrier function v satisfies the following equation—corresponding to [BDMP95, (5.12)]:

$$\partial_t v - \frac{1}{2}\partial_{xx} v + 2v = -3v^2 - v^3 + 2^{-1/4}\epsilon^{\gamma+1/2}\partial_{xt} W. \tag{20}$$

Consider now the function

$$V(x,t) = \int_0^t \exp(-2(t-s)) H_{t-s}^{(\epsilon\sqrt{2})}(x,y)\,\mathrm{d}W(y,s), \tag{21}$$

where $H_{t-s}^{(\epsilon\sqrt{2})}$ is the Green operator defined by

$$\exp(-2t) H_{t-s}^{(\epsilon\sqrt{2})} = \left(\partial_t - \frac{1}{2}\partial_{xx} + 2\,\mathrm{Id}\right)^{-1}, \tag{22}$$

with homogeneous boundary conditions on $\big(-1/(\sqrt{2}\epsilon), 1/(\sqrt{2}\epsilon)\big)$. By using equation [BDMP95, (5.2)] with $\lambda = \exp(-(\gamma+1/2))$ and adapting properly the proof of [BDMP95, Lemma 2.1] we can easily conclude that for each $b > 0$ there exist c_1 and $c_2 > 0$ such that

$$P^\epsilon\left(\sup_{t\leq T\epsilon^{-2}, x\in\mathbb{R}} \left|\epsilon^{\gamma+1/2} V(x,t)\right| > b\right) \leq c_1 \exp(-c_2/\epsilon^{1+2\gamma}). \tag{23}$$

The rest of the proof is now identical to that of [BDMP95, Proposition 5.2]. ∎

3. White noise approximation

In order to introduce a finite element method (FEM) that approximates a solution of (18), we first need to to obtain a *weak formulation* in the standard sense of PDE and FEM. This is not possible with the presence of the white noise, so we regularize first the problem by replacing the white noise with a smoother stochastic term. Our technique is inspired by that of Allen, Novosel & Zhang [ANZ98] for the linear heat equation.

3.1. **A piecewise constant approximation of the white noise.** Consider a tensor-product partition of the space-time domain, $\mathscr{D}_\sigma \times \mathscr{I}_\rho$, where $\sigma, \rho \in \mathbb{R}^+$ and

$$\begin{aligned} &\mathscr{D}_\sigma := \{D_m :\ D_m := (x_{m-1}, x_m),\ m \in [1:M]\}, \\ &\text{and } \mathscr{I}_\rho := \{I_n :\ I_n := [t_{n-1}, t_n),\ n \in [1:N]\}, \end{aligned} \tag{24}$$

are, respectively, a space-domain, and a time-domain, partition; each one of these partitions is uniform, that is

$$x_m - x_{m-1} = \sigma,\ \forall m \in [1:M] \text{ and } t_n - t_{n-1} = \rho,\ \forall n \in [1:N] \tag{25}$$

and $x_0 = -1$, $x_M = 1$, $t_0 = 0$ and $t_N = T$. Let us denote by $\chi_m = \mathbf{1}_{D_m}$ and $\varphi_n = \mathbf{1}_{I_n}$ the characteristic functions of the space subdomains and time subdomains respectively. We think of these as "basis functions" with respect to which we will construct the regularized approximation of the white noise through a projection operation.

The (piecewise constant) *approximation of white noise*, abbreviated by *AWN*, is given by the random space-time function

$$\partial_{xt}\bar{W}(x,t) = \sum_{n=1}^{N}\sum_{m=1}^{N} \bar{\eta}_{m,n}\chi_m(x)\varphi_n(t) \tag{26}$$

where the coefficients are the random variables defined by

$$\bar{\eta}_{m,n} := \frac{1}{\sigma\rho}\int_I\int_D \chi_m(x)\psi_n(t)\,\mathrm{d}W(x,t). \tag{27}$$

In the sequel we will use the shorthand

$$\int_0^t\int_D f(x,s)\,\mathrm{d}\bar{W}(x,s) = \int_0^t\int_D f(x,s)\partial_{xt}\bar{W}(x,s)\,\mathrm{d}x\,\mathrm{d}s, \tag{28}$$

despite the integral being a classical non-stochastic one.

3.2. **Lemma** (Moments and independence of the AWN coefficients) *The coefficients $\bar{\eta}_{m,n}$ defined in* (27) *are i.i.d.* $\mathrm{N}\left(0, 1/(\sigma\rho)\right)$ *variables.*

Proof From the definitions of $\bar{\eta}_{m,n}$ and property (9) we have

$$\begin{aligned}\bar{\eta}_{m,n} &= \frac{1}{\sigma\rho}\int_I\int_D \chi_m(x)\varphi_n(t)\,\mathrm{d}W(x,t)\\ &= \frac{1}{\sigma\rho}\int_{I_n}\int_{D_m}\mathrm{d}W(x,t) = \frac{W(I_n\times D_m)}{\sigma\rho}\\ &\in \mathrm{N}\left(0, \frac{|I_n\times D_m|}{\sigma^2\rho^2}\right) = \mathrm{N}\left(0,\frac{1}{\sigma\rho}\right).\end{aligned} \tag{29}$$

To show independence compute the covariances for $m, m' \in [1:M]$ and $n, n' \in [1:N]$, using (12), as follows

$$\begin{aligned}(\sigma\rho)^2\,\mathbb{E}[\bar{\eta}_{m,n}\bar{\eta}_{m',n'}] &= \mathbb{E}\left[\int_I\int_D \chi_m\varphi_n\,\mathrm{d}W\int_I\int_D\chi_{m'}\varphi_{n'}\,\mathrm{d}W\right]\\ &= \int_I\int_D \chi_m(x)\chi_{m'}(x)\varphi_n(t)\varphi_{n'}(t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \delta^m_{m'}\delta^n_{n'}\sigma\rho,\end{aligned} \tag{30}$$

where δ^i_j is the Kronecker symbol. ∎

The AWN satisfies two important technical properties that we state and prove next.

3.3. **Lemma** (Approximate Itô-type inequality) *For all deterministic functions $f \in \mathrm{L}_2(I\times D)$ the following holds true*

$$\mathbb{E}\left[\left(\int_I\int_D f(x,t)\,\mathrm{d}\bar{W}(x,t)\right)^2\right] \le \int_I\int_D f(x,t)^2\,\mathrm{d}x\,\mathrm{d}t. \tag{31}$$

Proof Lemma 3.2 and some manipulations yield

$$\mathbb{E}\left[\left(\int_I\int_D f(x,t)\,\mathrm{d}\bar{W}(x,t)\right)^2\right] = \mathbb{E}\left[\left(\int_I\int_D f(x,t)\sum_{mn}\bar{\eta}_{m,n}\chi_m(x)\phi_n(t)\,\mathrm{d}x\,\mathrm{d}t\right)^2\right]$$

$$= \mathbb{E}\left[\left(\sum_{mn}\bar{\eta}_{m,n}\int_{I_n}\int_{D_m} f(x,t)\,\mathrm{d}x\,\mathrm{d}t\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{nm}\bar{\eta}^2_{m,n}\left(\int_{I_n}\int_{D_m} f\right)^2 + 2\sum_{n\neq n', m\neq m'}\bar{\eta}_{m,n}\bar{\eta}'_{m'n}\left(\int_{I_n}\int_{D_m} f\right)\left(\int_{I'_n}\int_{D'_m} f\right)\right]$$

$$= \sum_{mn}\mathbb{E}\left[\bar{\eta}^2_{m,n}\right]\left(\int_{I_n}\int_{D_m} f\right)^2$$

$$= \sum_{mn}\frac{1}{\rho\sigma}\left(\int_{I_n}\int_{D_m} f\right)^2$$

$$\leq \sum_{mn}\int_{I_n}\int_{D_m} f^2 = \int_I\int_D f(x,t)^2\,\mathrm{d}x\,\mathrm{d}t.$$

In the next-to-last step we use the Cauchy–Schwarz inequality. ∎

3.4. **Remark** Lemma 3.3 and (10) imply that

$$\mathbb{E}\left[\left(\int_I\int_D f(x,t)\,\mathrm{d}\bar{W}(x,t)\right)^2\right] \leq \mathbb{E}\left[\left(\int_I\int_D f(x,t)\,\mathrm{d}W(x,t)\right)^2\right]. \tag{32}$$

In other words, the L_2-type regularity properties of the AWN will be, at the worse, the same as those of the white noise itself. In many cases the regularity properties of the AWN can be actually improved with respect to those of the white noise, as illustrated by the next result.

3.5. **Lemma** ($\mathrm{L}_\infty(0,T;\mathrm{L}_2(D))$ bound for the AWN) *For each $K > 0$ there exists an event $\Omega^2_K \subset \Omega$ that satisfies*

$$P(\Omega^2_K) \geq \left[1 - \frac{T}{\rho}\left(1 + \frac{K^2}{2}\rho\right)^{1/(2\sigma)-1}\exp\left(-\frac{K^2}{2}\rho\right)\right]^+ \tag{33}$$

and such that

$$\sup_{t\in[0,T]}\left\|\partial_{xt}\bar{W}(t)\right\|_{\mathrm{L}_2(D)} \leq K, \text{ on } \Omega^2_K. \tag{34}$$

Proof We proceed in steps.
Step 1. Recall that $M = 1/\sigma$ and $N = T/\rho$. By the definition of $\partial_{xt}\bar{W}$ we have, for each $t \in [0,T]$ and $n \in [1:N]$ such that $t \in I_n$, that

$$\left\|\partial_{xt}\bar{W}(t)\right\|^2_{\mathrm{L}_2(D)} = \sigma\sum_{m=1}^M \bar{\eta}^2_{m,n} = \frac{1}{\rho}\sum_{m=1}^M \eta^2_{m,n}, \tag{35}$$

where the $\eta_{m,n} \in \mathrm{N}(0,1)$. In order to conclude, we will obtain a condition on the right-hand side that makes it smaller than K^2, for all $n \in [1:N]$.
Step 2. For each $n \in [1:N]$ we consider the random variable

$$H_n := \sum_{m=1}^M \eta^2_{m,n}. \tag{36}$$

Notice that, in view of Lemma 3.2 for $n \neq n'$, H_n and $H_{n'}$ are independent. Let us fix n for a while and find an event for which $H_n \leq \rho K^2$. By Lemma 3.2 and a basic probability fact [Bil95, Pbm. 20.16], the random variable H_n has a chi-squared distribution with M degrees of freedom. Its density is given by

$$\frac{z^{M/2-1}\exp(-z/2)}{2^{M/2}\Gamma(M/2)}, \text{ for } z > 0, \tag{37}$$

and 0 for $z \leq 0$, where Γ is the Euler Gamma-function. Thus we have

$$P(H_n \leq \rho K^2) = \frac{1}{2^{M/2}\Gamma(M/2)} \int_0^{\rho K^2} z^{M/2-1}\exp(-z/2)\,\mathrm{d}z. \tag{38}$$

Step 3. We prove next a lower bound on this integral in the case where M is even, the odd case being similar. Let y play the role of ρK^2 and consider for each $k \in \mathbb{N}_0$ the integral

$$I_k := \int_0^y z^k \exp(-z/2)\,\mathrm{d}z. \tag{39}$$

An integration by parts yields the recursive expression

$$I_k = 2kI_{k-1} - 2y^k \exp(-y/2), \tag{40}$$

which allows, by an inductive argument, to see that

$$I_k = 2^{k+1}k! - 2\sum_{i=0}^{k} \frac{k!}{(k-i)!} y^{k-i} 2^i \exp(-y/2). \tag{41}$$

An easy manipulation with the binomial formula implies that

$$I_k \geq 2^{k+1}k!\left(1 - (1+y/2)^k \exp(-y/2)\right). \tag{42}$$

Taking $k = M/2 - 1$ in the above and recalling the definition of I_k and (37) it follows that

$$P(H_n \leq \rho K^2) \geq 1 - \left(1 + \frac{\rho K^2}{2}\right)^{M/2-1} \exp\left(-\frac{\rho K^2}{2}\right); \tag{43}$$

which implies

$$P(H_n \leq \rho K^2) \geq \left[1 - \left(1 + \frac{\rho K^2}{2}\right)^{M/2-1} \exp\left(-\frac{\rho K^2}{2}\right)\right]^+. \tag{44}$$

Step 4. To conclude the proof, we introduce the event

$$\Omega_K^2 = \bigcap_{n=1}^{N} \left\{H_n \leq \rho K^2\right\}, \tag{45}$$

and we observe that, in view of (35), on Ω_K^2 we have

$$\left\|\partial_{xt}\bar{W}(t)\right\|_{\mathrm{L}_2(D)} \leq K, \ \forall t \in [0,T]. \tag{46}$$

On the other hand, using the independence of H_n, $n \in [1:N]$, the simple fact that $(1-\xi)^N \geq 1 - N\xi$ for $\xi \leq 1$ and (44) we can estimate the probability

$$P(\Omega_K^2) = \prod_{n=1}^{N} P\left(H_n \leq \rho K^2\right)$$

$$\geq \left(\left[1 - \left(1 + \frac{\rho K^2}{2}\right)^{M/2-1} \exp\left(-\frac{\rho K^2}{2}\right)\right]^+\right)^N \tag{47}$$

$$\geq \left[1 - N\left(1 + \frac{\rho K^2}{2}\right)^{M/2-1} \exp\left(-\frac{\rho K^2}{2}\right)\right]^+.$$

By replacing $N = T/\rho$ and $M = 1/\sigma$ the lemma's assertion is obtained. ∎

3.6. **Remark** (interpretation of (33)) We may rewrite the term appearing in (33), as

$$\frac{T}{\rho}\left(1 + \frac{K^2}{2}\rho\right)^{1/(2\sigma)-1} \exp\left(-\frac{K^2}{2}\rho\right) \tag{48}$$
$$= T \exp\left(\log\frac{1}{\rho} + \log\left(1 + \frac{K^2}{2}\rho\right)\left(\frac{1}{2\sigma} - 1\right) - \frac{K^2}{2}\rho\right) =: T \exp F(\rho, \sigma, K).$$

The way we will use Lemma 3.5 will be by fixing first $\rho, \sigma \in \mathbb{R}^+$ and by requiring a big enough K such that $T \exp F(\rho, \sigma, K) \ll 0$. This is made possible by the fact that $\lim_{K\to\infty} F(\rho, \sigma, K) = -\infty$ for any fixed $\rho, \sigma \in \mathbb{R}^+$.

4. The regularized solution

We now introduce the regularized solution to problem (1), (5)–(6), which we obtain by replacing the white noise by the AWN in (1). The role of the regularized problem is pivotal in devising a numerical scheme to approximate the stochastic Allen-Cahn problem. We discuss the approximation properties of this regularization with respect to the original problem.

4.1. **Definition** (regularized solution) The *regularized solution*, $\bar{u}$, of the noisy Allen-Cahn problem is the unique continuous solution of the integral equation

$$\bar{u}(x,t) = -\int_0^t \int_D G_{t-s}(x,y) f_\epsilon(\bar{u}(y,s))\, \mathrm{d}y\, \mathrm{d}s \tag{49}$$
$$+ \int_D G_t(x,y) u_0(y)\, \mathrm{d}y + \epsilon^\gamma \int_0^t \int_D G_{t-s}(x,y)\, \mathrm{d}\bar{W}(y,s).$$

4.2. **Theorem** (maximum principle for regularized solutions) *There exist* $\delta_0, c_1, c_2 > 0$*, independent of* ϵ*, such that if* $\|u_0\|_{\mathrm{L}_\infty(D)} \leq 1 + \delta_0$ *then*

$$P\left\{\sup_{t\in[0,T]} \|\bar{u}(t)\|_{\mathrm{L}_\infty(D)} \leq 3\right\} \geq 1 - c_1 \exp(-c_2/\epsilon^{1+2\gamma}). \tag{50}$$

Proof We follow exactly the proof of Theorem 2.4, by observing that (32) ensures that all the estimates for the stochastic integrals of the white noise can be "translated" in corresponding estimates for the integrals of the approximate white noise. The constants appearing in this Theorem can be therefore taken to be the same that appear in §2.4. ∎

4.3. **Remark** (regularized solution is strong solution) Notice that the regularized solution $\bar{u}$ of (49) is in fact a weak solution in the PDE sense, i.e., $\bar{u}(t;\omega) \in \mathrm{H}^1(D)$

and $\partial_t \bar{u}(t,\omega) \in \mathrm{L}_2(D)$ for all $t \in (0,T]$ and $\omega \in \Omega$, and the following *weak formulation* is satisfied:

$$\begin{aligned}\langle \partial_t \bar{u}(t;\omega), \phi\rangle - \langle \partial_x \bar{u}(t;\omega), \partial_x \phi\rangle + \langle f_\epsilon(\bar{u}(t;\omega)), \phi\rangle \\ = \epsilon^\gamma \left\langle \partial_{xt} \bar{W}(t;\omega), \phi\right\rangle \quad \forall \phi \in \mathrm{H}^1_0(D), t \in (0,T] \\ \text{and } \bar{u}(0;\omega) = u_0,\end{aligned} \tag{51}$$

for each $\omega \in \Omega$ (the notation $\langle \cdot, \cdot\rangle$ indicating the inner product in $\mathrm{L}_2(D)$). Indeed, each one of the AWN's realizations, $\partial_{xt}\bar{W}(\omega)$, is a piecewise constant space-time function. For each such realization the usual regularity theory for semilinear parabolic equations with piecewise continuous data can be applied and the corresponding weak formulation written down [LSU68].

4.4. **Definition** (regularization error) Our next goal is to show that the regularized approximate solution converges to the solution u. For this we will estimate the *regularization error*

$$e(x,t) = u(x,t) - \bar{u}(x,t), \tag{52}$$

in terms of the white noise regularization parameters σ and ρ, and show that it converges to zero in an appropriate sense.

4.5. **Theorem** (convergence to the stochastic solution) *For a fixed T, there exist constants c_1, c_2, C_1 and C_2 such that for each $\epsilon \in (0,1)$ there correspond an event Ω^∞_ϵ and a constant $C_\epsilon > 0$ such that*

$$P(\Omega^\infty_\epsilon) \geq 1 - 2c_1 \exp(-c_2/\epsilon^{1+2\gamma}) \text{ and} \tag{53}$$

$$\int_{\Omega^\infty_\epsilon} \left(\int_0^T \int_D |\bar{u} - u|^2 \right) \mathrm{d}P \leq C_\epsilon \left(C_1 \rho^{1/2} + C_2 \frac{\sigma^2}{\rho^{1/2}} \right), \ \forall \sigma, \rho > 0. \tag{54}$$

Proof We proceed by steps.
Step 1. By the integral representations of u, (18), and $\bar{u}$, (49), we can represent the error as an integral too:

$$\begin{aligned} e(x,t) = \int_0^t \int_D G_{t-s}(x,y) \big(f(\bar{u}(y,s)) - f_\epsilon(u(y,s)) \big) \,\mathrm{d}y \,\mathrm{d}s \\ + \int_0^t \int_D G_{t-s}(x,y) \left(\mathrm{d}W(y,s) - \mathrm{d}\bar{W}(y,s) \right) \end{aligned} \tag{55}$$

for all $(x,t) \in D \times (0,T]$. So our task now is to bound the terms in the left-hand side of (55) in the appropriate norm.
Step 2. In view of the maximum principle for both the exact solution, §2.4, and the approximate solution, §4.2, there exists an event $\Omega^\infty_\epsilon \subset \Omega$ such that

$$P(\Omega^\infty_\epsilon) \geq 1 - 2c_1 \exp(-c_2/\epsilon^{1+2\gamma}) \tag{56}$$

and

$$\Omega^\infty_\epsilon \subset \left\{ \|u(t)\|_{\mathrm{L}_\infty(D)}, \|\bar{u}(t)\|_{\mathrm{L}_\infty(D)} \leq 3, \ \forall t \in [0,T] \right\}. \tag{57}$$

This and the local Lipschitz continuity of f imply that

$$|f_\epsilon(\bar{u}) - f_\epsilon(u)| \leq \frac{28}{\epsilon^2} |\bar{u} - u|, \ \text{ on } \Omega^\infty_\epsilon. \tag{58}$$

Step 3. Working now on the event Ω_ϵ^∞ and introducing the functions

$$\varepsilon(r) := \int_0^r \int_D e(x,t)^2 \,\mathrm{d}x\,\mathrm{d}t \tag{59}$$

$$\phi(r) := \int_0^r \int_D \left| \int_0^t \int_D G_{t-s}(x,y)(\,\mathrm{d}W(y,s) - \,\mathrm{d}\bar{W}(y,s)) \right|^2 \mathrm{d}x\,\mathrm{d}t \tag{60}$$

for all $r \in [0,T]$, we infer from (55) that

$$\varepsilon(r) \leq 2\int_0^r \int_D \left(\int_0^t \int_D |G_{t-s}(x,y)| \,\frac{28}{\epsilon^2} e(y,s)\,\mathrm{d}y\,\mathrm{d}s \right)^2 \mathrm{d}x\,\mathrm{d}t + 2\phi(r). \tag{61}$$

The integral in (61) can be bounded, using the Cauchy–Schwarz inequality, by

(62)

$$2\frac{28^2}{\epsilon^4} \int_0^r \int_D \left(\int_0^t \int_D |G_{t-s}(x,y)|^2 \,\mathrm{d}y\,\mathrm{d}s \int_0^t \int_D e(y,s)^2 \,\mathrm{d}y\,\mathrm{d}s \right) \mathrm{d}x\,\mathrm{d}t = \int_0^r z(t)\varepsilon(t)\,\mathrm{d}t$$

where

$$z(t) := 2\frac{28^2}{\epsilon^4} \int_D \int_0^t \int_D |G_{t-s}(x,y)|^2 \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x. \tag{63}$$

Inequality (61) implies

$$\varepsilon(r) \leq \phi(r) + \int_0^r z(t)\varepsilon(t)\,\mathrm{d}t, \tag{64}$$

for each $r \in I$. Applying the Gronwall lemma to this inequality we obtain

$$\varepsilon(T) \leq \exp\left(\int_0^T z(t)\,\mathrm{d}t \right) \phi(T) \leq C_\epsilon \phi(T), \tag{65}$$

where—by estimating the heat kernel—the constant is given by

$$C_\epsilon := \exp\left(\frac{28^2\,T}{12\,\epsilon^4} \right). \tag{66}$$

Step 4. By summing with respect to P on the event Ω_ϵ^∞ both members of this inequality we obtain

$$\int_{\Omega_\epsilon^\infty} \int_0^T \int_D |\bar{u} - u|^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}P \leq C_\epsilon \int_{\Omega_\epsilon^\infty} \phi(T)\,\mathrm{d}P \leq C_\epsilon\,\mathbb{E}[\phi(T)]. \tag{67}$$

We conclude by observing [ANZ98, Lem. 2.3] that there exist $C_1, C_2 > 0$, depending only on T, such that

$$\mathbb{E}[\phi(T)] \leq C_1 \rho^{1/2} + C_2 \frac{\sigma^2}{\rho^{1/2}}. \tag{68}$$

Thus we established that

$$\int_{\Omega_\epsilon^\infty} \int_0^T \int_D |\bar{u} - u|^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}P \leq C_\epsilon \left(C_1 \rho^{1/2} + C_2 \frac{\sigma^2}{\rho^{1/2}} \right), \tag{69}$$

as we claimed. ∎

4.6. **Remark** (About the constant C_ϵ) Theorem 4.5 insures that, for fixed T and ϵ, the approximate solution $\bar{u}$ converges to u as $\rho, \sigma \to 0$. The constant C_ϵ appearing in the estimate depends exponentially on both $1/\epsilon^4$ and T, thus for small ϵ, or large T, this might force us to take very small ρ and σ. This fact should to be taken into account in practice. The bound we have proved seems to be pessimistic though, as the choice of σ and ρ, used in our subsequent numerical experiments, indicates.

5. Convergence for weak noise

Notice that the regularized problem from previous section, for small yet non-vanishing ρ and σ, could have a value in modeling for its own sake. In other words, the stochastic PDE might be a rougher model than its regularization for some physical phenomena, such as small scale fluctuations. Therefore the analysis and the approximation of the regularized problem is important.

In this section we show that for weak noise intensity, ϵ^γ, the limit as $\epsilon \to 0$ coincides with the deterministic solution. Our proof makes use of a *spectrum estimate* result for the linearized Allen-Cahn operator [Che94, dMS95], which is recalled in Theorem 5.2. For technical reasons, we have to assume that $\gamma > 4$ for the result to hold. We believe that this threshold could be lowered, ideally down to $\gamma > 0$, but we do not know how to prove this at the moment.

We notice that the analogous result for the stochastic PDE is also valid, and has been proved independently by Funaki [Fun95] and Brassesco et al [BDMP95]. Their proofs, however are much more involved and require much more technical background in probability than ours.

5.1. **Definition** (Deterministic solution) Let q be the (classical) solution of the problem

(70) $$\partial_t q - \partial_{xx} q + f_\epsilon(q) = 0, \text{ in } D \times I$$

(71) $$q(0) = u_0, \text{ on } D$$

(72) $$\partial_x q(t,0) = \partial_x q(t,1),\ t \in I.$$

Recalling that the initial condition u_0 is considered to be a resolved profile solution, the linearization of the operator $u \mapsto -\partial_{xx} u + f_\epsilon(u)$ about q enjoys a rather striking spectral property given by the following result.

5.2. **Theorem** (Spectrum estimate [Che94, dMS95]) *There exists a constant $\lambda_0 > 0$ such that for any $\epsilon \in (0,1]$ we have*

(73) $$\|\partial_x \phi\|^2_{\mathrm{L}_2(D)} + \langle f'_\epsilon(q)\phi, \phi\rangle \geq -\lambda_0 \|\phi\|^2_{\mathrm{L}_2(D)},\ \forall \phi\, \mathrm{H}^1(D).$$

As a consequence of this estimate we can prove the following result.

5.3. **Theorem** (Convergence for weak noise) *Suppose $\gamma > 4$. For each choice of $T, K > 0$, there exist $\epsilon_0, c_0, \mu_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ there correspond an event $\Omega_\epsilon \subset \Omega$ such that the estimates*

(74) $$\sup_{[0,T]} \|\bar{u} - q\|_{\mathrm{L}_2(D)} \leq \frac{\exp(\mu_0 T) - 1}{\mu_0} K \epsilon^\gamma$$

(75) $$\sup_{[0,T]} \|\bar{u} - q\|_{\mathrm{L}_\infty(D)} \leq c_0 \epsilon^2$$

are satisfied with probability

(76) $$P(\Omega_\epsilon) \geq \left[1 - \frac{T}{\rho}\left(1 + \frac{K^2}{2}\rho\right)^{1/(2\sigma)-1} \exp\left(-\frac{K^2}{2}\rho\right) - c_1 \exp(-c_2/\epsilon^{1+2\gamma})\right]^+,$$

for all $\rho, \sigma > 0$.

Proof We divide the proof in several steps.
Step 1. We start by deriving an energy inequality for the error

(77) $$\bar{e} := \bar{u} - q.$$

Since $\bar{u}$ satisfies the weak formulation (51) and q is a classical solution, we can write the following PDE in its weak formulation for $\bar{e}$:

$$\langle \partial_t \bar{e}, \phi \rangle + \langle \partial_x \bar{e}, \partial_x \phi \rangle + \langle f'_\epsilon(q)\bar{e}, \phi \rangle = \epsilon^\gamma \left\langle \partial_{xt}\bar{W}, \phi \right\rangle - \frac{1}{\epsilon^2} \left\langle \bar{r}\bar{e}^2, \phi \right\rangle, \ \forall \phi \in \mathrm{H}^1(D), \tag{78}$$

where

$$\bar{r} := 3q + \bar{e} = 2q + \bar{u}. \tag{79}$$

Testing with $\bar{e}$ in (78) and using the spectrum estimate (73) we obtain

$$\begin{aligned} \|\bar{e}\|_{\mathrm{L}_2(D)} \, \mathrm{d}_t \|\bar{e}\|_{\mathrm{L}_2(D)} - \lambda_0 \|\bar{e}\|^2_{\mathrm{L}_2(D)} &\leq \langle \partial_t \bar{e}, \bar{e} \rangle + \|\partial_x \bar{e}\|^2_{\mathrm{L}_2(D)} + \langle f'_\epsilon(q)\bar{e}, \bar{e} \rangle \\ &\leq \epsilon^\gamma \left\langle \partial_{xt}\bar{W}, \bar{e} \right\rangle - \frac{1}{\epsilon^2} \left\langle \bar{r}, \bar{e}^3 \right\rangle. \end{aligned} \tag{80}$$

To shorten displays, let us introduce the following notation:

$$a(t) := \|\bar{e}(t)\|_{\mathrm{L}_2(D)}, \text{ and } b(t) := \|\bar{e}(t)\|_{\mathrm{L}_\infty(D)}. \tag{81}$$

Step 2. Our next objective is to bound the terms in the right-hand side of (80). For the first term, we use the $\mathrm{L}_\infty(0, T; \mathrm{L}_2(D))$-norm bound of the AWN of Lemma 3.5, according to which correspondingly to K there exists an event $\Omega^2_K \subset \Omega$ on which

$$\left\| \epsilon^\gamma \left\langle \partial_{xt}\bar{W}(t), \bar{e}(t) \right\rangle \right\| \leq \epsilon^\gamma K a(t), \ \forall t \in [0, T]. \tag{82}$$

To produce a bound on the second term of the right-hand side of (80) we use the maximum principle for $\bar{u}$, Lemma 4.2, which states that the event

$$\Omega^\infty_\epsilon := \left\{ \sup_{t \in [0,T]} \|\bar{u}(t)\|_{\mathrm{L}_\infty(D)} \leq 3 \right\}, \tag{83}$$

has probability $P(\Omega^\infty_\epsilon) = 1 - c_1 \exp(-c_2/\epsilon^{1+2\gamma})$. Using this fact and the expression for $\bar{r}$ in (79) we deduce that, on the event Ω^∞_ϵ,

$$\|\bar{r}(t)\|_{\mathrm{L}_\infty(D)} \leq 5, \ \forall t \in [0, T]. \tag{84}$$

Therefore, we have shown that, on the event Ω^∞_ϵ and for all times $t \in [0, T]$, we have

$$\left| \frac{1}{\epsilon^2} \left\langle \bar{r}(t)\bar{e}(t), \bar{e}(t)^2 \right\rangle \right| \leq \frac{5}{\epsilon^2} \|\bar{e}(t)\|_{\mathrm{L}_\infty(D)} \|\bar{e}(t)\|^2_{\mathrm{L}_2(D)} = \frac{5}{\epsilon^2} b(t) a(t)^2. \tag{85}$$

The inequalities (80), (82) and (85) imply now the following bound

$$\mathrm{d}_t a(t) - \lambda_0 a(t) \leq \epsilon^\gamma K + \frac{5}{\epsilon^2} b(t) a(t), \ \forall t \in I, \tag{86}$$

on the event

$$\Omega_\epsilon := \Omega^2_K \cap \Omega^\infty_\epsilon. \tag{87}$$

Notice that, in view of (33) and (50), the event Ω_ϵ satisfies

$$\begin{aligned} P(\Omega_\epsilon) &\geq \left[P(\Omega^2_K) + P(\Omega^\infty_\epsilon) - 1 \right]^+ \\ &\geq \left[1 - \frac{T}{\rho} \left(1 + \frac{K^2}{2}\rho \right)^{1/(2\sigma)-1} \exp\left(-\frac{K^2}{2}\rho \right) - c_1 \exp(-c_2/\epsilon^{1+2\gamma}) \right]^+. \end{aligned} \tag{88}$$

This establishes (76).

Step 3. In order to use inequality (86) to prove both (74) and (75), we need to show first the following pointwise bound on $\bar{e}$:

$$b(t) = \|\bar{e}(t)\|_{\mathrm{L}_\infty(D)} \le C_1 t^{3/4} \left(\epsilon^\gamma K + \epsilon^{-2} \sup_{[0,t]} \left((5b+2)\, a \right) \right) \text{ on } \Omega_\epsilon, \tag{89}$$

where C_1 is the constant appearing in the right-hand side of (119).

This bound is derived by using the integral representation for the error

$$\begin{aligned} \bar{e}(x,t) = \int_0^t \int_D & G_{t-s}(x,y) \\ & \times \left(\epsilon^\gamma \partial_{ys} \bar{W}(y,s) - f'_\epsilon(q(y,s))\bar{e}(y,s) - \frac{1}{\epsilon^2}\bar{r}(y,s)\bar{e}(y,s)^2 \right) \mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{90}$$

The result follows by applying Lemma A.1 and observing that the following bounds are valid on the event Ω_ϵ:

$$\epsilon^\gamma \left\| \partial_{xt} \bar{W} \right\|_{\mathrm{L}_2(D)} \le \epsilon^\gamma K \tag{91}$$

$$\|f'_\epsilon(q)\bar{e}\|_{\mathrm{L}_2(D)} \le 2\epsilon^{-2} \|\bar{e}\|_{\mathrm{L}_2(D)} = 2\epsilon^{-2} a \tag{92}$$

$$\epsilon^{-2} \left\| \bar{r}\bar{e}^2 \right\|_{\mathrm{L}_2(D)} \le 5\epsilon^{-2} \|\bar{e}\|_{\mathrm{L}_\infty(D)} \|\bar{e}\|_{\mathrm{L}_2(D)} = 5\epsilon^{-2} ba. \tag{93}$$

Step 4. We now conclude the proof, by employing a connectedness argument which is similar to the one used by de Mottoni & Schatzman [dMS95, §6]. Start by introducing the (arbitrary) constant $c_0 > 0$ and

$$\mu_0 := \lambda_0 + 5c_0. \tag{94}$$

Consider the set

$$\left\{ t \in [0,T] :\ b(s) \le c_0 \epsilon^2,\ \forall s \in [0,t] \right\}. \tag{95}$$

This set is non-empty for $b(0) = 0$, and it constitutes a closed interval—denoted as $[0, t_\star]$—for b is continuous. We will show $[0, t_\star]$ to be relatively open in $[0,T]$ which, in view of connectedness, will imply the bound

$$b(t) \le c_0 \epsilon^2,\ \forall t \in [0,T]. \tag{96}$$

To see that the interval $[0, t_\star]$ is open it is sufficient to show that the bound is strict at $t_\star$, i.e., that

$$b(t_\star) < c_0 \epsilon^2. \tag{97}$$

By (86) and the definition of $t_\star$ we have

$$\mathrm{d}_t a - \lambda_0 a \le \epsilon^\gamma K + 5c_0 a, \text{ on } [0, t_\star]. \tag{98}$$

It follows that

$$\mathrm{d}_t a - \mu_0 a \le \epsilon^\gamma K, \text{ on } [0, t_\star], \tag{99}$$

and by the Gronwall lemma that

$$a(t_\star) \le \epsilon^\gamma K \frac{\exp(\mu_0 t_\star) - 1}{\mu_0}. \tag{100}$$

We have used that $a(0) = 0$. From (89) it follows that

$$\begin{aligned} b(t_\star) &\le C_1 t_\star^{3/4} \left(\epsilon^\gamma K + \epsilon^{\gamma-2} \frac{\exp(\mu_0 t_\star) - 1}{\mu_0} \left(5c_0\epsilon^2 + 2 \right) \right) \\ &\le C_1 T^{3/4} \left(\left(K + 5c_0 \frac{\exp(\mu_0 T) - 1}{\mu_0} \right) \epsilon^{\gamma-2} + 2 \frac{\exp(\mu_0 T) - 1}{\mu_0} \epsilon^{\gamma-4} \right) \epsilon^2. \end{aligned} \tag{101}$$

Since $\gamma > 4$, the last term above can be made strictly smaller than $c_0\epsilon^2$ for all ϵ smaller than an appropriate ϵ_0. For instance, it is enough to choose ϵ_0 such that

$$\epsilon_0 < \left({c_0}^{-1}2C_1T^{3/4}\left(K + 5c_0(\exp(\mu_0 T) - 1)\mu_0^{-1}\right)\right)^{-1/(\gamma-2)} \tag{102}$$

and

$$\epsilon_0 < \left(c_0^{-1}4C_1T^{3/4}\left(\exp(\mu_0 T) - 1\right)\mu_0^{-1}\right)^{-1/(\gamma-4)}. \tag{103}$$

Notice that since $c_0 > 0$ is arbitrary, it can be chosen in such a way as to maximize the value of the possible ϵ_0. ∎

5.4. **Remark** (role of ρ and σ) The above result is basically a "weak noise" result in the sense that it gives a threshold under which the noise becomes negligible and for which the stochastic solution becomes quite close to the deterministic solution. Notice that the effect of the AWN parameters ρ and σ is seen only in the probability of the event Ω_ϵ and not in the estimates themselves, which reflects the fact that the noise is so weak as for the equation not to distinguish between the white noise and its approximation. Notice also that the choice of c_0, μ_0, ϵ_0 is independent of ρ and σ. However, equation (76) says also that for any given ρ and σ, K should be big enough in order to ensure a high probability for the estimates to hold, which means that the estimates may deteriorate as σ and ρ tend to zero.

5.5. **Remark** (relation to large deviation results) The previous result provides a theoretical benchmark for the validity of the approximation of (1) by (2), by showing that it is close in a suitable sense to the most probable deterministic trajectory of the stochastic Allen-Cahn equation (1), even in large time intervals where interfaces form and evolve according to macroscopic mean curvature evolutions. Note that the most probable path of (2) is given by its deterministic counterpart, (70), as obtained by large deviations results for stochastic reaction diffusion equations [FW98, Section 10.5]. Here this latter result is used not only as a benchmark for the validity of our AWN regularization (2) but it is further quantified in terms of detailed error estimates in Theorem 5.3.

5.6. **Remark** (relation to stochastic mean curvature flow) Finally, in view of well-known asymptotic limits in dimension 2 or higher of the deterministic Allen-Cahn equation to motion by mean curvature [BSS93, e.g.], Theorem 5.3 demonstrates that a higher dimensional white noise regularization, corresponding to the one presented here, should provide a stochastic approximation to motion by mean curvature evolutions. The influence of stochastic corrections to motion by mean curvature evolutions and in particular to its instabilities such as interface fattening was recently demonstrated in [KK01, DLN01, SY04]. Thus our results give a first rigorous indication that the approximation framework presented here may provide an accurate and potentially efficient algorithm (via FEM adaptivity) for simulating stochastic motion by mean curvature.

6. An Euler-Galerkin finite element scheme

We introduce now the finite element discretization of the regularized problem (51).

6.1. **Discretization partitions.**
Start by introducing the space-time partitions

$$\begin{aligned} &\mathscr{D}_h := \left\{D'_m : D'_m := (x'_{m-1}, x'_m),\ m \in [1 : M']\right\}, \\ &\text{and } \mathscr{I}_k := \left\{I'_n : I'_n := [t'_{n-1}, t'_n),\ n \in [1 : N']\right\}. \end{aligned} \tag{104}$$

These partitions do not necessarily coincide with the partitions $\mathscr{D}_\sigma$ and $\mathscr{I}_\rho$ used for the regularization procedure in §3.1. Bearing in mind that this setting could be further generalized, we limit ourselves here to the case where the numerical discretization partitions, $\mathscr{D}_h$ and $\mathscr{I}_k$, are refinements of the white noise regularization partitions $\mathscr{D}_\sigma$ and $\mathscr{I}_\rho$, respectively. I.e., for each $D'_m \in \mathscr{D}_h$ there exists $D_l \in \mathscr{D}_\sigma$ such that $D'_m \subset D_l$ etc; this determines a unique mapping $\mu : [0 : M'] \to [0 : M]$, such that $D'_m \subset D_{\mu(m)}$. For simplicity, we also assume that the partitions are uniform and that the *meshsize* and *timestep* are denoted respectively by h and k. The reason we do not make these partitions coincide is that for the finite element method's convergence analysis it may prove useful to have couplings of the type $h = \sigma^p$ and $k = \rho^q$, with $p, q \geq 1$. We observe that for all practical aspects coming up in §7, we will consider only the simplest situation possible where these partitions do coincide.

6.2. **Finite element space and the discrete scheme.** Let $\mathbb{V} \subset \mathrm{H}^1_0(D)$ be the space of continuous piecewise linear functions associated with the partition $\mathscr{D}_h$, we define the *(spatial) semi-discrete solution* as the time-dependent random finite element function $U : [0, T] \times \Omega \to \mathbb{V}$ which solves the SDE

$$\langle \partial_t U(t), V\rangle + \langle \partial_x U(t), \partial_x V\rangle + \langle f_\epsilon(U(t)), V\rangle = \left\langle \partial_{xt}\bar{W}, V\right\rangle, \forall V \in \mathbb{V},\ t \in [0, T]. \tag{105}$$

We discretize further this SDE in the time variable by taking a semi-implicit Euler scheme in time associated to the partition $I = \{t_0\} \cup \bigcup_m I_m$

$$\left\langle \frac{U^n - U^{n-1}}{k}, V\right\rangle + \langle \partial_x U^n, \partial_x V\rangle + \left\langle f_\epsilon(U^{n-1}), V\right\rangle = \left\langle \partial_{xt}\bar{W}, V\right\rangle, \forall V \in \mathbb{V},\ t \in [0, T]. \tag{106}$$

The adjective "semi-implicit" expresses the fact that the scheme is implicit in the linear part, while it is explicit in the nonlinearity. This means that at each timestep only a linear problem has to be solved.

In practice, we find it more useful to use a slightly modified version of (106) given by

$$\begin{aligned} &\left\langle \frac{U^n - U^{n-1}}{k}, V\right\rangle + \langle \partial_x U^n, \partial_x V\rangle + \left\langle f'_\epsilon(U^{n-1})U^n, V\right\rangle \\ &= \left\langle f'_\epsilon(U^{n-1})U^{n-1} - f_\epsilon(U^{n-1}), V\right\rangle + \epsilon^\gamma \left\langle \partial_{xt}\bar{W}, V\right\rangle,\ \forall V \in \mathbb{V},\ t \in [0, T]. \end{aligned} \tag{107}$$

which allows to take bigger timesteps k [KNS04]. Note that this is nothing but a linearization involving one step of the Newton method to solve the nonlinear (fully implicit) backward Euler scheme.

6.3. **The linear time-stepping system.** Let us indicate the basis functions of $\mathbb{V}$ by Φ_m, for $m \in [0 : M']$; that is the piecewise linear continuous function such that $\Phi_m(x_l) = \delta^m_l$, for $l \in [0 : M']$. If we indicate by $\boldsymbol{u}^n = (u^n_m)$ the vector of nodal values corresponding to the discrete solution U^n at time t^n, that is $U^n(x) = \sum_{m=0}^{M'} u^n_m \Phi_m(x)$, then, we can translate (107) in the following matrix form

$$\left[\frac{1}{k}\boldsymbol{M} + \boldsymbol{A} + \frac{1}{\epsilon^2}\boldsymbol{N}(\boldsymbol{u}^{n-1})\right]\boldsymbol{u}^n = \frac{1}{\epsilon^2}\boldsymbol{g}(\boldsymbol{u}^{n-1}) + \frac{1}{k}\boldsymbol{M}\boldsymbol{u}^{n-1} + \epsilon^\gamma \boldsymbol{w}, \tag{108}$$

where $\boldsymbol{M}$, $\boldsymbol{A}$ are the usual finite element mass and stiffness matrices, respectively, $\boldsymbol{N}(\boldsymbol{u}^{n-1})$ and $\boldsymbol{g}(\boldsymbol{u}^{n-1})$ are a "nonlinear" mass matrix and load vector, respectively, and $\boldsymbol{w} = (w_m)$ a random load vector generated at each time-step. A short calculation shows that for m an internal degree of freedom (node) we have

$$w_m = \frac{h}{2\sqrt{\sigma\rho}}(\eta_{\mu(m)-1} + \eta_{\mu(m)}) \tag{109}$$

where μ is the mapping introduced earlier in this section and η_l is a $\mathrm{N}(0,1)$ random number for $l \in [0:M]$, or zero for $l = -1, M+1$ (the boundary cases). If the partitions $\mathscr{D}_\sigma$ and $\mathscr{D}_h$ coincide, which will be the case in the next section, then $h = \sigma$ and (109) simplifies to

$$w_m = \frac{1}{2}\sqrt{\frac{h}{\rho}}(\eta_{m-1} + \eta_m). \tag{110}$$

This is the form that we employ in our computations below.

7. Computations

We state some computational results. The main issue here is benchmarking as, in contrast with the deterministic case, there is no explicit exact solution known. Therefore our benchmarking procedure is based on the theoretical results of Funaki [Fun95] and Brassesco et al [BDMP95].

7.1. **Monte Carlo simulations.** We run a series of Monte Carlo simulations. For each choice of parameters ϵ, γ and meshsize h, we choose the timestep $k = h^2$ and run 1600 times the code with different seeds for the random number generator. By "run" we mean the computation of a sample path from time 0 to a final time T which is fixed at 20 for all runs. At the beginning of each run, the random number generator is seeded and the subsequent η_m appearing in (110) are chosen according to this seed for all the run. The seed for each run is determined by the clock of the machine at the start of each run (these are also recorded for rerunning purposes). In this section we denote the numerical solution (which tacitly depends on ϵ, γ, h) by $(U^n_\omega)_{n\in[0:N]}$, where n corresponds to the timestep and ω is a discrete sample, i.e., the choice of the initial seed. Let us indicate by $\bar{\Omega}$ the discrete sample space, which can be thought of being the choice initial seeds.

Since it is well known that for the deterministic Allen-Cahn, the meshsize h has to be, at most, smaller than ϵ in order to resolve satisfactorily the transition level we choose $h \le c_0\epsilon$, $c_0 \le 1/2$, in all our computations. As comparisons are made with respect to different ϵ, we use the same meshsize h and the same seed ω for each run with all different values of ϵ.

7.2. **Benchmarking.** Our benchmarking procedure consists in tracking the center of the discrete solution. This benchmarking is used on a heuristic argument, based in part on rigorous results.

According to known results [Fun95, Thm. 8.1], we expect the center of $\bar{U}^n$, which is a piecewise linear function in the space variable $x \in D$, to perform a Brownian Motion, modulo perturbations of order $\mathrm{O}(\epsilon)$ and the numerical error. Namely, we expect

$$\lim_{\epsilon\searrow 0} P\left\{\max_{n\in[0:N]} \left\|U^n_\omega - X_{\xi^\epsilon_{t_n}}\right\|_{\mathrm{L}_2(D)} > \delta\right\} = 0, \tag{111}$$

for each fixed $\delta > 0$. Here the function $X_\xi(x)$ has value -1 for $x < \xi$ and 1 for $x > \xi$ and ξ^ϵ_t is the solution of a SDE, which converges in an appropriate sense to the Brownian Motion with diffusion coefficient squared equaling $3 \times 2^{-3/2}\epsilon^{1+2\gamma}$. It follows that for a fixed $\delta > 0$, for $\epsilon > 0$ small enough, there exists a set $\bar{\Omega}^\epsilon_\delta$ for which

$$\begin{aligned} &\max_n \int_D \left|\frac{1}{\#\bar{\Omega}^\epsilon_\delta}\sum_{\omega\in\bar{\Omega}^\epsilon_\delta}\left(U^n_\omega(x) - X_{\xi^\epsilon_{t_n}}\right)\right|^2 \mathrm{d}x \\ &\le \frac{1}{\#\bar{\Omega}^\epsilon_\delta}\sum_{\omega\in\bar{\Omega}^\epsilon_\delta}\max_n\left\|U^n_\omega - X_{\xi^\epsilon_{t_n}}\right\|^2_{\mathrm{L}_2(D)} \le \delta^2, \end{aligned} \tag{112}$$

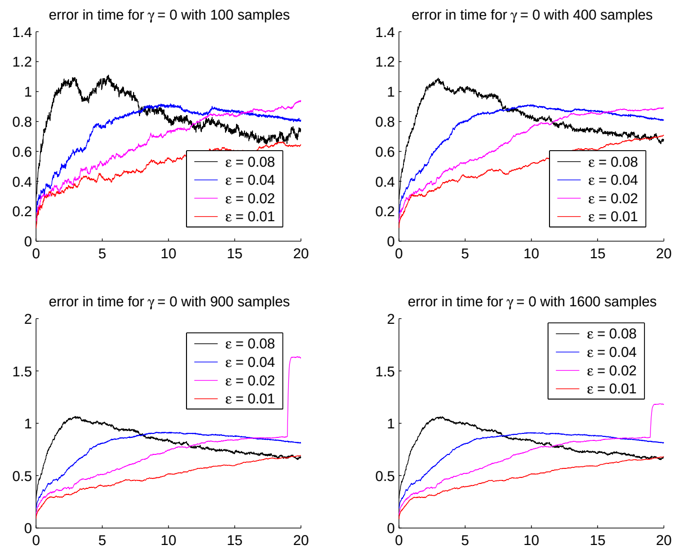

FIGURE 1. Numerical results for $\gamma = 0$, fixed $h \le c_0\epsilon$, $c_0 < 1/2$ and $k = h^2$. Here $\epsilon = \{0.08, 0.04, 0.02, 0.01\}$ and the number of samples $\#\bar{\Omega}$ increases. The abscissa represents the time interval $[0, 20]$ whereas the ordinate is $\left\|\bar{U}^n - \operatorname{erf}(\cdot/\sigma(t_n))\right\|_{\mathrm{L}_2(D)}$, as detailed in §7.2. For such low values of γ the convergence is rather poor; in fact, almost absent. It is interesting to notice in the two lower diagrams, that certain solutions can lead to a "shooting" of the error. In fact, a proper convergence check should exclude such solutions by the means of the maximum principle. This is explained in §7.2.

and

$$\lim_{\epsilon \searrow 0} \frac{\#\bar{\Omega}_\delta^\epsilon}{\#\bar{\Omega}} = 1. \tag{113}$$

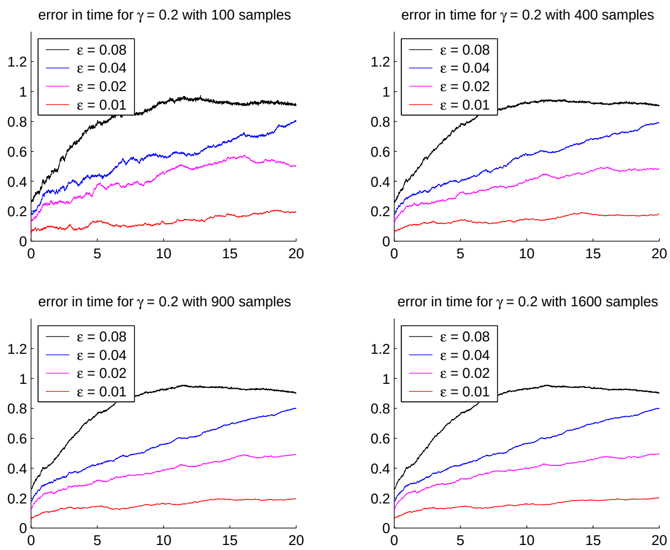

FIGURE 2. Numerical results for $\gamma = 0.2$, fixed $h \leq c_0\epsilon$, $c_0 < 1/2$ and $k = h^2$. Here $\epsilon = \{0.08, 0.04, 0.02, 0.01\}$ and the number of samples $\#\bar{\Omega}$ increases. The abscissa represents the time interval $[0, 20]$ whereas the ordinate is $\left\|\bar{U}^n - \mathrm{erf}(\cdot/\sigma(t_n))\right\|_{\mathrm{L}_2(D)}$, as detailed in §7.2. In this case, the convergence is much more detectable, with respect to figure 1. This is a "fortunate" case, in which all the samples do not violate the maximum principle, leading to no "shooting" of the diagrams.

Now, indicate by ξ_t a normalized Brownian Motion, and take the approximation

$$\frac{1}{\#\bar{\Omega}^\epsilon_\delta} \sum_{\omega \in \bar{\Omega}^\epsilon_\delta} X_{\xi^\epsilon_t} \approx \int_\Omega X_{\xi_t} \,\mathrm{d}P = \mathrm{erf}(x/\sigma(t)), \tag{114}$$

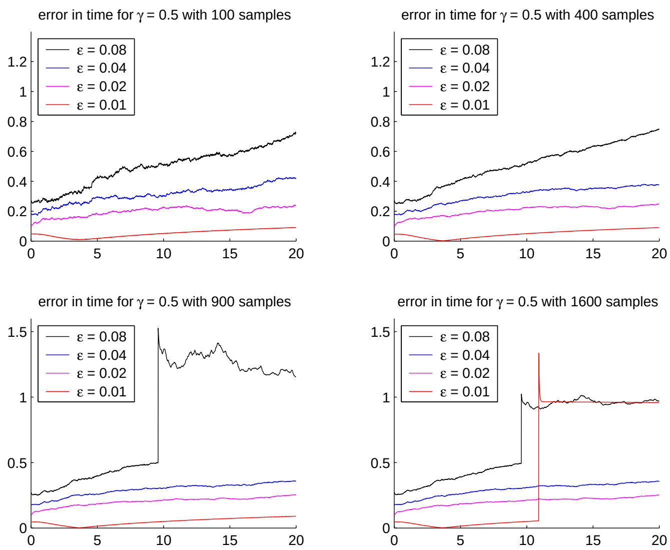

FIGURE 3. Numerical results for $\gamma = 0.5$, fixed $h \le c_0\epsilon$, $c_0 < 1/2$ and $k = h^2$. Here $\epsilon = \{0.08, 0.04, 0.02, 0.01\}$ and the number of samples $\#\bar\Omega$ increases. The abscissa represents the time interval $[0, 20]$ whereas the ordinate is $\left\|\bar U^n - \operatorname{erf}(\cdot/\sigma(t_n))\right\|_{\mathrm{L}_2(D)}$, as detailed in §7.2. It is clear that the convergence becomes faster as γ increases. Notice that also in this case we can still get solutions that lead to "shooting" of the error. This is further evidence that the discrete probabilistic maximum principle ought to be employed in practical results as much as it is in the theoretical ones.

where the *error function* is defined as

$$\operatorname{erf}(x) := \frac{2}{\pi}\int_0^x \exp(-y^2)\,\mathrm{d}y \tag{115}$$

and the variance is related to the diffusion coefficient

$$\sigma(t)^2 = 3 \times 2^{-3/2}\epsilon^{1+2\gamma}t. \tag{116}$$

In a first, and rather coarse, approach we replace the first average in the left hand side of (112) by the average over all $\bar{\Omega}$, i.e.,

$$\bar{U}^n := \frac{\sum_{\omega\in\bar{\Omega}} U^n_\omega}{\#\bar{\Omega}} \approx \frac{\sum_{\omega\in\bar{\Omega}^\epsilon_\delta} U^n_\omega}{\#\bar{\Omega}^\epsilon_\delta}. \tag{117}$$

This turns out to be too coarse and, as seen from figures 1 and 3, it may happen, very rarely though, that some solution makes the average diverge. This lack of convergence can be avoided by excluding solutions that violate the maximum principle. To be concise, if we introduce the discrete analog of Ω^∞, that is the set

$$\bar{\Omega}^\infty = \left\{\omega \in \bar{\Omega} : \max_n \|U^n_\omega\|_{\mathrm{L}_2(D)} \leq 3\right\}, \tag{118}$$

and replace $\bar{\Omega}$ by this subset in the above discussion, we get a converging pattern, such as the one in figure 2.

Figures 1 and 3 show very clearly that the convergence results cannot hold for the whole probability space. In particular, this is a computational evidence that the restriction to the events Ω^∞_ϵ and Ω_ϵ in Theorems 4.5 and 5.3, respectively, may be necessary to obtain convergence.

7.3. **Results.** Our results can be summarized by the figures and the main comments are reported in the captions. The main observation, besides that made in the previous paragraph about the sample paths to be excluded in the benchmarking, is that as γ becomes larger—i.e., as the noise intensity is dimmed out—the convergence as $\epsilon \searrow 0$ becomes faster. In fact, for $\gamma = 0$, there is basically no convergence visible from the numerics, whereas for $\gamma = 0.2$ and 0.5 there is a clear pattern that echoes the expectations from theoretical results.

Appendix A. Some useful facts

A.1. **Lemma** (Heat kernel estimate) *The heat kernel G defined by* (17) *satisfies the following estimate*

$$\left|\int_0^t \int_D G_{t-s}(x,y)v(y,s)\,\mathrm{d}y\,\mathrm{d}s\right| \leq C_1 t^{3/4} \sup_{[0,t]} \|v\|_{\mathrm{L}_2(D)}\,, \tag{119}$$

for any $v \in \mathrm{L}_\infty(0,t;\mathrm{L}_2(D))$, with

$$C_1 = \frac{2^{17/4}}{3\pi^{1/4}}. \tag{120}$$

Proof Let

$$f(x,t) := \int_0^t \int_D G_{t-s}(x,y)v(y,s)\,\mathrm{d}y\,\mathrm{d}s. \tag{121}$$

Then

$$|f(x,t)| \leq \int_0^t \left(\int_D |G_{t-s}(x,y)|^2\,\mathrm{d}y\right)^{1/2} \|v(s)\|_{\mathrm{L}_2(D)}\,\mathrm{d}s. \tag{122}$$

By the Parseval identity and an approximation of the series by an improper integral we obtain

(123)
$$\begin{aligned}\int_D |G_{t-s}(x,y)|^2 \, \mathrm{d}y &= 16 \sum_{k=0}^{\infty} (2-\delta_0^k)^2 \exp(-(k\pi)^2(t-s)/2) \cos^2(k\pi(x+1)/2) \\ &\le 64 \sum_{k=0}^{\infty} \exp(-(k\pi)^2(t-s)/2) \\ &\le 64 \int_0^{\infty} \exp(-(\kappa\pi)^2(t-s)/2) \, \mathrm{d}\kappa = \frac{16\sqrt{2}}{\sqrt{\pi(t-s)}}.\end{aligned}$$

Thus

$$\begin{aligned}|f(x,t)| &\le 4(2\pi^{-1})^{1/4} \int_0^t (t-s)^{-1/4} \|v(s)\|_{\mathrm{L}_2(D)} \, \mathrm{d}s \\ &\le \frac{2^{17/4}}{3\pi^{1/4}} t^{3/4} \sup_{s\in[0,t]} \|v(s)\|_{\mathrm{L}_2(D)} .\end{aligned} \tag{124}$$

∎

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