A finite element analysis of critical state models for type-II superconductivity in 3D

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We consider the numerical analysis of evolution variational inequalities which are derived from Maxwell's equations coupled with a nonlinear constitutive relation between the electric field and the current density and governing the magnetic field around a type-II bulk superconductor located in three dimensional space. The nonlinear Ohm's law is formulated using the sub-differential of a convex energy so the theory is applied to the Bean critical state model, a power law model and an extended Bean critical state model. The magnetic field in the nonconductive region is expressed as a gradient of a magnetic scalar potential in order to handle the curl free constraint. The variational inequalities are discretized in time implicitly and in space by Nédélec's curl conforming finite element of lowest order. The non-smooth energies are smoothed with a regularization parameter so that the fully discrete problem is a system of nonlinear algebraic equations at each time step. We prove various convergence results. Some numerical simulations under a uniform external magnetic field are presented.

Keywords: macroscopic models for superconductivity, variational inequality, Maxwell's equations, edge finite element, convergence, computational electromagnetism.

1. Introduction

In this paper we propose a finite element method to analyze critical state problems for type-II superconductivity numerically. Especially we are interested in analyzing the situation where a bulk superconductor is located in a 3D domain. Models of type-II superconductors use the eddy current version of Maxwell's equations together with nonlinear constitutive relations between the current and the electric field such as the Bean critical state model (Bean (1964)), the extended Bean critical state models (Bossavit (1994)), or the power law type relation (Rhyner (1993)) instead of the linear Ohm's law. The numerical study of the Bean critical state model based on a variational formulation without introducing a free boundary between the region of the critical current and of the subcritical current was initiated by Prigozhin (1996a,b). The approach of Prigozhin mathematically treats the electric field as a subdifferential of a critical energy density which takes either the value 0 if the current density does not exceed some critical value or infinity otherwise. By analyzing the subdifferential formulation, the magnetic penetration and the current distribution around the superconductor in 2D situation were intensively investigated by Prigozhin (1996b,1997,1998,2004). Adopting the variational formulation by Prigozhin, Elliott *et al.* (2004) reported a numerical analysis of the Bean critical state model modelling the magnetic field and the current density. The same authors also presented a finite element analysis of the current density - electric field variational formulation (see Elliott *et al.* (2005)). Also see Barnes *et al.* (1999) for engineering application of the Bean model to modelling electrical machine containing superconductors. In all these articles the problems are considered in 2D. The derivation of the Bean critical state model from various models of type-II superconductivity such as the Ginzburg-Landau equations was summarized by Chapman (2000).

Bossavit (1994) extended the Bean critical state model by allowing the current to exceed the critical value after the superconductor switched to the normal state. The numerical results using this extended Bean's model in 3D geometry were reported by Rubinacci *et al.* (2000,2002).

As an alternative model, the power law constitutive relation $\mathbf{E} = |\mathbf{J}|^p\mathbf{J}$ for large $p > 0$ is commonly used in the modeling of type-II superconductivity (see, eg, Rhyner (1993) for a theory with the power law, Brandt (1996) for 2D problems and Grilli *et al.* (2005) for a recent engineering application of 3D model, etc). It was mathematically proved that as $p \to \infty$ the solution of the power law formulation converges to the solution of the Bean critical state formulation (see Barrett & Prigozhin (2000) for 2D problem, Yin (2001), Yin *et al.* (2002) for 3D cases).

Software package to solve Maxwell's equations coupled with various nonlinear E-J relations modelling type-II superconductors in 3D for engineering application was developed by Pecher (2003).

While the numerical analysis of these critical state models in 2D has been developed by many authors, to the best of the authors' knowledge no article tackling the numerical analysis of 3D critical state problems is found in mathematical literature. The purpose of this paper is to define a finite element approximation in this setting and prove convergence. Following Prigozhin (1996a,b) we formulate the magnetic field around the bulk type-II superconductor as an unknown quantity in an evolution variational inequality obtained from the eddy current model and the subdifferential formulation of the critical state models.

The Bean type critical state model requires the current density not to exceed some critical value, which is a difficult constraint to attain in 3D numerical analysis. To avoid this difficulty we employ a penalty method which approximates the non-smooth energy with a smooth energy so that the electric field -current relation is monotone and single valued. The curl free constraint on the magnetic field in the nonconductive region coming from the eddy current model can be handled by introducing a magnetic scalar potential in the outside of the superconductor. This magnetic field - scalar potential hybrid formulation is an effective method to carry out the discretization in space for eddy current problems with an unknown magnetic field (see Bermúdez *et al.* (2002) for an application of this method), though it needs an additional treatment to assure a tangential continuity on the boundary between the conductor and the dielectric. Discretizing the problems in time variable yield an unconstrained optimization problem. The problem is then discretized in space by using curl conforming 'edge' element by Nédélec (1980) of lowest order on a tetrahedral mesh. The full discrete solution consisting of the minimizers of the optimization problem is proved to converge to the unique solution of the variational inequality formulation of the Bean critical state model. This convergence result is based on the compactness property of edge element firstly proved by Kikuchi (1989) and extended by Monk (2003). The power law constitutive relation can be viewed as a penalty method for the Bean model by letting the power become arbitrarily large. We carry out a numerical analysis of both the power law and the extended Bean model in their own right and as penalty methods for the Bean model.

The outline of this paper is as follows. In section 2 we recall the mathematical models of the eddy current problem and the critical state constitutive laws and formulate the models as evolution variational inequalities. In section 3 we formulate the discretization of the variational inequality formulations. In section 4 the convergence of the discretization to the analytical solution is proved. Finally in section 5 we describe the implementation and report some numerical results showing the behaviour of the magnetic field and the distribution of the current density flowing through a bulk cubic superconductor in an uniform applied magnetic field.

2. The models and the mathematical formulation

2.1 *The critical state models*

We consider the problem in a convex polyhedron $\Omega(\subset I\!R^3)$ with a boundary $\partial\Omega$. The bulk type-II superconductor Ω_s is a simply connected domain contained in Ω, with a connected Lipschitz boundary $\partial\Omega_s$. Let Ω_d denote the dielectric region $\Omega\backslash\overline{\Omega_s}$.

The model is based on the Maxwell's equations, where the displacement current is neglected. These equations are called the eddy current model:-

$$\partial_t \mathbf{B} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \text{ (Faraday)}\,, \tag{2.1}$$
$$\mathbf{curl}\,\mathbf{H} = \mathbf{J} \text{ (Ampère)}\,, \tag{2.2}$$
$$\operatorname{div}\mathbf{B} = 0 \text{ (Gauss)}\,, \tag{2.3}$$

where $\partial_t\mathbf{B}$ denotes $\partial\mathbf{B}/\partial t$, and

$$\begin{aligned}
&\mathbf{B} : \Omega\times[0,T]\to I\!R^3 \text{ denotes the magnetic flux density,}\\
&\mathbf{E} : \Omega\times[0,T]\to I\!R^3 \text{ denotes the electric field intensity,}\\
&\mathbf{H} : \Omega\times[0,T]\to I\!R^3 \text{ denotes the magnetic field intensity,}\\
&\mathbf{J} : \Omega\times[0,T]\to I\!R^3 \text{ denotes the electric current density.}
\end{aligned}$$

We assume that the constitutive relation between $\mathbf{B}$ and $\mathbf{H}$ is

$$\mathbf{B} = \mu\mathbf{H}, \tag{2.4}$$

where the magnetic permeability is denoted by $\mu : \Omega \to I\!R_{>0}$, which is positive, piecewise constant and defined by

$$\mu = \begin{cases} \mu_s & \text{in } \Omega_s, \\ \mu_d & \text{in } \Omega_d, \end{cases}$$

for constants $\mu_s, \mu_d > 0$.

We assume that there are no current sources so that outside of superconductor

$$\mathbf{J} = \mathbf{0} \text{ in } \Omega_d. \tag{2.5}$$

We study the problem in a physical situation where an external time varying source magnetic field $\mathbf{H}_s$ is applied. We impose the boundary condition

$$\mathbf{n}\times\mathbf{H} = \mathbf{n}\times\mathbf{h}_s \text{ on } \partial\Omega, \tag{2.6}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and $\mathbf{h}_s = \mathbf{H}_s|_{\partial\Omega}$. Since the source magnetic field $\mathbf{H}_s$ is induced by a generator outside of the domain Ω, we extend $\mathbf{H}_s$ into Ω so that the superconductor is absent from the field $\mathbf{H}_s$ and $\mathbf{H}_s$ satisfies the curl free condition in the domain. Using a source magnetic flux density $\mathbf{B}_s$, we suppose that the following equations hold.

$$\mathbf{curl}\,\mathbf{H}_s = \mathbf{0} \text{ in } \Omega, \tag{2.7}$$
$$\mathbf{B}_s = \mu_d\mathbf{H}_s \text{ in } \Omega, \tag{2.8}$$
$$\operatorname{div}\mathbf{B}_s = 0 \text{ in } \Omega, \tag{2.9}$$
$$\mathbf{n}\times\mathbf{H}_s = \mathbf{n}\times\mathbf{h}_s \text{ on } \partial\Omega. \tag{2.10}$$

Next we state the critical constitutive law between the electric field $\mathbf{E}$ and the supercurrent $\mathbf{J}$ in the superconductor Ω_s. In this paper we always assume the following nonlinear constitutive law

$$\mathbf{E} \in \partial\gamma(\mathbf{J}), \tag{2.11}$$

where $\gamma : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ is a convex functional and $\partial\gamma(\cdot)$ is the subdifferential of γ defined by

$$\partial\gamma(\mathbf{v}) := \{\mathbf{q} \in \mathbb{R}^3 \mid \langle \mathbf{q}, \mathbf{p}\rangle + \gamma(\mathbf{v}) \leqslant \gamma(\mathbf{v}+\mathbf{p}) \text{ for } \forall \mathbf{p} \in \mathbb{R}^3\}.$$

As the convex functional γ we consider the following energy densities.

The Bean critical state model's energy density:

$$\gamma(\mathbf{v}) = \gamma^B(\mathbf{v}) := \begin{cases} 0 & \text{if } |\mathbf{v}| \leqslant \mathscr{J}_c, \\ +\infty & \text{otherwise}, \end{cases} \tag{2.12}$$

where the positive constant $\mathscr{J}_c > 0$ is a critical current density.

The modified Bean critical state model's energy density:

$$\gamma(\mathbf{v}) = \gamma_\varepsilon^{mB}(\mathbf{v}) := \begin{cases} 0 & \text{if } |\mathbf{v}| \leqslant \mathscr{J}_c, \\ \dfrac{1}{2\varepsilon}(|\mathbf{v}|^2 - \mathscr{J}_c^2) & \text{otherwise}, \end{cases} \tag{2.13}$$

where $\varepsilon > 0$ is a positive constant. More generally, we consider a class of energy densities of the type $\gamma(\mathbf{v}) = g(|\mathbf{v}|)/\varepsilon$, where

$$\begin{aligned} & g : \mathbb{R} \to \mathbb{R} \text{ is convex}, \\ & g(x) = 0 \text{ if } x \leqslant \mathscr{J}_c,\ g(x) > 0 \text{ if } x > \mathscr{J}_c, \\ & A_1 x^2 - A_2 \leqslant g(x) \text{ for } \forall x \in \mathbb{R}_{\geqslant 0}, \\ & g(x + \mathscr{J}_c) \leqslant A_3 x^2 + A_4 x \text{ for } \forall x \in \mathbb{R}, \end{aligned} \tag{2.14}$$

where $A_i > 0$ $(i = 1-3)$ are positive constants and $A_4 \geqslant 0$ is a nonnegative constant. Note that γ_ε^{mB} is one example of these $g(|\cdot|)/\varepsilon$ with $A_4 > 0$.

The power law model's energy density:

$$\gamma(\mathbf{v}) = \gamma_p^P(\mathbf{v}) := \frac{\mathscr{J}_c}{p} |\mathbf{v}/\mathscr{J}_c|^p, \tag{2.15}$$

where $p \geqslant 2$.

Let us introduce a new quantity $\widehat{\mathbf{H}}$ by

$$\widehat{\mathbf{H}} = \mathbf{H} - \mathbf{H}_s. \tag{2.16}$$

Substituting (2.4), (2.7) and (2.16) into (2.1)-(2.3), we reach a system of p.d.es.

$$\mu\partial_t\widehat{\mathbf{H}} + \mu\partial_t\mathbf{H}_s + \mathbf{curl}\,\mathbf{E} = \mathbf{0}, \tag{2.17}$$

$$\mathbf{curl}\,\widehat{\mathbf{H}} = \mathbf{J}, \tag{2.18}$$

$$\mathrm{div}(\mu\widehat{\mathbf{H}} + \mu\mathbf{H}_s) = 0. \tag{2.19}$$

We couple the critical state constitutive relation (2.11) with the eddy current model (2.17)-(2.19) and (2.7)-(2.10) to derive the equation for the unknown field $\widehat{\mathbf{H}}$.

For well-posedness of the model, let us give initial boundary conditions for $\widehat{\mathbf{H}}$. At the beginning of the time evolution we assume that no source magnetic field is applied to the domain. Hence there is no induced current in the superconductor and the initial condition of $\widehat{\mathbf{H}}$ is the zero field.

$$\widehat{\mathbf{H}}|_{t=0} = \mathbf{H}_s|_{t=0} = \mathbf{0}. \tag{2.20}$$

It follows from (2.6) and (2.10) that

$$\mathbf{n} \times \widehat{\mathbf{H}} = \mathbf{0} \text{ on } \partial\Omega. \tag{2.21}$$

2.2 *Characterization of the nonlinear constitutive laws*

To see the nonlinearity of the constitutive relation (2.11) clearly, let us characterize (2.11) for each energy density.

PROPOSITION 2.1 For vectors $\mathbf{E}, \mathbf{J} \in \mathbb{R}^3$, the inclusion $\mathbf{E} \in \partial\gamma^B(\mathbf{J})$ holds if and only if there is a constant $\rho \geqslant 0$ such that the following relations hold.

$$\mathbf{E} = \rho\mathbf{J}, \tag{2.22}$$

$$|\mathbf{J}| \leqslant \mathscr{J}_c, \tag{2.23}$$

$$|\mathbf{J}| < \mathscr{J}_c \Longrightarrow \mathbf{E} = \mathbf{0}. \tag{2.24}$$

Proof. Firstly note by definition the inclusion $\mathbf{E} \in \partial\gamma^B(\mathbf{J})$ is equivalent to the inequality

$$\langle \mathbf{E}, \mathbf{p} \rangle + \gamma^B(\mathbf{J}) \leqslant \gamma^B(\mathbf{J} + \mathbf{p}) \tag{2.25}$$

for all $\mathbf{p} \in \mathbb{R}^3$.

Assume (2.22)-(2.24). Fix any $\mathbf{p} \in \mathbb{R}^3$. If $|\mathbf{J} + \mathbf{p}| > \mathscr{J}_c$ holds, then the inequality (2.25) is trivial by the definition of γ^B. If $|\mathbf{J} + \mathbf{p}| \leqslant \mathscr{J}_c$ and $|\mathbf{J}| < \mathscr{J}_c$, then by the relation (2.24) the inequality (2.25) holds since in this case the both side is zero. If $|\mathbf{J}| = \mathscr{J}_c$, then the inequality $|\mathbf{J} + \mathbf{p}| \leqslant \mathscr{J}_c$ yields

$$2\langle \mathbf{J}, \mathbf{p} \rangle \leqslant -|\mathbf{p}|^2 \leqslant 0. \tag{2.26}$$

Multiplying (2.26) by $\rho/2$ we have $\langle \mathbf{E}, \mathbf{p} \rangle \leqslant 0$, which is (2.25).

Conversely we show that the inequality (2.25) leads to the relations (2.22)-(2.24). If $|\mathbf{J}| > \mathscr{J}_c$ happens, then by substituting $\mathbf{p} = -\mathbf{J}$ into (2.25) we arrive at $+\infty \leqslant 0$, which is a contradiction. Thus the inequality (2.23) must always hold.

Suppose $|\mathbf{J}| < \mathscr{J}_c$ and $\mathbf{E} \neq \mathbf{0}$. Then taking a large constant $C > 0$ satisfying $|\mathbf{J} + \mathbf{E}/C| \leqslant \mathscr{J}_c$ and plugging $\mathbf{p} = \mathbf{E}/C$ in (2.25), we have $|\mathbf{E}|^2/C \leqslant 0$, a contradiction. Therefore the relation (2.24) is valid.

Finally we show (2.22). Taking $\mathbf{p} = \mathbf{q} - \mathbf{J}$ for all $\mathbf{q} \in \mathbb{R}^3$ with $|\mathbf{q}| \leqslant \mathscr{J}_c$, we obtain

$$\langle \mathbf{E}, \mathbf{q} - \mathbf{J} \rangle \leqslant 0. \tag{2.27}$$

If $\mathbf{E} = \mathbf{0}$ then (2.22) is true for $\rho = 0$. Let $\mathbf{E}$ be nonzero, then by (2.24) $|\mathbf{J}| = \mathscr{J}_c$. Suppose that the vector $\mathbf{E}$ is not parallel to the vector $\mathbf{J}$. Let us consider the plane A containing the vectors $\mathbf{E}$ and $\mathbf{J}$. Draw the line L which passes through the point $\mathbf{J}$ and is perpendicular to the vector $\mathbf{E}$ on A. Then the line L divides the plane A into two domains. Take any point $\mathbf{q}$ belonging to one of these domains containing the point $\mathbf{E}$ to satisfy $|\mathbf{q}| \leqslant \mathscr{J}_c$, $\mathbf{q} \neq \mathbf{J}$ (see Figure 1). Then we obviously see that $\langle \mathbf{E}, \mathbf{q} - \mathbf{J} \rangle > 0$, which contradicts (2.27). Thus the vector $\mathbf{E}$ must be parallel to the vector $\mathbf{J}$ and we can write $\mathbf{E} = \rho \mathbf{J}$. If $\rho < 0$, then by taking $\mathbf{q} = \mathbf{0}$ in (2.27) we have $-\rho \mathscr{J}_c^2 \leqslant 0$, which is a contradiction. Therefore (2.22) is correct.

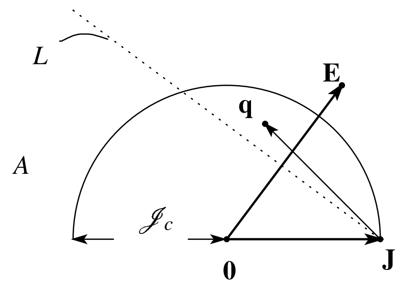

FIG.1.

□

REMARK 2.1 As proposition 2.1 shows, the subdifferential formulation $\mathbf{E} \in \partial \gamma^B(\mathbf{J})$ requires the parallel condition $\mathbf{E} = \rho \mathbf{J}$ for $\rho \geqslant 0$. From a modelling perspective, this relation is accepted if the superconductor Ω_s is axially symmetric (see Prigozhin (1996b)) or a thin film (see Prigozhin (1998)) in a perpendicular external field. However, in the full 3D configuration the direction of the current flowing through the superconductor is not yet settled (see Prigozhin (1996a,b), Chapman (2000) where this issue is argued from the point of view of mathematical modelling). However, the power law characteristic is a popular model based on experimental measurements of superconductors (see Rhyner (1993) and its reference). Furthermore, the Bean type critical state type constitutive law is a limit case of the power law even in 3D situations (see Yin (2001), Yin *et al.* (2002), or proposition 2.6). Thus it is an interesting topic to consider the numerical analysis of the Bean type critical state model $\mathbf{E} \in \partial \gamma^B(\mathbf{J})$ for a general class of 3D type-II superconductor.

PROPOSITION 2.2 For vectors $\mathbf{E}, \mathbf{J} \in \mathbb{R}^3$, the inclusion $\mathbf{E} \in \partial \gamma_\varepsilon^{mB}(\mathbf{J})$ holds if and only if the following relations hold.

$$\mathbf{E} = \begin{cases} \mathbf{0} & \text{if } |\mathbf{J}| < \mathscr{J}_c, \\ \dfrac{1}{\varepsilon}\mathbf{J} & \text{if } |\mathbf{J}| \geqslant \mathscr{J}_c. \end{cases} \tag{2.28}$$

Proof. This equivalence was proved by Bossavit (1994). We sketch the proof.

By definition $\mathbf{E} \in \partial \gamma_\varepsilon^{mB}(\mathbf{J})$ is equivalently written as

$$\langle \mathbf{E}, \mathbf{p} \rangle + \gamma_\varepsilon^{mB}(\mathbf{J}) \leqslant \gamma_\varepsilon^{mB}(\mathbf{J} + \mathbf{p}) \tag{2.29}$$

for all $\mathbf{p} \in \mathbb{R}^3$. By elementary calculation we can check that (2.28) yields (2.29). Let us assume (2.29). If $|\mathbf{J}| < \mathscr{J}_c$, the inequality (2.29) yields that for all $\mathbf{p} \in \mathbb{R}^3$ with $|\mathbf{p}| \leqslant \mathscr{J}_c$,

$$\langle \mathbf{E}, \mathbf{p} - \mathbf{J} \rangle \leqslant 0. \tag{2.30}$$

Taking a large $C >$ such that $|\mathbf{J} + \mathbf{E}/C| \leqslant \mathscr{J}_c$ and substituting $\mathbf{p} = \mathbf{J} + \mathbf{E}/C$ into (2.30), we obtain $|\mathbf{E}|^2/C \leqslant 0$, thus $\mathbf{E} = \mathbf{0}$, which is (2.28).

Assume $|\mathbf{J}| > \mathscr{J}_c$. Take any $\mathbf{p} \in \mathbb{R}^3$. Choosing a small $\delta > 0$ such that $|\mathbf{J} + \delta \mathbf{p}| > \mathscr{J}_c$ and substituting $\delta \mathbf{p}$ into (2.29), we have

$$\delta \langle \mathbf{E}, \mathbf{p} \rangle \leqslant (2\delta \langle \mathbf{J}, \mathbf{p} \rangle + \delta^2 |\mathbf{p}|^2)/(2\varepsilon).$$

Dividing the both side by δ and sending $\delta \searrow 0$, we have

$$\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle \leqslant 0. \tag{2.31}$$

Similarly taking a negative $\delta < 0$ such that $|\mathbf{J} + \delta \mathbf{p}| > \mathscr{J}_c$, we can derive

$$\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle \geqslant 0. \tag{2.32}$$

By (2.31) and (2.32) we have $\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle = 0$ for all $\mathbf{p} \in \mathbb{R}^3$, or $\mathbf{E} = \mathbf{J}/\varepsilon$, which is (2.28).

If $|\mathbf{J}| = \mathscr{J}_c$, take any $\mathbf{p} \in \mathbb{R}^3$ such that $\langle \mathbf{p}, \mathbf{J} \rangle > 0$. Then for all $\delta > 0$, we see $|\mathbf{J} + \delta \mathbf{p}| > \mathscr{J}_c$. Thus, by substituting $\delta \mathbf{p}$ into (2.29) and sending $\delta \searrow 0$ we obtain that for all $\mathbf{p} \in \mathbb{R}^3$ with $\langle \mathbf{p}, \mathbf{J} \rangle > 0$

$$\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle \leqslant 0.$$

This implies that there is $C \geqslant 0$ such that

$$\mathbf{E} - \mathbf{J}/\varepsilon = -C\mathbf{J}. \tag{2.33}$$

Similarly take any $\mathbf{p} \in \mathbb{R}^3$ such that $\langle \mathbf{p}, \mathbf{J} \rangle < 0$. Then for all $\delta < 0$, we see that $|\mathbf{J} + \delta \mathbf{p}| > \mathscr{J}_c$. By substituting $\delta \mathbf{p}$ into (2.29) and sending $\delta \nearrow 0$ we have that for all $\mathbf{p} \in \mathbb{R}^3$ with $\langle \mathbf{p}, \mathbf{J} \rangle < 0$

$$\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle \geqslant 0,$$

which implies that there is $C' \geqslant 0$ such that

$$\mathbf{E} - \mathbf{J}/\varepsilon = C'\mathbf{J}. \tag{2.34}$$

By (2.33) and (2.34) we obtain $\mathbf{E} - \mathbf{J}/\varepsilon = \mathbf{0}$. Therefore, the relation (2.28) holds. □

REMARK 2.2 The model (2.28) proposed by Bossavit (1994) is a modification of the Bean type model (2.22)-(2.24) in the sense that if the current density $|\mathbf{J}|$ exceeds the critical value then $\mathbf{E} - \mathbf{J}$ relation is switched to be the linear Ohm's law.

PROPOSITION 2.3 For vectors $\mathbf{E}, \mathbf{J} \in \mathbb{R}^3$, the inclusion $\mathbf{E} \in \partial \gamma_p^P(\mathbf{J})$ holds if and only if $\mathbf{E} = \mathscr{J}_c^{1-p} |\mathbf{J}|^{p-2} \mathbf{J}$.

Proof. If a convex function is differentiable, its subdifferential is always equal to the derivative of the function (see, eg, Barbu & Precupanu (1986)). Since now $\gamma_p^P(\cdot)$ is differentiable, this equivalence is immediate. □

2.3 *Mathematical formulation of the magnetic field $\widehat{\mathbf{H}}$ via variational inequality*

We formulate Faraday's law (2.17) in an integral form. Take a function $\phi : \Omega \to \mathbb{R}^3$ with $\mathbf{curl}\, \phi = \mathbf{0}$ in Ω_d and $\mathbf{n} \times \phi = \mathbf{0}$ on $\partial \Omega$. Then the equation (2.17) yields

$$\int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}} + \partial_t \mathbf{H}_s, \phi \rangle d\mathbf{x} + \int_{\Omega_s} \langle \mathbf{E}, \mathbf{curl}\, \phi \rangle d\mathbf{x} = 0. \tag{2.35}$$

Let us combine the weak form (2.35) with the constitutive relation (2.11). Substituting $\mathbf{p} = \mathbf{curl}\, \phi(\mathbf{x})$ for all $\phi : \Omega \to \mathbb{R}^3$ satisfying that $\mathbf{curl}\, \phi = \mathbf{0}$ in Ω_d and $\mathbf{n} \times \phi = \mathbf{0}$ on $\partial \Omega$ to the definition of the subdifferential $\partial \gamma$ and recalling the equality (2.18) we see that

$$\begin{aligned} -\int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}}(\mathbf{x},t) + \partial_t \mathbf{H}_s(\mathbf{x},t), \phi(\mathbf{x}) \rangle d\mathbf{x} + \int_{\Omega_s} \gamma(\mathbf{curl}\, \widehat{\mathbf{H}}(\mathbf{x},t)) d\mathbf{x} \\ \leqslant \int_{\Omega_s} \gamma(\mathbf{curl}\, \widehat{\mathbf{H}}(\mathbf{x},t) + \mathbf{curl}\, \phi(\mathbf{x})) d\mathbf{x}, \end{aligned}$$

or equivalently by taking $\phi - \widehat{\mathbf{H}}$ as ϕ above, we reach

$$\int_{\Omega} \mu \langle \partial_t \widehat{\mathbf{H}}(\mathbf{x},t) + \partial_t \mathbf{H}_s(\mathbf{x},t), \phi(\mathbf{x}) - \widehat{\mathbf{H}}(\mathbf{x},t)\rangle d\mathbf{x} + \int_{\Omega_s} \gamma(\mathbf{curl}\,\phi(\mathbf{x}))d\mathbf{x} - \int_{\Omega_s} \gamma(\mathbf{curl}\,\widehat{\mathbf{H}}(\mathbf{x},t))d\mathbf{x} \geqslant 0.$$

Thus we formally obtained a variational inequality formulation of the unknown magnetic field $\widehat{\mathbf{H}}$.

2.3.1 *Functional spaces* To complete the formulation, let us introduce the functional spaces where we analyze the problem mathematically,

$$H(\mathbf{curl};\Omega) := \{\phi \in L^2(\Omega;\mathbb{R}^3) \mid \mathbf{curl}\,\phi \in L^2(\Omega;\mathbb{R}^3)\}$$

with the norm $\|\phi\|_{H(\mathbf{curl};\Omega)} := (\|\phi\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{curl}\,\phi\|^2_{L^2(\Omega;\mathbb{R}^3)})^{1/2}$,

$$H^1(\mathbf{curl};\Omega) := \{\phi \in H^1(\Omega;\mathbb{R}^3) \mid \mathbf{curl}\,\phi \in H^1(\Omega;\mathbb{R}^3)\}$$

with the norm $\|\phi\|_{H^1(\mathbf{curl};\Omega)} := (\|\phi\|^2_{H^1(\Omega;\mathbb{R}^3)} + \|\mathbf{curl}\,\phi\|^2_{H^1(\Omega;\mathbb{R}^3)})^{1/2}$, and

$$H(\mathrm{div};\Omega) := \{\phi \in L^2(\Omega;\mathbb{R}^3) \mid \mathrm{div}\,\phi \in L^2(\Omega)\}$$

with the norm $\|\phi\|_{H(\mathrm{div};\Omega)} := (\|\phi\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathrm{div}\,\phi\|^2_{L^2(\Omega)})^{1/2}$.

Next we define the traces of functions in $H(\mathbf{curl};\Omega)$ and $H(\mathrm{div};\Omega)$. Note that for all $\phi \in H^1(\Omega;\mathbb{R}^N)$ $(N = 1,3)$ $\phi|_{\partial\Omega} \in H^{1/2}(\partial\Omega;\mathbb{R}^N)$, where $H^{1/2}(\partial\Omega;\mathbb{R}^N)$ is a Sobolev space with the norm

$$\|\phi\|_{H^{1/2}(\partial\Omega;\mathbb{R}^N)} := \left(\|\phi\|^2_{L^2(\partial\Omega;\mathbb{R}^N)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|\phi(\mathbf{x}) - \phi(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^3} dA(\mathbf{x})dA(\mathbf{y})\right)^{1/2}.$$

Let $H^{-1/2}(\partial\Omega;\mathbb{R}^N)$ be the dual space of $H^{1/2}(\partial\Omega;\mathbb{R}^N)$ with respect to the inner product of $L^2(\partial\Omega;\mathbb{R}^N)$ with the norm

$$\|\phi\|_{H^{-1/2}(\partial\Omega;\mathbb{R}^N)} := \sup_{\psi \in H^{1/2}(\partial\Omega;\mathbb{R}^N)} \frac{|\langle \phi, \psi\rangle_{L^2(\partial\Omega;\mathbb{R}^N)}|}{\|\psi\|_{H^{1/2}(\partial\Omega;\mathbb{R}^N)}}.$$

For all $\phi \in H(\mathbf{curl};\Omega)$, the trace $\mathbf{n}\times\phi$ on $\partial\Omega$ is well-defined in $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, in the sense that

$$\langle \mathbf{n}\times\phi, \psi\rangle_{L^2(\partial\Omega;\mathbb{R}^3)} := \langle \mathbf{curl}\,\phi, \psi\rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle \phi, \mathbf{curl}\,\psi\rangle_{L^2(\Omega;\mathbb{R}^3)}$$

for all $\phi \in H^1(\Omega;\mathbb{R}^3)$. For all $\phi \in H(\mathrm{div};\Omega)$ the trace $\mathbf{n}\cdot\phi$ on $\partial\Omega$ is well-defined in $H^{-1/2}(\partial\Omega)$ in the sense that

$$\langle \mathbf{n}\cdot\phi, f\rangle_{L^2(\partial\Omega)} := \langle \mathrm{div}\,\phi, f\rangle_{L^2(\Omega)} + \langle \phi, \nabla f\rangle_{L^2(\Omega;\mathbb{R}^3)}$$

for all $f \in H^1(\Omega)$.

Set the subspace $V(\Omega)$ of $H(\mathbf{curl};\Omega)$ by

$$V(\Omega) := \{\phi \in H(\mathbf{curl};\Omega) \mid \mathbf{curl}\,\phi = \mathbf{0} \text{ in } \Omega_d,\ \mathbf{n}\times\phi = \mathbf{0} \text{ on } \partial\Omega\}.$$

The subspace $V_p(\Omega)$ of $V(\Omega)$ $(p \geqslant 2)$ is defined by

$$V_p(\Omega) := \{\phi \in V(\Omega) \mid \mathbf{curl}\,\phi|_{\Omega_s} \in L^p(\Omega_s;\mathbb{R}^3)\}.$$

The subset S of $V(\Omega)$ is defined by

$$S := \{\phi \in V(\Omega) \mid |\mathbf{curl}\,\phi| \leqslant \mathscr{J}_c \text{ a.e. in } \Omega_s\}.$$

The subspace $X^{(\mu)}(\Omega)$ of $H(\mathbf{curl};\Omega)$ consisting of divergence free functions for the magnetic permeability μ is defined by

$$X^{(\mu)}(\Omega) := \{\phi \in H(\mathbf{curl};\Omega) \mid \operatorname{div}(\mu\phi) = 0 \text{ in } \mathscr{D}'(\Omega)\},$$

where $\mathscr{D}'(\Omega)$ denotes the space of Schwartz distribution.

The spaces $L^q(0,T;B)$ ($q = 2$ or ∞), $H^1(0,T;B)$, $C([0,T];B)$ and $C^{1,1}([0,T];B)$ for a Banach space B are defined in the usual way.

2.3.2 *External magnetic field* In this section we discuss the external magnetic field $\mathbf{H}_s$ solving (2.7)-(2.10). We assume that the boundary value $\mathbf{h}_s : \partial\Omega \times [0,T] \to \mathbb{R}^3$ satisfies

$$\mathbf{h}_s \in C^{1,1}([0,T];H^{1/2}(\partial\Omega;\mathbb{R}^3)) \text{ and } \mathbf{h}_s(0) = \mathbf{0}, \tag{2.36}$$

and for all $t \in [0,T]$ and $\phi \in H(\mathbf{curl};\Omega)$ with $\mathbf{curl}\,\phi = \mathbf{0}$

$$\int_{\partial\Omega} \langle \mathbf{h}_s(t), \mathbf{n} \times \phi \rangle dA = 0. \tag{2.37}$$

LEMMA 2.1 On the assumptions (2.36) and (2.37) there uniquely exists
$\mathbf{H}_s \in C^{1,1}([0,T];H^1(\mathbf{curl};\Omega))$ such that $\mathbf{H}_s$ satisfies the system (2.7)-(2.10) in the weak sense for all $t \in [0,T]$ and $\mathbf{H}_s(0) = \mathbf{0}$. Moreover, the following inequalities hold for all $t \in [0,T]$,

$$\|\mathbf{H}_s(t)\|_{H^1(\mathbf{curl};\Omega)} \leqslant C\|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega;\mathbb{R}^3)}, \tag{2.38}$$

$$\|\partial_t \mathbf{H}_s(t)\|_{H^1(\mathbf{curl};\Omega)} \leqslant C\|\partial_t \mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega;\mathbb{R}^3)}. \tag{2.39}$$

Proof. The proof of the unique existence follows Auchmuty & Alexander (2005) where the unique solvability theory for general div-**curl** systems assuming C^2 class boundary was developed. Fix any $t \in [0,T]$. We use the following Helmholtz decomposition
(see, eg, (Cessenat, 1996, Theorem 10', Chapter 2)).

$$L^2(\Omega;\mathbb{R}^3) = \nabla H_0^1(\Omega) \oplus \mathbf{curl}\,H^1(\Omega;\mathbb{R}^3), \tag{2.40}$$

$$L^2(\Omega;\mathbb{R}^3) = \nabla H^1(\Omega) \oplus \mathbf{curl}\,H_0^1(\Omega;\mathbb{R}^3). \tag{2.41}$$

We will find $\mathbf{H}_s(t) \in L^2(\Omega;\mathbb{R}^3)$ such that

$$\mathbf{curl}\,\mathbf{H}_s(t) = \mathbf{0} \text{ in } \Omega, \tag{2.42}$$

$$\operatorname{div}\mathbf{H}_s(t) = 0 \text{ in } \Omega, \tag{2.43}$$

$$\mathbf{n} \times \mathbf{H}_s(t) = \mathbf{n} \times \mathbf{h}_s(t) \text{ on } \partial\Omega. \tag{2.44}$$

By the decomposition (2.40) we can write $\mathbf{H}_s(t) = \nabla f + \mathbf{curl}\,\mathbf{H}_1$ with $f \in H_0^1(\Omega)$ and $\mathbf{H}_1 \in H^1(\Omega;\mathbb{R}^3)$. The condition (2.43) implies $f \equiv 0$. Thus, our problem is equivalent to find $\mathbf{H}_1 \in H^1(\Omega;\mathbb{R}^3)$ such that

$$\mathbf{curl}(\mathbf{curl}\,\mathbf{H}_1) = \mathbf{0} \text{ in } \Omega, \tag{2.45}$$

$$\mathbf{n} \times \mathbf{curl}\,\mathbf{H}_1 = \mathbf{n} \times \mathbf{h}_s(t) \text{ on } \partial\Omega. \tag{2.46}$$

The weak form of (2.45)-(2.46) is

$$\int_{\Omega} \langle \mathbf{curl}\,\mathbf{H}_1, \mathbf{curl}\,\phi \rangle d\mathbf{x} + \int_{\partial\Omega} \langle \mathbf{n} \times \mathbf{h}_s(t), \phi \rangle dA = 0, \tag{2.47}$$

for all $\phi \in H^1(\Omega; I\!R^3)$.

For all $\phi \in H^1(\Omega; I\!R^3)$, the decomposition (2.41) implies that there uniquely exist $\hat{f} \in H^1(\Omega)$ and $\mathbf{H}_2 \in \mathbf{curl}\, H_0^1(\Omega; I\!R^3)$ such that $\phi = \nabla \hat{f} + \mathbf{H}_2$. Note that $\mathbf{H}_2 \cdot \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. Therefore by the assumption (2.37) the problem (2.47) is equal to the problem; find $\mathbf{H}_2 \in X_1$ such that

$$\int_{\Omega} \langle \mathbf{curl}\,\mathbf{H}_2, \mathbf{curl}\,\phi \rangle d\mathbf{x} - \int_{\partial\Omega} \langle \mathbf{h}_s(t), \mathbf{n} \times \phi \rangle dA = 0, \tag{2.48}$$

for all $\phi \in X_1$, where the space X_1 is defined by

$$X_1 := \{\phi \in H(\mathbf{curl}; \Omega) \mid \operatorname{div} \phi = 0 \text{ in } \Omega,\ \mathbf{n} \cdot \phi = 0 \text{ on } \partial\Omega\}$$

equipped with the norm of $H(\mathbf{curl}; \Omega)$.

Let us define a functional $F : X_1 \to I\!R$ by

$$F(\phi) := \frac{1}{2} \int_{\Omega} |\,\mathbf{curl}\,\phi|^2 d\mathbf{x} - \int_{\partial\Omega} \langle \mathbf{h}_s(t), \mathbf{n} \times \phi \rangle dA.$$

Then we see that

$$\begin{aligned} F(\phi) &\geqslant \frac{1}{2} \|\,\mathbf{curl}\,\phi\|^2_{L^2(\Omega; I\!R^3)} - \|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega; I\!R^3)} \|\mathbf{n} \times \phi\|_{H^{-1/2}(\partial\Omega; I\!R^3)} \\ &\geqslant \frac{1}{2} \|\,\mathbf{curl}\,\phi\|^2_{L^2(\Omega; I\!R^3)} - C \|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega; I\!R^3)} \|\,\mathbf{curl}\,\phi\|_{L^2(\Omega; I\!R^3)}, \end{aligned} \tag{2.49}$$

where we have used a fact that the map $\phi \mapsto \mathbf{n} \times \phi : H(\mathbf{curl}; \Omega) \to H^{-1/2}(\partial\Omega; I\!R^3)$ is continuous and the Friedrichs inequality (see Girault & Raviart (1986))

$$\|\phi\|_{L^2(\Omega; I\!R^3)} \leqslant C \|\,\mathbf{curl}\,\phi\|_{L^2(\Omega; I\!R^3)}$$

for all $\phi \in X_1$. Therefore by noting the convexity of F and (2.49), we can show the unique existence of $\mathbf{H}_2 \in X_1$ satisfying

$$F(\mathbf{H}_2) = \min_{\phi \in X_1} F(\phi), \tag{2.50}$$

which is equivalent to the problem (2.48). Hence, the unique existence of the solution of (2.42)-(2.44) has been proved.

Next we will show that $\mathbf{H}_s \in C^{1,1}([0,T]; H^1(\mathbf{curl}; \Omega))$ and the inequalities (2.38) and (2.39). Fix $t \in [0,T]$. Let $\xi^t \in H^1(\Omega; I\!R^3)$ be a weak solution of the following elliptic problem.

$$\begin{aligned} &\Delta \xi^t = \mathbf{0} \text{ in } \Omega, \\ &\xi^t = \mathbf{h}_s(t) \text{ on } \partial\Omega. \end{aligned} \tag{2.51}$$

Then, we have

$$\|\xi^t\|_{H^1(\Omega; I\!R^3)} \leqslant C \|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega; I\!R^3)}. \tag{2.52}$$

Since Ω is convex, the space X_2 defined by

$$X_2 := \{\phi \in H(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega) \mid \mathbf{n} \times \phi = \mathbf{0} \text{ on } \partial\Omega\}$$

equipped with the inner product

$$\langle \phi, \psi \rangle_{X_2} := \langle \phi, \psi \rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle \mathbf{curl}\,\phi, \mathbf{curl}\,\psi \rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle \mathrm{div}\,\phi, \mathrm{div}\,\psi \rangle_{L^2(\Omega)}$$

is continuously imbedded in $H^1(\Omega;\mathbb{R}^3)$ (see (Amrouche *et al.*, 1998, Proposition 2.17)). Noting that $\mathbf{H}_s(t) - \xi^t \in X_2$ we have

$$\begin{aligned}
\|\mathbf{H}_s(t)\|_{H^1(\Omega;\mathbb{R}^3)} &\leqslant \|\mathbf{H}_s(t) - \xi^t\|_{H^1(\Omega;\mathbb{R}^3)} + \|\xi^t\|_{H^1(\Omega;\mathbb{R}^3)} \\
&\leqslant C_1(\|\mathbf{H}_s(t) - \xi^t\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{curl}\,\xi^t\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathrm{div}\,\xi^t\|_{L^2(\Omega)}) + \|\xi^t\|_{H^1(\Omega;\mathbb{R}^3)} \\
&\leqslant C_2\|\mathbf{n} \times \mathbf{h}_s(t)\|_{L^2(\partial\Omega;\mathbb{R}^3)} + C_3\|\xi^t\|_{H^1(\Omega;\mathbb{R}^3)} \\
&\leqslant C_4\|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega;\mathbb{R}^3)},
\end{aligned}$$

where we have used the inequality (2.52) and the Friedrichs inequality (see Girault & Raviart (1986))

$$\|\mathbf{H}_s(t)\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant C\|\mathbf{n} \times \mathbf{h}_s(t)\|_{L^2(\partial\Omega;\mathbb{R}^3)}.$$

Since $\mathbf{h}_s : [0,T] \to H^{1/2}(\partial\Omega;\mathbb{R}^3)$ is continuous, we have $\mathbf{H}_s \in C([0,T];H^1(\Omega;\mathbb{R}^3))$. By repeating the same argument for $\partial_t\mathbf{H}_s$, we can show that $\partial_t\mathbf{H}_s : [0,T] \to H^1(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and the inequality (2.39). $\square$

From now on, the magnetic field $\mathbf{H}_s$ is the one proved in lemma 2.1 on assumptions (2.36) and (2.37).

2.3.3 *Variational inequality formulations* Now we are ready to propose our mathematical formulation of (2.17)-(2.21) coupled with the nonlinear constitutive law (2.11) as the initial value problem of the evolution variational inequality for unknown $\widehat{\mathbf{H}}$. The first one is the formulation with the Bean's model:-

($\mathbf{P}^B\mathbf{1}$) Find $\widehat{\mathbf{H}} \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ such that $\widehat{\mathbf{H}}(t) \in S$ for a.e. $t \in [0,T]$,

$$\int_\Omega \mu\langle \partial_t\widehat{\mathbf{H}}(\mathbf{x},t) + \partial_t\mathbf{H}_s(\mathbf{x},t), \phi(\mathbf{x}) - \widehat{\mathbf{H}}(\mathbf{x},t)\rangle d\mathbf{x} \geqslant 0 \text{ for a.e. } t \in (0,T] \tag{2.53}$$

holds for all $\phi \in S$ and $\widehat{\mathbf{H}}(\mathbf{x},0) = \mathbf{0}$ in Ω.

PROPOSITION 2.4 The solution $\widehat{\mathbf{H}}$ of ($\mathbf{P}^B\mathbf{1}$) uniquely exists. Moreover, the solution $\widehat{\mathbf{H}}(t) : [0,T] \to L^2(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and satisfies $\widehat{\mathbf{H}}(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$.

Proof. The proof essentially follows (Prigozhin, 1996a, Theorem 2) where the magnetic permeability μ was assumed to be constant and the problem was formulated in the whole space $\mathbb{R}^3$. Let $L^2_\mu(\Omega;\mathbb{R}^3)$ denote a Hilbert space $L^2(\Omega;\mathbb{R}^3)$ equipped with the inner product $\langle \mu\cdot,\cdot\rangle_{L^2(\Omega;\mathbb{R}^3)}$. The problem ($\mathbf{P}^B\mathbf{1}$) becomes an evolution problem in $L^2_\mu(\Omega;\mathbb{R}^3)$ as follows.

$$\begin{cases} d_t\widehat{\mathbf{H}}(t) + \partial_t\mathbf{H}_s(t) \in -\partial E(\widehat{\mathbf{H}}(t)) \text{ a.e. } t \in (0,T], \\ \widehat{\mathbf{H}}(0) = \mathbf{0}, \end{cases} \tag{2.54}$$

where the energy functional $E : L^2_\mu(\Omega;\mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ is an indicator functional of the nonempty closed convex set $S \subset L^2_\mu(\Omega;\mathbb{R}^3)$ so that

$$E(\phi) := \begin{cases} 0 & \text{if } \phi \in S, \\ +\infty & \text{otherwise}. \end{cases} \tag{2.55}$$

Since E is convex and lower semicontinuous, not identically $+\infty$, its subdiffrential ∂E is a maximal monotone operator in $L^2_\mu(\Omega;\mathbb{R}^3)$. Therefore by the Lipschitz continuity of the given data $\partial_t \mathbf{H}_s(t)$ the standard theorem from nonlinear semigroup theory (see, eg, (Brezis, 1971, Theorem 21)) assures the unique existence of $\widehat{\mathbf{H}} \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ satisfying $\widehat{\mathbf{H}}(t) \in S$ for all $t \in [0,T]$, (2.54) and the Lipschitz continuity on $[0,T]$.

We show that $\widehat{\mathbf{H}}(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$. Take any $f \in \mathscr{D}(\Omega)$ and any $\delta \in \mathbb{R}$. Substituting $\delta\nabla f + \widehat{\mathbf{H}}(t) \in S$ into (2.53), we obtain

$$\delta \int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}}(t) + \partial_t \mathbf{H}_s(t), \nabla f \rangle d\mathbf{x} \geqslant 0.$$

By separately taking positive and negative δ, we have

$$\int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}}(t) + \partial_t \mathbf{H}_s(t), \nabla f \rangle d\mathbf{x} = 0 \tag{2.56}$$

for a.e $t \in (0,T]$. Since now $\widehat{\mathbf{H}} + \mathbf{H}_s$ is Lipschitz continuous, by integrating (2.56) over $[0,t]$ we reach $\langle \mu(\widehat{\mathbf{H}}(t) + \mathbf{H}_s(t)), \nabla f \rangle_{L^2(\Omega;\mathbb{R}^3)} = 0$ for all $t \in [0,T]$. □

Formulating the modified Bean model with the energy density $g(|\cdot|)/\varepsilon$ in the same way as $(\mathbf{P}^B\mathbf{1})$ leads to the initial value problem $(\mathbf{P}^{mB}_\varepsilon\mathbf{1})$:-

$(\mathbf{P}^{mB}_\varepsilon\mathbf{1})$ Find $\widehat{\mathbf{H}}_\varepsilon \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ such that $\widehat{\mathbf{H}}_\varepsilon(t) \in V(\Omega)$ for all $t \in [0,T]$,

$$\begin{aligned} &\int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}}_\varepsilon(\mathbf{x},t) + \partial_t \mathbf{H}_s(\mathbf{x},t), \phi(\mathbf{x}) - \widehat{\mathbf{H}}_\varepsilon(\mathbf{x},t) \rangle d\mathbf{x} \\ &\qquad + \frac{1}{\varepsilon}\int_\Omega g(|\mathbf{curl}\,\phi(\mathbf{x})|)d\mathbf{x} - \frac{1}{\varepsilon}\int_\Omega g(|\mathbf{curl}\,\widehat{\mathbf{H}}_\varepsilon(\mathbf{x},t)|)d\mathbf{x} \geqslant 0 \text{ for a.e. } t \in (0,T] \end{aligned} \tag{2.57}$$

holds for all $\phi \in V(\Omega)$ and $\widehat{\mathbf{H}}_\varepsilon(\mathbf{x},0) = \mathbf{0}$ in Ω.

PROPOSITION 2.5 The solution $\widehat{\mathbf{H}}_\varepsilon$ of $(\mathbf{P}^{mB}_\varepsilon\mathbf{1})$ uniquely exists. The solution $\widehat{\mathbf{H}}_\varepsilon : [0,T] \to L^2(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and satisfies $\widehat{\mathbf{H}}_\varepsilon(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$. Moreover, the following convergences to the solution $\widehat{\mathbf{H}}$ of $(\mathbf{P}^B\mathbf{1})$ hold. As $\varepsilon \searrow 0$,

$$\begin{aligned} &\widehat{\mathbf{H}}_\varepsilon \to \widehat{\mathbf{H}} \text{ strongly in } C([0,T];L^2(\Omega;\mathbb{R}^3)), \\ &\partial_t \widehat{\mathbf{H}}_\varepsilon \to \partial_t \widehat{\mathbf{H}} \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \\ &\frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_\varepsilon(t)|)d\mathbf{x} \to 0 \text{ uniformly in } [0,T]. \end{aligned} \tag{2.58}$$

Proof. Define $E_\varepsilon(\cdot) : L^2_\mu(\Omega;\mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by

$$E_\varepsilon(\phi) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_{\Omega_s} g(|\mathbf{curl}\,\phi|)d\mathbf{x} & \text{if } \phi \in V(\Omega), \\ +\infty & \text{otherwise}. \end{cases}$$

Then the problem ($\mathbf{P}_\varepsilon^{mB}\mathbf{1}$) is written as an evolution equation

$$d_t\widehat{\mathbf{H}}_\varepsilon(t)+\partial_t\mathbf{H}_s(t)\in-\partial E_\varepsilon(\widehat{\mathbf{H}}_\varepsilon(t)) \tag{2.59}$$

for a.e $t\in(0,T]$ and $\widehat{\mathbf{H}}_\varepsilon(0)=\mathbf{0}$.

The energy E_ε is convex and not identically $+\infty$. We show that E_ε is lower semicontinuous. Suppose $\phi_n\to\phi$ in $L^2_\mu(\Omega;\mathbb{R}^3)$ as $n\to+\infty$ and $E_\varepsilon(\phi_n)\leqslant\lambda$ for all $n\in\mathbb{N}$. Since $\phi_n\in V(\Omega)$ and $V(\Omega)$ is a closed subspace of $L^2_\mu(\Omega;\mathbb{R}^3)$, we have $\phi\in V(\Omega)$. By (2.14) we have

$$A_1\|\mathbf{curl}\,\phi_n\|^2_{L^2(\Omega_s;\mathbb{R}^3)}/\varepsilon\leqslant\lambda+A_2/\varepsilon,$$

for all $n\in\mathbb{N}$, which means that $\{\mathbf{curl}\,\phi_n\}_{n=1}^\infty$ is bounded in $L^2(\Omega_s;\mathbb{R}^3)$. Thus, by taking a subsequence and its convex combination still denoted by $\{\mathbf{curl}\,\phi_n\}_{n=1}^\infty$ we have

$$\mathbf{curl}\,\phi_n(\mathbf{x})\to\mathbf{curl}\,\phi(\mathbf{x})\ \text{a.e}\ \mathbf{x}\in\Omega_s$$

as $n\to+\infty$. Thus by the convexity of $g(\cdot)$ and Fatou's lemma we obtain

$$E_\varepsilon(\phi)\leqslant\liminf_{n\to+\infty}E_\varepsilon(\phi_n)\leqslant\lambda.$$

Therefore, E_ε is lower semicontinuous in $L^2_\mu(\Omega;\mathbb{R}^3)$. Hence, by the Lipschitz continuity of $\partial_t\mathbf{H}_s:[0,T]\to L^2_\mu(\Omega;\mathbb{R}^3)$, the evolution equation (2.59) has the unique solution $\widehat{\mathbf{H}}_\varepsilon(\in H^1(0,T;L^2(\Omega;\mathbb{R}^3)))$ which satisfies that $\widehat{\mathbf{H}}_\varepsilon(t)\in V(\Omega)$ for all $t\in[0,T]$ and $\widehat{\mathbf{H}}_\varepsilon:[0,T]\to L^2(\Omega;\mathbb{R}^3)$ is Lipschitz continuous. The property $\widehat{\mathbf{H}}_\varepsilon(t)+\mathbf{H}_s(t)\in X^{(\mu)}(\Omega)$ for all $t\in[0,T]$ can be proved in the same way as proposition 2.4.

We will prove the convergences (2.58). Take any sequence $\{\varepsilon_i\}_{i=1}^\infty$ satisfying $\varepsilon_i\searrow0$ as $i\to+\infty$. By (Attouch, 1978, Theorem 2.1) it is sufficient to prove that the sequence of energies E_{ε_i} converges to the energy E defined in (2.55) in the sense of Mosco as $i\to+\infty$, that is

(i) If $\phi_{\varepsilon_i}\rightharpoonup\phi$ weakly in $L^2_\mu(\Omega;\mathbb{R}^3)$ as $i\to+\infty$, $E(\phi)\leqslant\liminf_{i\to+\infty}E_{\varepsilon_i}(\phi_{\varepsilon_i})$ holds.

(ii) For any $\phi\in L^2_\mu(\Omega;\mathbb{R}^3)$ with $E(\phi)<+\infty$, there exists a sequence $\{\phi_{\varepsilon_i}\}_{i=1}^\infty$ such that $\phi_{\varepsilon_i}\to\phi$ strongly in $L^2_\mu(\Omega;\mathbb{R}^3)$ and $E_{\varepsilon_i}(\phi_{\varepsilon_i})\to E(\phi)$ as $i\to+\infty$.

Since $E_{\varepsilon_i}(\phi)=E(\phi)=0$ for all $\phi\in S$ and all i, (ii) is true. Assume $\phi_{\varepsilon_i}\rightharpoonup\phi$ weakly in $L^2_\mu(\Omega;\mathbb{R}^3)$ and $E_{\varepsilon_i}(\phi_{\varepsilon_i})\leqslant\lambda$ for all $i\in\mathbb{N}$. By convexity of $g(\cdot)$ and Fatou's lemma we see that by taking a subsequence and its convex combination denoted by $\{\sum_{i=1}^n\phi_{\varepsilon_i}/n\}_{n=1}^\infty$

$$\int_{\Omega_s}g(|\mathbf{curl}\,\phi|)d\mathbf{x}\leqslant\liminf_{n\to+\infty}\int_{\Omega_s}g(|\frac{1}{n}\sum_{i=1}^n\mathbf{curl}\,\phi_{\varepsilon_i}|)d\mathbf{x}\leqslant\liminf_{n\to+\infty}\frac{1}{n}\sum_{i=1}^n\varepsilon_i\lambda=0,$$

which yields $|\mathbf{curl}\,\phi(\mathbf{x})|\leqslant\mathscr{J}_c$ a.e in Ω_s. Therefore $\phi\in S$ and $E(\phi)=0\leqslant\lambda$, which means that (i) is correct. The desired convergences are proved by applying (Attouch, 1978, Theorem 2.1). $\square$

The variational inequality formulation with the power law constitutive relation $\mathbf{E}\in\partial\gamma_p^P(\mathbf{J})$ is stated as follows.

($\mathbf{P}_p^P\mathbf{1}$) Find $\widehat{\mathbf{H}}_p \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ such that $\widehat{\mathbf{H}}_p(t) \in V_p(\Omega)$ for all $t \in [0,T]$,

$$\begin{aligned}\int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}}_p(\mathbf{x},t) + \partial_t \mathbf{H}_s(\mathbf{x},t), \phi(\mathbf{x}) - \widehat{\mathbf{H}}_p(\mathbf{x},t)\rangle d\mathbf{x} \\ + \int_\Omega \gamma_p^P(\mathbf{curl}\,\phi(\mathbf{x}))d\mathbf{x} - \int_\Omega \gamma_p^P(\mathbf{curl}\,\widehat{\mathbf{H}}_p(\mathbf{x},t))d\mathbf{x} \geqslant 0 \text{ for a.e. } t \in (0,T]\end{aligned} \tag{2.60}$$

holds for all $\phi \in V_p(\Omega)$ and $\widehat{\mathbf{H}}_p(\mathbf{x},0) = \mathbf{0}$ in Ω.

PROPOSITION 2.6 The solution $\widehat{\mathbf{H}}_p$ of ($\mathbf{P}_p^P\mathbf{1}$) uniquely exists. The solution $\widehat{\mathbf{H}}_p : [0,T] \to L^2(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and satisfies $\widehat{\mathbf{H}}_p(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$. Moreover, the following convergences to the solution $\widehat{\mathbf{H}}$ of ($\mathbf{P}^B\mathbf{1}$) hold. As $p \to +\infty$,

$$\begin{aligned}&\widehat{\mathbf{H}}_p \to \widehat{\mathbf{H}} \text{ strongly in } C([0,T];L^2(\Omega;\mathbb{R}^3)),\\ &\partial_t\widehat{\mathbf{H}}_p \to \partial_t\widehat{\mathbf{H}} \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)),\\ &\int_{\Omega_s} \gamma_p^P(\mathbf{curl}\,\widehat{\mathbf{H}}_p(t))d\mathbf{x} \to 0 \text{ uniformly in } [0,T].\end{aligned} \tag{2.61}$$

Proof. Let us define the energy functional $E_p : L^2_\mu(\Omega;\mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by

$$E_p(\phi) := \begin{cases} \int_{\Omega_s} \gamma_p^P(\mathbf{curl}\,\phi)d\mathbf{x} & \text{if } \phi \in V_p(\Omega),\\ +\infty & \text{otherwise .}\end{cases}$$

The convexity of E_p is obvious. By noting that $L^p(\Omega;\mathbb{R}^3)$ is reflexive and Mazur's theorem (see, eg, Yosida (1980)) we can check that E_p is lower semicontinuous. Therefore, by the Lipschitz continuity of $\mathbf{H}_s : [0,T] \to L^2_\mu(\Omega;\mathbb{R}^3)$ the evolution variational inequality (2.60) has the unique solution $\widehat{\mathbf{H}}_p$ satisfying that $\widehat{\mathbf{H}}_p(t) \in V_p(\Omega)$ for all $t \in [0,T]$ and $\widehat{\mathbf{H}}_p(t) : [0,T] \to L^2_\mu(\Omega;\mathbb{R}^3)$ is Lipschitz continuous. The condition $\widehat{\mathbf{H}}_p(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ can be conformed in the same way as proposition 2.4.

To show the convergences (2.61) we show that E_{p_i} converges to E in the sense of Mosco as $i \to +\infty$ for any sequence $\{p_i\}_{i=1}^\infty \subset \mathbb{R}_{\geqslant 2}$ satisfying $p_i \nearrow +\infty$. Let us check the condition (ii) of Mosco convergence stated in the proof of proposition 2.5 first. Take any $\phi \in S$. We see that

$$0 \leqslant E_{p_i}(\phi) - E(\phi) \leqslant \mathscr{J}_c|\Omega_s|/p_i \to 0$$

as $i \to +\infty$. Thus, (ii) holds. To show (i), assume that $\phi_i \rightharpoonup \phi$ weakly in $L^2_\mu(\Omega;\mathbb{R}^3)$ as $i \to +\infty$ and $E_{p_i}(\phi_i) \leqslant \lambda$ for any $i \in \mathbb{N}$, i.e,

$$\frac{\mathscr{J}_c}{p_i}\int_{\Omega_s} |\mathbf{curl}\,\phi_i/\mathscr{J}_c|^{p_i} d\mathbf{x} \leqslant \lambda. \tag{2.62}$$

Fix p_i and take $q \in [2, p_i]$. Then by applying Hölder's inequality to (2.62) we have

$$\int_{\Omega_s} |\mathbf{curl}\,\phi_i/\mathscr{J}_c|^q d\mathbf{x} \leqslant \left(\int_{\Omega_s} |\mathbf{curl}\,\phi_i/\mathscr{J}_c|^{p_i} d\mathbf{x}\right)^{q/p_i} |\Omega_s|^{1-q/p_i} \leqslant (\lambda p_i/\mathscr{J}_c)^{q/p_i}|\Omega_s|^{1-q/p_i}. \tag{2.63}$$

By taking $q = 2$ in (2.63) we obtain

$$\int_{\Omega_s} |\mathbf{curl}\,\phi_i/\mathscr{J}_c|^2 d\mathbf{x} \leqslant (\lambda p_i/\mathscr{J}_c)^{2/p_i}|\Omega_s|^{1-2/p_i}. \tag{2.64}$$

Since $\lim_{i\to\infty}(\lambda p_i/\mathscr{J}_c)^{2/p_i}|\Omega_s|^{1-2/p_i} = |\Omega_s|$ (2.64) implies that $\{\mathbf{curl}\,\phi_i\}_{i=1}^{\infty}$ is bounded in $L^2(\Omega;\mathbb{R}^3)$. Therefore by extracting a subsequence still denoted by $\{\mathbf{curl}\,\phi_i\}_{i=1}^{\infty}$ we observe that ϕ_i weakly converges to ϕ in $H(\mathbf{curl};\Omega)$ as $i\to\infty$ and $\phi\in V(\Omega)$. We show $|\mathbf{curl}\,\phi|\leqslant\mathscr{J}_c$ in Ω_s. We can choose a subsequence of $\{\phi_i\}_{i=1}^{\infty}$ so that its convex combination denoted by $\{\sum_{j=1}^{i}\phi_j/i\}_{i=1}^{\infty}$ strongly converges to ϕ in $H(\mathbf{curl};\Omega)$ as $i\to+\infty$. Thus if necessary by taking a subsequence we see that $\mathbf{curl}(\sum_{j=1}^{i}\phi_j(\mathbf{x})/i)$ converges to $\mathbf{curl}\,\phi(\mathbf{x})$ a.e. in Ω as $i\to+\infty$. By applying Fatou's lemma to (2.63) we have

$$\int_{\Omega_s}|\mathbf{curl}\,\phi/\mathscr{J}_c|^q d\mathbf{x}\leqslant\liminf_{i\to+\infty}\left(\sum_{j=1}^{i}(\lambda p_j/\mathscr{J}_c)^{q/p_j}|\Omega_s|^{1-q/p_j}/i\right)=|\Omega_s|,$$

or $\|\mathbf{curl}\,\phi/\mathscr{J}_c\|_{L^q(\Omega_s;\mathbb{R}^3)}\leqslant|\Omega_s|^{1/q}$. By sending $q\to\infty$ we obtain $\|\mathbf{curl}\,\phi/\mathscr{J}_c\|_{L^\infty(\Omega_s;\mathbb{R}^3)}\leqslant 1$. Therefore $\phi\in S$ and $E(\phi)=0\leqslant\lambda$. Thus, (ii) has been proved. This Mosco convergence immediately shows the desired convergences by (Attouch, 1978, Theorem 2.1). □

We will use the following statement, which can be proved in the same way as the proof above, in section 4.

COROLLARY 2.1 Let $\{p_n\}_{n=1}^{\infty}$ be a sequence satisfying that $p_n\geqslant 2$ for any $n\in\mathbb{N}$ and $p_n\to+\infty$ as $n\to+\infty$.

(1) If a sequence $\{\psi_n\}_{n=1}^{\infty}\subset L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ satisfies that for a.e. $t\in(0,T)$ and any $n\in\mathbb{N}$

$$\frac{1}{p_n}\int_{\Omega}|\psi_n(t)|^{p_n}d\mathbf{x}\leqslant\lambda,$$

where $\lambda\geqslant 0$, then $\{\psi_n\}_{n=1}^{\infty}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$.

(2) For a sequence $\{\phi_n\}_{n=1}^{\infty}\subset L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ with $\phi_n\to\phi$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ assume

$$\frac{1}{p_n}\int_0^T\int_{\Omega_s}|\mathbf{curl}\,\phi_n/\mathscr{J}_c|^{p_n}d\mathbf{x}dt\leqslant\lambda$$

for all $n\in\mathbb{N}$. Then $|\mathbf{curl}\,\phi(\mathbf{x},t)|\leqslant\mathscr{J}_c$ a.e. in $\Omega_s\times(0,T)$.

2.4 *Magnetic field - magnetic scalar potential hybrid formulation*

The curl free constraint in the nonconductive region Ω_d can be enforced by expressing the magnetic field as a magnetic scalar potential. This hybrid formulation was recently applied to time harmonic eddy current models with input current intensities on the boundary of the domain in Bermúdez *et al.* (2002). We adopt this method to rewrite the variational inequality formulation ($\mathbf{P}^B\mathbf{1}$) in an equivalent form without the constraint.

Let us prepare some notations. For $\mathbf{u_1}\in L^2(\Omega_s;\mathbb{R}^3)$ and $\mathbf{u_2}\in L^2(\Omega_d;\mathbb{R}^3)$, $(\mathbf{u_1}|\mathbf{u_2})\in L^2(\Omega;\mathbb{R}^3)$ is defined by

$$(\mathbf{u_1}|\mathbf{u_2}):=\begin{cases}\mathbf{u_1} & \text{in }\Omega_s,\\ \mathbf{u_2} & \text{in }\Omega_d.\end{cases}$$

We define a linear space $W(\Omega)$ and its subspace $W_p(\Omega)$ $(p\geqslant 2)$ by

$$W(\Omega):=\{(\phi|\nabla v)\in L^2(\Omega;\mathbb{R}^3)\mid(\phi,v)\in L^2(\Omega_s;\mathbb{R}^3)\times H^1(\Omega_d),(\phi|\nabla v)\in H(\mathbf{curl};\Omega),v=0\text{ on }\partial\Omega\}.$$
$$W_p(\Omega):=\{(\phi|\nabla v)\in W(\Omega)\mid\mathbf{curl}\,\phi\in L^p(\Omega_s;\mathbb{R}^3)\}.$$

The space $W(\Omega)$ is endowed with the inner product of $H(\mathbf{curl};\Omega)$.

PROPOSITION 2.7 The space $W(\Omega)$ is isomorphic to $V(\Omega)$ as a Hilbert space.

Proof. For any $\mathbf{H} \in V(\Omega)$ there exists a scalar potential $v_H \in H^1(\Omega_d)$ such that $\mathbf{H}|_{\Omega_d} = \nabla v_H$ and v_H is unique up to an additive constant since $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ in the simply connected domain Ω_d (see, eg, (Monk, 2003, Theorem 3.37)). The boundary condition $\mathbf{n} \times \mathbf{H} = \mathbf{0}$ on $\partial\Omega$ implies that the surface gradient of v_H on $\partial\Omega$ is zero, therefore v_H is constant on $\partial\Omega$. By choosing v_H to be zero on $\partial\Omega$ we can uniquely determine v_H satisfying $\mathbf{H}|_{\Omega_d} = \nabla v_H$. The linear map $\mathbf{H} \mapsto (\mathbf{H}|_{\Omega_s}|\nabla v_H)$ from $V(\Omega)$ to $W(\Omega)$ is thus well-defined and gives the desired isomorphism. □

This proposition allows us to reform the problem ($\mathbf{P}^B\mathbf{1}$) in a problem where the curl free constraint imposed on test functions is eliminated. Define a convex set $R \subset W(\Omega)$ by

$$R := \{(\phi|\nabla v) \in W(\Omega) \mid |\mathbf{curl}\,\phi| \leqslant \mathscr{J}_c \text{ a.e. in } \Omega_s\}.$$

The hybrid problem ($\mathbf{P}^B\mathbf{2}$) is proposed as follows.

($\mathbf{P}^B\mathbf{2}$) Find $\psi : [0,T] \to H(\mathbf{curl};\Omega_s)$ and $u : [0,T] \to H^1(\Omega_d)$ such that $(\psi|\nabla u) \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$, $(\psi|\nabla u)(t) \in R$ for all $t \in [0,T]$,

$$\int_{\Omega_s} \mu_s \langle \partial_t \psi(\mathbf{x},t) + \partial_t \mathbf{H}_s(\mathbf{x},t), \phi(\mathbf{x}) - \psi(\mathbf{x},t)\rangle d\mathbf{x} + \int_{\Omega_d} \mu_d \langle \partial_t \nabla u(\mathbf{x},t) + \partial_t \mathbf{H}_s(\mathbf{x},t), \nabla v(\mathbf{x}) - \nabla u(\mathbf{x},t)\rangle d\mathbf{x} \geqslant 0 \text{ for a.e. } t \in (0,T] \tag{2.65}$$

holds for all $(\phi|\nabla v) \in R$ and $(\psi|\nabla u)(\mathbf{x},0) = \mathbf{0}$ in Ω.

By the equivalence between $V(\Omega)$ and $W(\Omega)$ and proposition 2.4, the unique existence of the solution $(\psi|\nabla u)$ of ($\mathbf{P}^B\mathbf{2}$) satisfying that $(\psi|\nabla u) : [0,T] \to L^2(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and $(\psi|\nabla u)(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$ is immediately proved. It is also possible to rewrite the problems ($\mathbf{P}_\varepsilon^{mB}\mathbf{1}$) and ($\mathbf{P}_p^P\mathbf{1}$) in the hybrid problems with the magnetic scalar potential.

3. Discretization

In this section we discretize our variational inequality formulations ($\mathbf{P}_\varepsilon^{mB}\mathbf{1}$) and ($\mathbf{P}_p^P\mathbf{1}$) to construct discrete solutions converging to the analytical solutions of ($\mathbf{P}^B\mathbf{1}$) and ($\mathbf{P}^B\mathbf{2}$). Let us precisely set the geometry. The domain Ω ($\subset \mathbb{R}^3$) is a convex polyhedron. The bulk type-II superconductor Ω_s ($\subset \Omega$) is a simply connected polyhedral domain with a connected boundary $\partial\Omega_s$ satisfying $\partial\Omega_s \cap \partial\Omega = \emptyset$. Moreover, we assume that the domain Ω_s is starshape for a point $y_0 \in \Omega_s$ in the sense that

$$\text{for any } z \in \overline{\Omega_s},\ \alpha(z - y_0) + y_0 \in \Omega_s\ (\forall \alpha \in [0,1)). \tag{3.1}$$

Let Ω_d denote the nonconductive region $\Omega \backslash \overline{\Omega_s}$. Note that in this situation Ω_d ($= \Omega \backslash \overline{\Omega_s}$) is simply connected, and Ω and Ω_s can be meshed by tetrahedra (see Figure 2).

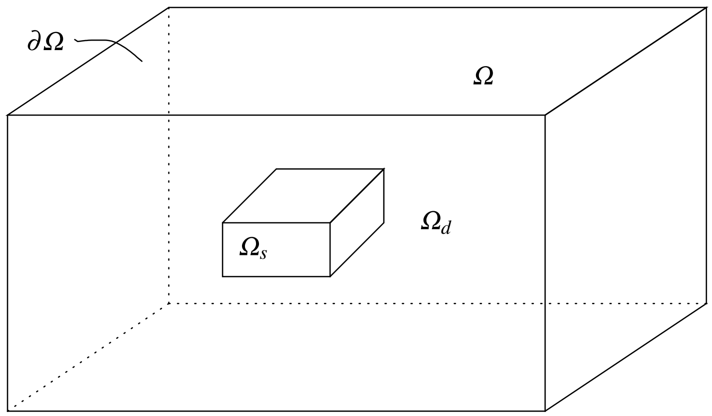

FIG.2. Domain of the problem

3.1 *Finite element approximation*

Let τ_h be tetrahedral mesh covering Ω, satisfying $h = \max\{h_K \mid K \in \tau_h\}$, where h_K is the diameter of the smallest sphere containing $\overline{K}$. The mesh τ_h is assumed to be regular in the sense that there are constants $C > 0$ and $h_0 > 0$ such that

$$h_K/\rho_K \leqslant C \text{ for } \forall K \in \tau_h, 0 < \forall h \leqslant h_0, \tag{3.2}$$

where ρ_K is the diameter of the largest sphere contained in $\overline{K}$. Moreover the mesh τ_h is quasiuniform on $\partial\Omega$ in the sense that there is a constant $C' > 0$ such that

$$h/h_f \leqslant C' \text{ for any face } f \subset \partial\Omega \text{ and } 0 < \forall h \leqslant h_0, \tag{3.3}$$

where h_f is the diameter of the smallest circle containing f (see Monk (2003)). We assume that each element $K \in \tau_h$ belongs either to $\overline{\Omega_s}$ or to $\overline{\Omega_d}$.

Set the space R_1 of vector polynomials of degree 1 by

$$R_1 := \{\mathbf{a} + \mathbf{b} \times \mathbf{x} \mid \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}.$$

The curl conforming finite element space $U_h(\Omega)$ by Nédélec (1980) of the lowest order on tetrahedra mesh is defined by

$$U_h(\Omega) := \{\phi_h \in H(\mathbf{curl}; \Omega) \mid \phi_h|_K \in R_1 \text{ for } \forall K \in \tau_h\},$$

with the degree of freedom

$$M_e(\phi_h) := \int_e \langle \phi_h, \tau \rangle ds,$$

where e is an edge of $K \in \tau_h$ and τ is a unit tangent to e. The interpolation $r_h(\phi) \in U_h(\Omega)$ of a sufficiently smooth function ϕ is defined by $M_e(\phi - r_h(\phi)) = 0$ for all edges e. For more details of the edge element see Girault & Raviart (1986) or Monk (2003). To make the argument clear let us state one lemma proved in (Girault & Raviart, 1986, Chapter III, Lemma 5.7), (Monk, 2003, Lemma 5.35).

LEMMA 3.1 For $\phi_h \in U_h(\Omega)$ and a face $f \subset K$ $(K \in \tau_h)$, the tangential component of ϕ_h on f is zero if and only if $M_{e_i}(\phi_h) = 0$ $(i = 1, 2, 3)$, where e_i $(i = 1, 2, 3)$ are the edges of f.

We define the finite dimensional subspace $V_h(\Omega)$ of $V(\Omega)$ by

$$V_h(\Omega) := \{\phi_h \in U_h(\Omega) \mid \mathbf{curl}\,\phi_h = \mathbf{0} \text{ in } \Omega_d,\ \mathbf{n}\times\phi_h = \mathbf{0} \text{ on } \partial\Omega\}.$$

Note that the boundary condition $\mathbf{n}\times\phi_h = \mathbf{0}$ on $\partial\Omega$ is attained by taking all the degrees of freedom associated with the edges on $\partial\Omega$ to be zero by lemma 3.1.

To define a discrete space satisfying discrete divergence free condition and a discrete subspace of the space $W(\Omega)$ we need to use the standard H^1 conforming finite element space $Z_h(\Omega)$ of the lowest order on tetrahedra mesh.

$$Z_h(\Omega) := \{f_h \in H^1(\Omega) \mid f_h|_K \in P_1 \text{ for } \forall K \in \tau_h\},$$

where $P_1 := \{a_0 + a_1x + a_2y + a_3z \mid a_i \in I\!R, i = 0\text{-}3\}$. The degrees of freedom $m_v(f_h)$ of $Z_h(\Omega)$ is defined by

$$m_v(f_h) := f_h(\mathbf{x}_v),$$

where $\mathbf{x}_v$ $(\in I\!R^3)$ is the coordinate of the vertex v. Similarly let us define the finite element space $Z_{0,h}(\Omega)$ by

$$Z_{0,h}(\Omega) := \{f_h \in Z_h(\Omega) \mid f_h|_{\partial\Omega} = 0\}.$$

The boundary condition $f_h|_{\partial\Omega} = 0$ is attained by taking $m_v(f_h)$ for each vertex v on $\partial\Omega$ to be zero.

The space of discrete divergence free functions $X_h^{(\mu)}(\Omega)$ is defined by

$$X_h^{(\mu)}(\Omega) := \{\phi_h \in U_h(\Omega) \mid \langle \mu\phi_h, \nabla f_h\rangle_{L^2(\Omega;I\!R^3)} = 0 \text{ for } \forall f_h \in Z_{0,h}(\Omega)\}.$$

The discrete subspace $W_h(\Omega)$ of $W(\Omega)$ is defined by

$$\begin{aligned} W_h(\Omega) := \{(\phi_h|\nabla u_h) \in L^2(\Omega;I\!R^3) \mid\ & (\phi_h, u_h) \in U_h(\Omega_s)\times Z_h(\Omega_d),\\ & (\phi_h|\nabla u_h) \in U_h(\Omega),\ u_h|_{\partial\Omega} = 0\}, \end{aligned}$$

where $U_h(\Omega_s) := \{\phi_h|_{\Omega_s} \mid \phi_h \in U_h(\Omega)\}$ and $Z_h(\Omega_d) := \{u_h|_{\Omega_d} \mid u_h \in Z_h(\Omega)\}$.

The following proposition is the discrete analogue of proposition 2.7.

PROPOSITION 3.1 The space $W_h(\Omega)$ is isomorphic to $V_h(\Omega)$ as a Hilbert space.

Proof. Take any $\phi_h \in V_h(\Omega)$. Similarly as proposition 2.7 there uniquely exists $v_{\phi_h} \in H^1(\Omega_d)$ such that $\phi_h|_{\Omega_d} = \nabla v_{\phi_h}$ and $v_{\phi_h} = 0$ on $\partial\Omega$. We will show that $v_{\phi_h} \in Z_h(\Omega_d)$.

Take any $K \in \tau_h$ with $K \subset \overline{\Omega_d}$. We can write

$$\phi_h|_K = \mathbf{a} + \mathbf{b}\times\mathbf{x},\ (\mathbf{a} = (a_1,a_2,a_3)^T, \mathbf{b} \in I\!R^3).$$

The condition $\mathbf{curl}\,\phi_h|_K = \mathbf{0}$ and an explicit calculation lead to $\mathbf{b} = \mathbf{0}$. Therefore, we see that $\phi_h|_K = \nabla v_{\phi_h}|_K = \mathbf{a}$, or

$$v_{\phi_h}|_K = \text{constant} + a_1x + a_2y + a_3z \in R_1,$$

which means $v_{\phi_h} \in Z_h(\Omega_d)$.

Thus the linear map $\phi_h \mapsto (\phi_h|_{\Omega_s}|\nabla v_{\phi_h})$ from $V_h(\Omega)$ to $W_h(\Omega)$ is well-defined. This map gives the isomorphism. □

Let Λ denote a bounded subset of $I\!R_{>0}$ which has the only accumulation point 0. Our assumptions on μ, Ω, τ_h enable us to apply the following discrete compactness result proved in (Monk, 2003, Chapter 7). Especially, the quasiuniform property (3.3) of τ_h on $\partial\Omega$ is assumed only to apply this lemma.

LEMMA 3.1 Let $\{\phi_h\}_{h\in\Lambda}$ satisfy $\phi_h \in X_h^{(\mu)}(\Omega)$ for all $h \in \Lambda$. The following statements hold.

(i) If $\|\phi_h\|_{H(\mathbf{curl};\Omega)} \leqslant C$ for all $h \in \Lambda$, there exist a subsequence $\{\phi_{h_n}\}_{n=1}^{\infty} \subset \{\phi_h\}_{h\in\Lambda}$ and $\phi \in X^{(\mu)}(\Omega)$ such that as $n \to +\infty$

$$\phi_{h_n} \to \phi \text{ strongly in } L^2(\Omega;\mathbb{R}^3),$$
$$\phi_{h_n} \rightharpoonup \phi \text{ weakly in } H(\mathbf{curl};\Omega).$$

(ii) There is a constant $\hat{C} > 0$ such that for any $h \in \Lambda$,

$$\|\phi_h\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant \hat{C}(\|\mathbf{curl}\,\phi_h\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{n}\times\phi_h\|_{L^2(\partial\Omega;\mathbb{R}^3)}).$$

The following lemma is from (Girault & Raviart, 1986, Chapter III, Theorem 5.4), (Monk, 2003, Theorem 5.41).

LEMMA 3.2 There is a constant $C > 0$ such that

$$\|\phi - r_h(\phi)\|_{H(\mathbf{curl};\Omega)} \leqslant Ch\|\phi\|_{H^1(\mathbf{curl};\Omega)},$$

for any $\phi \in H^1(\mathbf{curl};\Omega)$.

By the similar argument as (Girault & Raviart, 1986, Chapter III, Theorem 5.4), (Monk, 2003, Theorem 5.41) we can prove the following estimates.

LEMMA 3.3 There exists a constant $C > 0$ depending only on the constant appearing in (3.2) such that

$$\|\phi - r_h(\phi)\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqslant Ch\|\nabla\phi\|_{L^\infty(\Omega;\mathbb{R}^9)},$$
$$\|\mathbf{curl}\,\phi - \mathbf{curl}\,r_h(\phi)\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqslant Ch\|\nabla\,\mathbf{curl}\,\phi\|_{L^\infty(\Omega;\mathbb{R}^9)},$$

for any $\phi \in C^2(\overline{\Omega};\mathbb{R}^3)$.

We need one more lemma where the assumption (3.1) is used. Let $W^{p,q}(\Omega;\mathbb{R}^3)$ ($p \in \mathbb{N}\cup\{0\}$, $1 \leqslant q \leqslant +\infty$) denote a Sobolev space defined as usual.

LEMMA 3.4 For any $\phi \in C([0,T];L^2(\Omega;\mathbb{R}^3))$ with $\phi(t) \in S$ for all $t \in [0,T]$ there exists a sequence $\{\phi_l\}_{l=1}^{\infty} \subset C([0,T];W^{p,q}(\Omega;\mathbb{R}^3))$ for all $p \in \mathbb{N}\cup\{0\}$ and $1 \leqslant q \leqslant +\infty$ with $\phi_l(t) \in S \cap C_0^\infty(\Omega;\mathbb{R}^3)$ for all $t \in [0,T]$ such that

$$\phi_l \to \phi \text{ strongly in } L^2(0,T;H(\mathbf{curl};\Omega)) \tag{3.4}$$

as $l \to +\infty$.

Proof. Take any $\phi \in C([0,T];L^2(\Omega;\mathbb{R}^3))$ with $\phi(t) \in S$ for all $t \in [0,T]$. Fix any $t \in [0,T]$. Noting $\mathbf{n}\times\phi = \mathbf{0}$ on $\partial\Omega$, define $\widetilde{\phi}(t) \in H(\mathbf{curl};\mathbb{R}^3)$ by

$$\widetilde{\phi}(t) := \begin{cases} \phi(t) & \text{in } \Omega, \\ \mathbf{0} & \text{in } \mathbb{R}^3\backslash\Omega. \end{cases}$$

For $\theta \in (0,1)$ set $\widetilde{\phi}_\theta(t) \in H(\mathbf{curl};\mathbb{R}^3)$ by

$$\widetilde{\phi}_\theta(\mathbf{x},t) := \theta\widetilde{\phi}\left(\frac{\mathbf{x}-\mathbf{y}_0}{\theta} + \mathbf{y}_0, t\right),$$

where $\mathbf{y}_0 \in \Omega_s$ is the point appearing in the assumption (3.1). Then we see that $\mathrm{supp}(\mathbf{curl}\,\widetilde{\phi}_\theta(t)) \subset \Omega_s$. Indeed, if $\mathrm{supp}(\mathbf{curl}\,\widetilde{\phi}_\theta(t)) \neq \emptyset$, for any $\hat{\mathbf{x}} \in \mathrm{supp}(\mathbf{curl}\,\widetilde{\phi}_\theta(t))$ there is a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset \mathbb{R}^3$ such that $\mathbf{x}_n \to \hat{\mathbf{x}}$ as $n \to +\infty$ and $\mathbf{curl}\,\widetilde{\phi}_\theta(\mathbf{x}_n, t) \neq \mathbf{0}$. By the definition of $\widetilde{\phi}_\theta$ we obtain

$$\frac{\mathbf{x}_n - \mathbf{y}_0}{\theta} + \mathbf{y}_0 \in \Omega_s.$$

By sending $n \to +\infty$, we have

$$\frac{\hat{\mathbf{x}} - \mathbf{y}_0}{\theta} + \mathbf{y}_0 \in \overline{\Omega_s}.$$

The assumption (3.1) yields

$$\hat{\mathbf{x}} = \theta\left(\frac{\hat{\mathbf{x}} - \mathbf{y}_0}{\theta} + \mathbf{y}_0 - \mathbf{y}_0\right) + \mathbf{y}_0 \in \Omega_s.$$

Since Ω is convex we can similarly show $\mathrm{supp}(\widetilde{\phi}_\theta(t)) \subset \Omega$, which implies $\mathbf{n} \times \widetilde{\phi}_\theta(t) = \mathbf{0}$ on $\partial\Omega$. Moreover the inequality $|\mathbf{curl}\,\widetilde{\phi}_\theta(\mathbf{x},t)| \leqslant \mathscr{J}_c$ holds a.e. in Ω_s.

For any $\theta \in (0,1)$ we can choose $\varepsilon = \varepsilon(\theta) > 0$ sufficiently small so that we can have $\rho_\varepsilon * \widetilde{\phi}_\theta(t)|_\Omega \in S \cap C_0^\infty(\Omega;\mathbb{R}^3)$, where $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^3)$ is the mollifier. By the standard properties of the mollifier it is seen that $\rho_\varepsilon * \widetilde{\phi}_\theta|_\Omega \to \phi$ strongly in $L^2(0,T;H(\mathbf{curl};\Omega))$ as $\theta \nearrow 1$, $\varepsilon(\theta) \searrow 0$.

For any multi-index $\alpha \in (\mathbb{N} \cup \{0\})^3$

$$|\partial \mathbf{x}^\alpha(\rho_\varepsilon * \widetilde{\phi}_\theta)(\mathbf{x},t)| \leqslant C(\varepsilon,\alpha)\|\widetilde{\phi}_\theta(t)\|_{L^2(\Omega;\mathbb{R}^3)} = C(\varepsilon,\alpha)\theta^{5/2}\|\phi(t)\|_{L^2(\Omega;\mathbb{R}^3)},$$

which implies that $\rho_\varepsilon * \widetilde{\phi}_\theta|_\Omega \in C([0,T];W^{p,q}(\Omega;\mathbb{R}^3))$ for all $p \in \mathbb{N} \cup \{0\}$ and $1 \leqslant q \leqslant +\infty$. □

Take $N \in \mathbb{N}$ and set $\Delta t := T/N$. By using a function ϕ_l proved in lemma 3.4, we define a piecewise constant in time function $\overline{\phi_{l,h}} : [0,T] \to V_h(\Omega)$ by

$$\overline{\phi_{l,h}}(t) := \begin{cases} r_h(\phi_l(\Delta t i)) & \text{in } (\Delta t(i-1), \Delta t i],\ (i = 1,\cdots,N), \\ r_h(\phi_l(0)) & \text{on } \{t = 0\}. \end{cases}$$

The following properties will be useful in section 4.

COROLLARY 3.1 There is a constant $C > 0$ independent of $l \in \mathbb{N}$, $h \in \Lambda$ and Δt such that

$$\|\mathbf{curl}\,\overline{\phi_{l,h}}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} \leqslant Ch\|\nabla\,\mathbf{curl}\,\phi_l\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))} + \mathscr{J}_c. \tag{3.5}$$

Moreover assume that the time step size Δt depends on h and satisfies $\lim_{h \searrow 0, h \in \Lambda} \Delta t(h) = 0$. Then the following convergences hold as $h \searrow 0$.

$$\overline{\phi_{l,h}} \to \phi_l \text{ strongly in } L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3)), \tag{3.6}$$

$$\mathbf{curl}\,\overline{\phi_{l,h}} \to \mathbf{curl}\,\phi_l \text{ strongly in } L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3)). \tag{3.7}$$

Proof. These statements can be proved by noting lemma 3.3 and lemma 3.4. We only give a proof for (3.5).

$$\begin{aligned}\|\mathbf{curl}\,\overline{\phi_{l,h}}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} &\leqslant \|\mathbf{curl}\,\overline{\phi_{l,h}} - \mathbf{curl}\,\overline{\phi_l}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} + \|\mathbf{curl}\,\overline{\phi_l}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} \\ &\leqslant Ch\|\nabla\,\mathbf{curl}\,\phi_l\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))} + \mathscr{J}_c.\end{aligned}$$

□

3.2 *Full discretization of the evolution problem*

Now we are going to discretize the problems in time implicitly and in space by the finite element introduced in section 3.1.

We need to prepare a few more notations. Let $\mathbf{H}_{s,n}$ denote $\mathbf{H}_s(\Delta tn)$ and $\mathbf{H}_{s,h,n}$ denote $r_h(\mathbf{H}_{s,n})$ for $n = 0, 1, \cdots, N(= T/\Delta t)$. Note that since $\mathbf{H}_{s,n} \in H^1(\mathbf{curl}; \Omega)$ by lemma 2.1 the interpolation is well-defined. Define the functional $F_{h,n,\varepsilon}$ $(n = 1, \cdots, N)$ on the full discrete space $U_h(\Omega)$ by

$$
\begin{aligned}
&F_{h,n,\varepsilon}(\phi_h) \\
&:= \frac{1}{2\Delta t}\int_\Omega \mu|\phi_h|^2 d\mathbf{x} + \frac{1}{\Delta t}\int_\Omega \mu\langle -\widehat{\mathbf{H}}_{h,n-1,\varepsilon} + \mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1}, \phi_h\rangle d\mathbf{x} + \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\phi_h|)d\mathbf{x},
\end{aligned}
$$

where $\widehat{\mathbf{H}}_{h,0,\varepsilon} = r_h(\widehat{\mathbf{H}}_0) = \mathbf{0}$.

We consider the following optimization problems in the finite dimensional space.

$(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1})$ For $n = 1 \to N$, find $\widehat{\mathbf{H}}_{h,n,\varepsilon} \in V_h(\Omega)$ such that

$$
F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon}) = \min_{\phi_h \in V_h(\Omega)} F_{h,n,\varepsilon}(\phi_h),
$$

where $\widehat{\mathbf{H}}_{h,0,\varepsilon} = \mathbf{0}$.

Equivalently,

$(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ For $n = 1 \to N$, find $(\psi_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon}) \in W_h(\Omega)$ such that

$$
F_{h,n,\varepsilon}((\psi_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon})) = \min_{(\phi_h|\nabla v_h) \in W_h(\Omega)} F_{h,n,\varepsilon}((\phi_h|\nabla v_h)),
$$

where $(\psi_{h,0,\varepsilon}|\nabla u_{h,0,\varepsilon}) = \mathbf{0}$.

PROPOSITION 3.2 There uniquely exists the minimizer $\widehat{\mathbf{H}}_{h,n,\varepsilon} \in V_h(\Omega)$ of $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1})$. Moreover $\widehat{\mathbf{H}}_{h,n,\varepsilon} \in V_h(\Omega)$ satisfies the discrete divergence free condition

$$
\widehat{\mathbf{H}}_{h,n,\varepsilon} + \mathbf{H}_{s,h,n} \in X_h^{(\mu)}(\Omega) \tag{3.8}
$$

and the discrete variational inequality

$$
\begin{aligned}
&\int_\Omega \mu\langle(\widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon} + \mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1})/\Delta t, \phi_h - \widehat{\mathbf{H}}_{h,n,\varepsilon}\rangle d\mathbf{x} \\
&\qquad + \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\phi_h|)d\mathbf{x} - \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,n,\varepsilon}|)d\mathbf{x} \geqslant 0
\end{aligned} \tag{3.9}
$$

for all $\phi_h \in V_h(\Omega)$.

Proof. The unique existence of the minimizer $\widehat{\mathbf{H}}_{h,n,\varepsilon} \in V_h(\Omega)$ is standard. We show (3.8).

Since $\nabla Z_{0,h}(\Omega) \subset V_h(\Omega)$ (see Monk (2003)), for any $w_h \in Z_{0,h}(\Omega)$ and $\delta > 0$, $\widehat{\mathbf{H}}_{h,n,\varepsilon} + \delta\nabla w_h \in V_h(\Omega)$ and

$$
\begin{aligned}
&\lim_{\delta\downarrow 0}\{(F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon} + \delta\nabla w_h) - F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon}))/\delta\} \\
&\quad = \frac{1}{\Delta t}\int_\Omega \mu\langle\widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon} + \mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1}, \nabla w_h\rangle d\mathbf{x} \\
&\quad = \frac{1}{\Delta t}\int_\Omega \mu\langle\widehat{\mathbf{H}}_{h,n,\varepsilon} + \mathbf{H}_{s,h,n}, \nabla w_h\rangle d\mathbf{x} \geqslant 0.
\end{aligned} \tag{3.10}
$$

Here we have used the assumption $\widehat{\mathbf{H}}_{h,n-1,\varepsilon}+\mathbf{H}_{s,h,n-1}\in X_h^{(\mu)}(\Omega)$. Similarly by calculating $\lim_{\delta\downarrow 0}\{(F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon}-\delta\nabla w_h)-F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon}))/\delta\}$ we have

$$\frac{1}{\Delta t}\int_\Omega \mu\langle \widehat{\mathbf{H}}_{h,n,\varepsilon}+\mathbf{H}_{s,h,n},\nabla w_h\rangle d\mathbf{x}\leqslant 0. \tag{3.11}$$

Combining (3.10) with (3.11) we obtain (3.8).

We derive (3.9). The inequality $F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon})\leqslant F_{h,n,\varepsilon}(\phi_h)$ is equivalent to the inequality

$$\begin{aligned}\frac{1}{2\Delta t}\int_\Omega \mu|\phi-\widehat{\mathbf{H}}_{h,n,\varepsilon}|^2 d\mathbf{x}+\frac{1}{\Delta t}\int_\Omega \mu\langle \widehat{\mathbf{H}}_{h,n,\varepsilon}-\widehat{\mathbf{H}}_{h,n-1,\varepsilon}+\mathbf{H}_{s,h,n}-\mathbf{H}_{s,h,n-1},\phi_h-\widehat{\mathbf{H}}_{h,n,\varepsilon}\rangle d\mathbf{x}\\ +\frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\phi_h|)d\mathbf{x}-\frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,n,\varepsilon}|)d\mathbf{x}\geqslant 0.\end{aligned} \tag{3.12}$$

Take any $\psi_h\in V_h(\Omega)$ and $\alpha\in(0,1)$. Substituting $\phi_h=\alpha\psi_h+(1-\alpha)\widehat{\mathbf{H}}_{h,n,\varepsilon}\in V_h(\Omega)$ into (3.12), dividing the both side by α and sending $\alpha\downarrow 0$, we obtain the inequality (3.9). □

By proposition 3.1 we immediately see the following statement.

COROLLARY 3.2 There uniquely exists the minimizer $(\psi_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon})\in W_h(\Omega)$ of $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$. Moreover $(\psi_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon})+\mathbf{H}_{s,h,n}\in X_h^{(\mu)}(\Omega)$ and the inequality (3.9) for $\widehat{\mathbf{H}}_{h,n,\varepsilon}=(\psi_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon})$ hold.

Similarly we define the functional $G_{h,n,p}$ on $U_h(\Omega)$ by

$$\begin{aligned}&G_{h,n,p}(\phi_h)\\ &:=\frac{1}{2\Delta t}\int_\Omega \mu|\phi_h|^2 d\mathbf{x}+\frac{1}{\Delta t}\int_\Omega \mu\langle -\widehat{\mathbf{H}}_{h,n-1,p}+\mathbf{H}_{s,h,n}-\mathbf{H}_{s,h,n-1},\phi_h\rangle d\mathbf{x}+\frac{\mathscr{J}_c}{p}\int_{\Omega_s}|\mathbf{curl}\,\phi_h/\mathscr{J}_c|^p d\mathbf{x},\end{aligned}$$

where $\widehat{\mathbf{H}}_{h,0,p}=r_h(\widehat{\mathbf{H}}_0)=\mathbf{0}$.

The full discrete formulation of $(\mathbf{P}^P_p\mathbf{1})$ is proposed as

$(\mathbf{P}^P_{h,\Delta t,p}\mathbf{1})$ For $n=1\to N$, find $\widehat{\mathbf{H}}_{h,n,p}\in V_h(\Omega)$ such that

$$G_{h,n,p}(\widehat{\mathbf{H}}_{h,n,p})=\min_{\phi_h\in V_h(\Omega)} G_{h,n,p}(\phi_h),$$

where $\widehat{\mathbf{H}}_{h,0,p}=\mathbf{0}$.

Equivalently we can propose the full discretization of $(\mathbf{P}^P_p\mathbf{2})$ as

$(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$ For $n=1\to N$, find $(\psi_{h,n,p}|\nabla u_{h,n,p})\in W_h(\Omega)$ such that

$$G_{h,n,p}((\psi_{h,n,p}|\nabla u_{h,n,p}))=\min_{(\phi_h|\nabla v_h)\in W_h(\Omega)} G_{h,n,p}((\phi_h|\nabla v_h)),$$

where $(\psi_{h,0,p}|\nabla u_{h,0,p})=\mathbf{0}$.

The unique existence of the minimizers of the problems $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{1})$ and $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$ can be stated in the same way as proposition 3.2 and corollary 3.2. Note that the hybrid problems $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ and $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$ are rather useful for practical computation since the curl free constraint is automatically fulfilled by the scalar potential.

4. Convergence of discrete solutions

In this section we will show the convergence of the discrete solutions constructed by using the minimizers of the optimization problems proposed in the previous section to the unique solution of the evolution variational inequality formulation.

4.1 *Convergence of the discrete solutions solving* ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1}$), ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$)

We will show that the discrete solutions made of the minimizers of ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1}$) and ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$) converge to the solution of ($\mathbf{P}^{B}\mathbf{1}$) and ($\mathbf{P}^{B}\mathbf{2}$) respectively. We define the piecewise linear in time functions $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}$, $\widehat{\mathbf{H}}_{s,h,\Delta t}$, and the piecewise constant in time functions $\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}$, $\overline{\mathbf{H}}_{s,h,\Delta t}$ by

$$
\begin{aligned}
\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) &:= \frac{t-\Delta t(n-1)}{\Delta t}\widehat{\mathbf{H}}_{h,n,\varepsilon} + \frac{\Delta tn - t}{\Delta t}\widehat{\mathbf{H}}_{h,n-1,\varepsilon} \text{ in } [\Delta t(n-1), \Delta tn],\\
\widehat{\mathbf{H}}_{s,h,\Delta t}(t) &:= \frac{t-\Delta t(n-1)}{\Delta t}\mathbf{H}_{s,h,n} + \frac{\Delta tn - t}{\Delta t}\mathbf{H}_{s,h,n-1} \text{ in } [\Delta t(n-1), \Delta tn],\\
\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) &:= \begin{cases} \widehat{\mathbf{H}}_{h,n,\varepsilon} & \text{in } (\Delta t(n-1), \Delta tn],\\ \widehat{\mathbf{H}}_{h,0,\varepsilon} & \text{on } \{t=0\}, \end{cases}\\
\overline{\mathbf{H}}_{s,h,\Delta t}(t) &:= \begin{cases} \mathbf{H}_{s,h,n} & \text{in } (\Delta t(n-1), \Delta tn],\\ \mathbf{H}_{s,h,0} & \text{on } \{t=0\}, \end{cases}
\end{aligned}
$$

for $n = 1, \cdots, N$, where $\widehat{\mathbf{H}}_{h,n,\varepsilon}$ is the minimizers of ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1}$) and $\widehat{\mathbf{H}}_{h,0,\varepsilon} = \mathbf{0}$.

By definition we easily see that $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}, \widehat{\mathbf{H}}_{s,h,\Delta t} \in C([0,T]; H(\mathbf{curl};\Omega))$, $\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}, \overline{\mathbf{H}}_{s,h,\Delta t} \in L^\infty(0,T;H(\mathbf{curl};\Omega))$, and $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t), \overline{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) \in V_h(\Omega)$ for all $t \in [0,T]$. The discrete analogue of (2.19) holds in the sense that $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) + \widehat{\mathbf{H}}_{s,h,\Delta t}(t)$, $\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) + \overline{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ for all $t \in [0,T]$ by (3.8).

LEMMA 4.1 The following estimates hold.

$$
\|\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon} - \overline{\mathbf{H}}_{h,\Delta t,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \leqslant \Delta t \|\partial_t \widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}, \tag{4.1}
$$

$$
\|\partial_t \widehat{\mathbf{H}}_{s,h,\Delta t}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leqslant Ch\|\partial_t \mathbf{h}_s\|_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} + C\|\mathbf{n}\times\partial_t \mathbf{h}_s\|_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}. \tag{4.2}
$$

The following convergences also hold as $h \searrow 0$ and $\Delta t \searrow 0$.

$$
\widehat{\mathbf{H}}_{s,h,\Delta t} \to \mathbf{H}_s \text{ strongly in } C([0,T];L^2(\Omega;\mathbb{R}^3)), \tag{4.3}
$$

$$
\overline{\mathbf{H}}_{s,h,\Delta t} \to \mathbf{H}_s \text{ strongly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.4}
$$

$$
\partial_t \widehat{\mathbf{H}}_{s,h,\Delta t} \to \partial_t \mathbf{H}_s \text{ strongly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{4.5}
$$

Proof. To show (4.1), (4.3)-(4.5) is standard. We only give a proof for (4.2). By using lemma 3.2, we observe that

$$
\begin{aligned}
\|\partial_t \widehat{\mathbf{H}}_{s,h,\Delta t}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} &\leqslant \sum_{i=1}^{N} \frac{1}{\Delta t} \|\mathbf{H}_{s,h,i} - \mathbf{H}_{s,h,i-1}\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\leqslant 2\sum_{i=1}^{N} \frac{1}{\Delta t} \|r_h(\mathbf{H}_{s,i} - \mathbf{H}_{s,i-1}) - \mathbf{H}_{s,i} + \mathbf{H}_{s,i-1}\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2\sum_{i=1}^{N} \frac{1}{\Delta t} \|\mathbf{H}_{s,i} - \mathbf{H}_{s,i-1}\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\leqslant Ch^2 \sum_{i=1}^{N} \frac{1}{\Delta t} \|\mathbf{H}_{s,i} - \mathbf{H}_{s,i-1}\|^2_{H^1(\Omega;\mathbb{R}^3)} + 2\sum_{i=1}^{N} \frac{1}{\Delta t} \|\mathbf{H}_{s,i} - \mathbf{H}_{s,i-1}\|^2_{L^2(\Omega;\mathbb{R}^3)} \\
&\leqslant Ch^2 \sum_{i=1}^{N} \int_{\Delta t(i-1)}^{\Delta t i} \|\partial_t \mathbf{H}_s(t)\|^2_{H^1(\Omega;\mathbb{R}^3)} dt + 2\sum_{i=1}^{N} \int_{\Delta t(i-1)}^{\Delta t i} \|\partial_t \mathbf{H}_s(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} dt \\
&\leqslant Ch^2 \|\partial_t \mathbf{H}_s\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^3))} + 2\|\partial_t \mathbf{H}_s\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}.
\end{aligned}
$$

By combining this inequality with (2.39) and the Friedrichs inequality

$$
\|\partial_t \mathbf{H}_s\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leqslant C\|\mathbf{n} \times \partial_t \mathbf{h}_s\|_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))},
$$

we obtain (4.2). □

Moreover, we observe

PROPOSITION 4.1 Take any $\tau \in (0,1)$. The following bounds hold. For any $h \in \Lambda$, $\varepsilon > 0$, $\Delta t \in (0,\tau]$,

$$
\|\partial_t \widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leqslant C \frac{\max\{\mu_d,\mu_s\}}{\min\{\mu_d,\mu_s\}} (h^2 \|\partial_t \mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} + \|\mathbf{n} \times \partial_t \mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}), \tag{4.6}
$$

$$
\begin{aligned}
\operatorname*{esssup}_{t\in[0,T]} \left\{ \int_{\Omega_s} g(|\mathbf{curl}\, \overline{\mathbf{H}}_{h,\Delta t,\varepsilon}(t)|) d\mathbf{x} \right\} \leqslant & C\varepsilon \max\{\mu_d,\mu_s\} (h^2 \|\partial_t \mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} \\
& + \|\mathbf{n} \times \partial_t \mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}),
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
\|\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \leqslant \frac{C}{1-\tau} e^{T/(1-\tau)} \frac{\max\{\mu_d,\mu_s\}}{\min\{\mu_d,\mu_s\}} & (h^2 \|\partial_t \mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} \\
& + \|\mathbf{n} \times \partial_t \mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}),
\end{aligned} \tag{4.8}
$$

where $C > 0$ is a positive constant independent of $h, \varepsilon, \Delta t, \mu$.

Proof. By substituting $\phi_h = \widehat{\mathbf{H}}_{h,n-1,\varepsilon}$ into (3.9) we have

$$
\begin{aligned}
\Delta t \int_\Omega \mu |(\widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon})/\Delta t|^2 d\mathbf{x} &+ \frac{1}{\varepsilon} \int_{\Omega_s} g(|\mathbf{curl}\, \widehat{\mathbf{H}}_{h,n,\varepsilon}|) d\mathbf{x} - \frac{1}{\varepsilon} \int_{\Omega_s} g(|\mathbf{curl}\, \widehat{\mathbf{H}}_{h,n-1,\varepsilon}|) d\mathbf{x} \\
&\leqslant \int_\Omega \mu \langle (\mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1})/\Delta t, \widehat{\mathbf{H}}_{h,n-1,\varepsilon} - \widehat{\mathbf{H}}_{h,n,\varepsilon} \rangle d\mathbf{x} \\
&\leqslant \frac{\Delta t}{2} \int_\Omega \mu |(\mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1})/\Delta t|^2 d\mathbf{x} + \frac{\Delta t}{2} \int_\Omega \mu |(\widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon})/\Delta t|^2 d\mathbf{x}.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\frac{\min\{\mu_d,\mu_s\}}{2} \int_{\Delta t(n-1)}^{\Delta t n} \int_\Omega |\partial_t \widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}|^2 d\mathbf{x} dt &+ \frac{1}{\varepsilon} \int_{\Omega_s} g(|\mathbf{curl}\, \widehat{\mathbf{H}}_{h,n,\varepsilon}|) d\mathbf{x} \\
- \frac{1}{\varepsilon} \int_{\Omega_s} g(|\mathbf{curl}\, \widehat{\mathbf{H}}_{h,n-1,\varepsilon}|) d\mathbf{x} &\leqslant \frac{\max\{\mu_d,\mu_s\}}{2} \int_{\Delta t(n-1)}^{\Delta t n} \int_\Omega |\partial_t \widehat{\mathbf{H}}_{s,h,\Delta t}|^2 d\mathbf{x} dt.
\end{aligned} \tag{4.9}
$$

Summing (4.9) over $n = 1 \to m\ (\leqslant N)$ we obtain

$$\begin{aligned}\frac{\min\{\mu_d,\mu_s\}}{2}\int_0^{\Delta tm}\int_\Omega |\partial_t \widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}|^2 d\mathbf{x}dt + \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,m,\varepsilon}|)d\mathbf{x} \\ \leqslant \frac{\max\{\mu_d,\mu_s\}}{2}\int_0^{\Delta tm}\int_\Omega |\partial_t \widehat{\mathbf{H}}_{s,h,\Delta t}|^2 d\mathbf{x}dt.\end{aligned} \tag{4.10}$$

Combining the inequality (4.10) with (4.2) we have (4.6) and (4.7).

On the other hand, substituting $\phi_h = \mathbf{0}$ into (3.9) and noting an equality $\langle \mathbf{p}-\mathbf{q},\mathbf{p}\rangle = |\mathbf{p}-\mathbf{q}|^2/2 + (|\mathbf{p}|^2 - |\mathbf{q}|^2)/2$, we have

$$\begin{aligned}\frac{\Delta t}{2}\int_\Omega \mu|(\widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon})/\Delta t|^2 d\mathbf{x} + \frac{1}{2\Delta t}\int_\Omega \mu|\widehat{\mathbf{H}}_{h,n,\varepsilon}|^2 d\mathbf{x} - \frac{1}{2\Delta t}\int_\Omega \mu|\widehat{\mathbf{H}}_{h,n-1,\varepsilon}|^2 d\mathbf{x} \\ \leqslant \frac{1}{2}\int_\Omega \mu|(\mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1})/\Delta t|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega \mu|\widehat{\mathbf{H}}_{h,n,\varepsilon}|^2 d\mathbf{x}.\end{aligned} \tag{4.11}$$

Multiplying (4.11) by Δt and summing over $n = 1 \to m(\leqslant N)$, we have

$$\int_\Omega \mu|\widehat{\mathbf{H}}_{h,m,\varepsilon}|^2 d\mathbf{x} \leqslant \int_0^{\Delta tm}\int_\Omega \mu|\partial_t \mathbf{H}_{s,h}|^2 d\mathbf{x}dt + \sum_{n=0}^{m}\Delta t\int_\Omega \mu|\widehat{\mathbf{H}}_{h,n,\varepsilon}|^2 d\mathbf{x},$$

which is equivalent to

$$\int_\Omega \mu|\widehat{\mathbf{H}}_{h,m,\varepsilon}|^2 d\mathbf{x} \leqslant \frac{1}{1-\Delta t}\int_0^{\Delta tm}\int_\Omega \mu|\partial_t \mathbf{H}_{s,h}|^2 d\mathbf{x}dt + \sum_{n=0}^{m-1}\frac{\Delta t}{1-\Delta t}\int_\Omega \mu|\widehat{\mathbf{H}}_{h,n,\varepsilon}|^2 d\mathbf{x}. \tag{4.12}$$

By applying the discrete Gronwall's inequality (see, eg, (Thomée, 1997, Lemma 10.5)) to (4.12) and combining (4.2) we obtain (4.8). □

To reduce the parameters, we assume that Δt and ε are positive functions of h satisfying

$$\sup_{h\in\Lambda}\Delta t(h) < 1,\ \lim_{h\searrow 0, h\in\Lambda}\Delta t(h) = \lim_{h\searrow 0, h\in\Lambda}\varepsilon(h) = \lim_{h\searrow 0, h\in\Lambda}\frac{h^{2-\mathrm{sgn}A_4}}{\varepsilon(h)} = 0, \tag{4.13}$$

where $A_4 \geqslant 0$ is a constant in the assumption (2.14) and $\mathrm{sgn}A_4 = 0$ if $A_4 = 0$, 1 if $A_4 > 0$.

We are now ready to state the convergence result.

THEOREM 4.1 The piecewise linear in time approximation $\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)}$ and the piecewise constant in time approximation $\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)}$ converge to the unique solution $\widehat{\mathbf{H}}$ of $(\mathbf{P}^B\mathbf{1})$ in the following sense.

$$\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \to \widehat{\mathbf{H}} \text{ strongly in } C([0,T];L^2(\Omega;\mathbb{R}^3)), \tag{4.14}$$

$$\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \rightharpoonup \widehat{\mathbf{H}} \text{ weakly}* \text{ in } L^\infty(0,T;H(\mathbf{curl};\Omega)), \tag{4.15}$$

$$\partial_t\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \rightharpoonup \partial_t\widehat{\mathbf{H}} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.16}$$

$$\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \to \widehat{\mathbf{H}} \text{ strongly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.17}$$

$$\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \rightharpoonup \widehat{\mathbf{H}} \text{ weakly}* \text{ in } L^\infty(0,T;H(\mathbf{curl};\Omega)), \tag{4.18}$$

as $h \searrow 0$, $h \in \Lambda$.

Proof. To simplify the notation let $\widehat{\mathbf{H}}_h$, $\overline{\mathbf{H}}_h$, $\widehat{\mathbf{H}}_{s,h}$ denote $\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)}$, $\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)}$, $\widehat{\mathbf{H}}_{s,h,\Delta t(h)}$, respectively.

(Step 1) We show that there exist subsequences $\{\widehat{\mathbf{H}}_{h_n}\}_{n=1}^{\infty}$ and $\{\overline{\mathbf{H}}_{h_n}\}_{n=1}^{\infty}$ of $\{\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ and $\{\overline{\mathbf{H}}_h\}_{h\in\Lambda}$ respectively, and $\widehat{\mathbf{H}} \in L^\infty(0,T;H(\mathbf{curl};\Omega))\cap H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ with $\widehat{\mathbf{H}}(t)\in V(\Omega)$ for all $t\in[0,T]$ such that the convergences (4.14)-(4.18) hold for $\widehat{\mathbf{H}}_{h_n}$, $\overline{\mathbf{H}}_{h_n}$ and $\widehat{\mathbf{H}}$ as $n\to+\infty$.

By (4.7) and (4.8) we see that $\{\overline{\mathbf{H}}_h\}_{h\in\Lambda}$ is bounded in $L^\infty(0,T;H(\mathbf{curl};\Omega))$. Thus, so is $\{\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ in $L^\infty(0,T;H(\mathbf{curl};\Omega))$ by definition. Moreover by (4.6) $\{\partial_t\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ is bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$. Therefore by extracting subsequences $\{\widehat{\mathbf{H}}_{h_n}\}_{n=1}^{\infty}$, $\{\overline{\mathbf{H}}_{h_n}\}_{n=1}^{\infty}$ of $\{\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ and $\{\overline{\mathbf{H}}_h\}_{h\in\Lambda}$ respectively we observe the weak($*$) convergences (4.15),(4.16) and (4.18) to some $\widehat{\mathbf{H}}$ ($\in L^\infty(0,T;H(\mathbf{curl};\Omega))\cap H^1(0,T;L^2(\Omega;\mathbb{R}^3))$) with $\widehat{\mathbf{H}}(t)\in V(\Omega)$ a.e. $t\in(0,T)$.

We show the strong convergences (4.14), (4.17). Fix any $t\in[0,T]$. Since $\{\widehat{\mathbf{H}}_{h_n}(t)+\widehat{\mathbf{H}}_{s,h_n}(t)\}_{n=1}^{\infty}$ is bounded in $H(\mathbf{curl};\Omega)$ and $\widehat{\mathbf{H}}_{h_n}(t)+\widehat{\mathbf{H}}_{s,h_n}(t)\in X_{h_n}^{(\mu)}(\Omega)$ for any $n\in\mathbb{N}$, we can apply lemma 3.1 (1) to see that $\{\widehat{\mathbf{H}}_{h_n}(t)+\widehat{\mathbf{H}}_{s,h_n}(t)\}_{n=1}^{\infty}$ contains a subsequence strongly converging in $L^2(\Omega;\mathbb{R}^3)$. This means that $\{\widehat{\mathbf{H}}_{h_n}(t)+\widehat{\mathbf{H}}_{s,h_n}(t)\}_{n=1}^{\infty}$ is relatively compact in $L^2(\Omega;\mathbb{R}^3)$ for any $t\in[0,T]$.

For any $s,t\in[0,T]$ with $s\leqslant t$ we see that by using the inequalities (4.2), (4.6)

$$
\begin{aligned}
\|\widehat{\mathbf{H}}_{h_n}(t)+\widehat{\mathbf{H}}_{s,h_n}(t)-(\widehat{\mathbf{H}}_{h_n}(s)+\widehat{\mathbf{H}}_{s,h_n}(s))\|_{L^2(\Omega;\mathbb{R}^3)} &= \|\int_s^t(\partial_t\widehat{\mathbf{H}}_{h_n}(\tau)+\partial_t\widehat{\mathbf{H}}_{s,h_n}(\tau))d\tau\|_{L^2(\Omega;\mathbb{R}^3)}\\
&\leqslant (\|\partial_t\widehat{\mathbf{H}}_{h_n}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}+\|\partial_t\widehat{\mathbf{H}}_{s,h_n}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))})|t-s|^{1/2}\\
&\leqslant C(\|\partial_t\mathbf{h}_s\|_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))}+\|\mathbf{n}\times\partial_t\mathbf{h}_s\|_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))})|t-s|^{1/2},
\end{aligned}
$$

where $C>0$ is a constant independent of h_n. Therefore $\{\widehat{\mathbf{H}}_{h_n}(t)+\widehat{\mathbf{H}}_{s,h_n}(t)\}_{n=1}^{\infty}$ is equicontinuous. By applying Ascoli-Arzela's theorem for $C([0,T];L^2(\Omega;\mathbb{R}^3))$ we see that there exists $\mathbf{w}\in C([0,T];L^2(\Omega;\mathbb{R}^3))$ such that by choosing a subsequence

$$
\widehat{\mathbf{H}}_{h_n}+\widehat{\mathbf{H}}_{s,h_n}\to\mathbf{w} \text{ strongly in } C([0,T];L^2(\Omega;\mathbb{R}^3))
$$

as $n\to+\infty$. Moreover by noting (4.1), (4.3) and (4.6) we can check that $\mathbf{w}=\widehat{\mathbf{H}}+\mathbf{H}_s$, $\widehat{\mathbf{H}}_{h_n}$ strongly converges to $\widehat{\mathbf{H}}$ in $C([0,T];L^2(\Omega;\mathbb{R}^3))$ and $\overline{\mathbf{H}}_{h_n}$ strongly converges to $\widehat{\mathbf{H}}$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ as $n\to+\infty$.

(Step 2) We will show that the limit $\widehat{\mathbf{H}}$ is the unique solution of ($\mathbf{P}^B\mathbf{1}$). Take any $\phi\in C([0,T];L^2(\Omega;\mathbb{R}^3))$ with $\phi(t)\in S$ for all $t\in[0,T]$. Let $\{\phi_l\}_{l=1}^{\infty}$ be the sequence satisfying the properties stated in lemma 3.4. Define a function $\overline{\phi_{l,h_n}}$ as corollary 3.1. Substituting $\phi_{l,h_n}(\Delta ti)$ into (3.9), multiplying by Δt and summing over $i=1\to N$, we obtain

$$
\begin{aligned}
&\int_0^T\int_\Omega\mu\langle\partial_t\widehat{\mathbf{H}}_{h_n}+\partial_t\widehat{\mathbf{H}}_{s,h_n},\overline{\phi_{l,h_n}}-\overline{\mathbf{H}}_{h_n}\rangle d\mathbf{x}dt\\
&+\frac{1}{\varepsilon}\int_0^T\int_{\Omega_s}g(|\mathbf{curl}\,\overline{\phi_{l,h_n}}|)d\mathbf{x}dt-\frac{1}{\varepsilon}\int_0^T\int_{\Omega_s}g(|\mathbf{curl}\,\overline{\mathbf{H}}_{h_n}|)d\mathbf{x}dt\geqslant 0.
\end{aligned}\tag{4.19}
$$

By the properties (2.14) of g, (3.5) and (4.13) we observe

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_0^T\int_{\Omega_s}g(|\mathbf{curl}\,\overline{\phi_{l,h_n}}|)d\mathbf{x}dt &\leqslant \frac{T|\Omega_s|}{\varepsilon}g(Ch_n\|\nabla\,\mathbf{curl}\,\phi_l\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))}+\mathscr{J}_c)\\
&\leqslant \frac{T|\Omega_s|C_l}{\varepsilon}(A_3h_n^2+A_4h_n)\to 0
\end{aligned}\tag{4.20}
$$

as $n \to +\infty$. Thus, by neglecting the last negative term in the left side of (4.19), passing $n \to +\infty$, and noting the convergences (4.5), (4.16), (4.17), (3.6) and (4.20) we obtain

$$\int_0^T \int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}} + \partial_t \mathbf{H}_s, \phi_l - \widehat{\mathbf{H}} \rangle d\mathbf{x} dt \geqslant 0.$$

By sending $l \to +\infty$ we arrive at

$$\int_0^T \int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}} + \partial_t \mathbf{H}_s, \phi - \widehat{\mathbf{H}} \rangle d\mathbf{x} dt \geqslant 0. \tag{4.21}$$

for all $\phi \in C([0,T]; L^2(\Omega; I\!R^3))$ with $\phi(t) \in S$.

Note that by the weak lower semicontinuity of the functional $\int_{\Omega_s} g(|\mathbf{curl} \cdot|) d\mathbf{x}$ in $H(\mathbf{curl}; \Omega)$ and sending $n \to \infty$ in (4.7) we obtain that

$$\int_{\Omega_s} g(|\mathbf{curl}\, \widehat{\mathbf{H}}(t)|) dx = 0 \text{ for a.e. } t \in [0,T],$$

which implies $\widehat{\mathbf{H}}(t) \in S$ for all $t \in [0,T]$. By taking $v \in C^\infty([0,T])$ with $0 \leqslant v \leqslant 1$ and replacing ϕ by $v\phi + (1-v)\widehat{\mathbf{H}}$ in (4.21), we can derive

$$\int_0^T v \int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}} + \partial_t \mathbf{H}_s, \phi - \widehat{\mathbf{H}} \rangle d\mathbf{x} dt \geqslant 0,$$

which implies that

$$\int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}} + \partial_t \mathbf{H}_s, \phi - \widehat{\mathbf{H}} \rangle d\mathbf{x} \geqslant 0$$

for a.e. $t \in (0,T)$ and any $\phi \in S$. Therefore $\widehat{\mathbf{H}}$ is the solution of ($\mathbf{P}^B\mathbf{1}$) and the unique solvability of ($\mathbf{P}^B\mathbf{1}$) assures the convergences (4.14)-(4.18) without extracting a subsequence of Λ. We have thus completed the proof. $\square$

Let us define the discrete functions $\widehat{(\psi|\nabla u)}_{h,\Delta t,\varepsilon} \in C([0,T]; H(\mathbf{curl}; \Omega))$ and $\overline{(\psi|\nabla u)}_{h,\Delta t,\varepsilon} \in L^\infty(0,T; H(\mathbf{curl}; \Omega))$ made of the minimizers of the hybrid optimization problem ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$) by

$$\widehat{(\psi|\nabla u)}_{h,\Delta t,\varepsilon}(t) := \frac{t - \Delta t(n-1)}{\Delta t}(\psi_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon}) + \frac{\Delta t n - t}{\Delta t}(\psi_{h,n-1,\varepsilon}|\nabla u_{h,n-1,\varepsilon}) \text{ in } [\Delta t(n-1), \Delta t n],$$
$$\overline{(\psi|\nabla u)}_{h,\Delta t,\varepsilon}(t) := \begin{cases} (\psi_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon}) & \text{in } (\Delta t(n-1), \Delta t n], \\ (\psi_{h,0,\varepsilon}|\nabla u_{h,0,\varepsilon}) & \text{on } \{t = 0\}, \end{cases}$$

for $n = 1, \cdots, N$, where $(\psi_{h,0,\varepsilon}|\nabla u_{h,0,\varepsilon}) = \mathbf{0}$. We see that $\widehat{(\psi|\nabla u)}_{h,\Delta t,\varepsilon}(t) + \widehat{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ and $\overline{(\psi|\nabla u)}_{h,\Delta t,\varepsilon}(t) + \overline{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ for all $t \in [0,T]$.

On the assumption (4.13) proposition 3.1 and theorem 4.1 immediately yield

COROLLARY 4.1 The discrete approximations $\widehat{(\psi|\nabla u)}_{h,\Delta t(h),\varepsilon(h)}$, $\overline{(\psi|\nabla u)}_{h,\Delta t(h),\varepsilon(h)}$ converge to the unique solution $(\psi|\nabla u)$ of ($\mathbf{P}^B\mathbf{2}$) in the same sense as (4.14)-(4.18) for $\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} = \widehat{(\psi|\nabla u)}_{h,\Delta t(h),\varepsilon(h)}$, $\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} = \overline{(\psi|\nabla u)}_{h,\Delta t(h),\varepsilon(h)}$ and $\widehat{\mathbf{H}} = (\psi|\nabla u)$ as $h \searrow 0$, $h \in \Lambda$.

REMARK 4.1 In the case that the penalty coefficient $\varepsilon > 0$ is fixed and it is assume that Δt depends on h, satisfying $\sup_{h\in\Lambda}\Delta t(h) < 1$ and $\lim_{h\searrow 0, h\in\Lambda}\Delta t(h) = 0$, by using (3.7) we can similarly prove the convergence of the discrete solutions $\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon}, \partial_t\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon}$ and $\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon}$ to the solution $\widehat{\mathbf{H}}_\varepsilon$ of $(\mathbf{P}^{mB}_\varepsilon\mathbf{1})$ in the same sense as (4.14)-(4.18) for $\widehat{\mathbf{H}} = \widehat{\mathbf{H}}_\varepsilon$.

4.2 *Convergence of the discrete solutions solving* $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{1})$,$(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$

We will prove that the discrete solutions consisting of the minimizers of $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{1})$ and $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$ converge to the solution of $(\mathbf{P}^B\mathbf{1})$ and $(\mathbf{P}^B\mathbf{2})$ respectively.

We define the piecewise linear in time functions $\widehat{\mathbf{H}}_{h,\Delta t,p} \in C([0,T];H(\mathbf{curl};\Omega))$, and the piecewise constant in time function $\overline{\mathbf{H}}_{h,\Delta t,p} \in L^\infty(0,T;H(\mathbf{curl};\Omega))$ in the same way as $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}$ and $\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}$ by using the minimizer $\widehat{\mathbf{H}}_{h,n,p}$ of $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{1})$. Note that $\widehat{\mathbf{H}}_{h,\Delta t,p}(t), \overline{\mathbf{H}}_{h,\Delta t,p}(t) \in V_h(\Omega)$ and $\widehat{\mathbf{H}}_{h,\Delta t,p}(t) + \widehat{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ and $\overline{\mathbf{H}}_{h,\Delta t,p}(t) + \overline{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ for all $t \in [0,T]$. By the same calculation as proposition 4.1 we can prove following bounds.

PROPOSITION 4.2 Take any $\tau \in (0,1)$. The following inequalities hold. For any $h \in \Lambda$, $p \geqslant 2$, $\Delta t \in (0,\tau]$,

$$\begin{aligned}\|\partial_t\widehat{\mathbf{H}}_{h,\Delta t,p}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leqslant C\frac{\max\{\mu_d,\mu_s\}}{\min\{\mu_d,\mu_s\}}(h^2\|\partial_t\mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} \\ + \|\mathbf{n}\times\partial_t\mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}),\end{aligned} \tag{4.22}$$

$$\begin{aligned}\operatorname*{esssup}_{t\in[0,T]}\left\{\int_{\Omega_s}\frac{\mathscr{J}_c}{p}|\mathbf{curl}\,\overline{\mathbf{H}}_{h,\Delta t,p}(t)/\mathscr{J}_c|^p d\mathbf{x}\right\} \leqslant C\max\{\mu_d,\mu_s\}(h^2\|\partial_t\mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} \\ + \|\mathbf{n}\times\partial_t\mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}),\end{aligned} \tag{4.23}$$

$$\begin{aligned}\|\overline{\mathbf{H}}_{h,\Delta t,p}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \leqslant \frac{C}{1-\tau}e^{T/(1-\tau)}\frac{\max\{\mu_d,\mu_s\}}{\min\{\mu_d,\mu_s\}}(h^2\|\partial_t\mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} \\ + \|\mathbf{n}\times\partial_t\mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}),\end{aligned} \tag{4.24}$$

where $C > 0$ is a constant independent of $h, p, \Delta t, \mu$.

Let us assume that Δt and p are positive functions of h satisfying

$$\begin{aligned}&\sup_{h\in\Lambda}\Delta t(h) < 1,\ \lim_{h\searrow 0,h\in\Lambda}\Delta t(h) = \lim_{h\searrow 0,h\in\Lambda} 1/p(h) = 0, \\ &\inf_{h\in\Lambda} p(h) \geqslant 2,\ \sup_{h\in\Lambda} hp(h) < +\infty.\end{aligned} \tag{4.25}$$

THEOREM 4.2 The piecewise linear in time approximation $\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)}$ and the piecewise constant in time approximation $\overline{\mathbf{H}}_{h,\Delta t(h),p(h)}$ converge to the unique solution $\widehat{\mathbf{H}}$ of $(\mathbf{P}^B\mathbf{1})$ in the following sense.

$$\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)} \to \widehat{\mathbf{H}} \text{ strongly in } C([0,T];L^2(\Omega;\mathbb{R}^3)), \tag{4.26}$$

$$\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)} \rightharpoonup \widehat{\mathbf{H}} \text{ weakly} * \text{ in } L^\infty(0,T;H(\mathbf{curl};\Omega)), \tag{4.27}$$

$$\partial_t\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)} \rightharpoonup \partial_t\widehat{\mathbf{H}} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.28}$$

$$\overline{\mathbf{H}}_{h,\Delta t(h),p(h)} \to \widehat{\mathbf{H}} \text{ strongly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.29}$$

$$\overline{\mathbf{H}}_{h,\Delta t(h),p(h)} \rightharpoonup \widehat{\mathbf{H}} \text{ weakly} * \text{ in } L^\infty(0,T;H(\mathbf{curl};\Omega)), \tag{4.30}$$

as $h \searrow 0$, $h \in \Lambda$.

Proof. To simplify the notation let $\widehat{\mathbf{H}}_h$, $\overline{\mathbf{H}}_h$, $\widehat{\mathbf{H}}_{s,h}$ denote $\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)}$, $\overline{\mathbf{H}}_{h,\Delta t(h),p(h)}$, $\widehat{\mathbf{H}}_{s,h,\Delta t(h)}$, respectively.

By corollary 2.1 (1) and the bound (4.23), we see that $\{\mathbf{curl}\,\overline{\mathbf{H}}_h\}_{h\in\Lambda}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ and by (4.22) $\{\partial_t\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ is bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$. Thus, by taking a subsequence $\{h_n\}_{n=1}^\infty \subset \Lambda$, the weak($*$) convergences (4.27), (4.28), (4.30) hold true for some $\widehat{\mathbf{H}} \in L^\infty(0,T;H(\mathbf{curl};\Omega)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ satisfying $\widehat{\mathbf{H}}(t) \in V(\Omega)$ a.e. $t \in [0,T]$. Moreover, using lemma 3.1 (1) and the same argument as theorem 4.1 we can apply Ascoli-Arzera's theorem to prove the strong convergences (4.26) and (4.29).

We show that the limit $\widehat{\mathbf{H}}$ is the solution of $(\mathbf{P}^B\mathbf{1})$. By substituting $\phi_{h_n} = \phi_{l,h_n}(\Delta ti)$ into the inequality corresponding to (3.9), multiplying by Δt and summing over $i = 1 \to N$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \mu\langle \partial_t\widehat{\mathbf{H}}_{h_n} + \partial_t\widehat{\mathbf{H}}_{s,h_n}, \overline{\phi_{l,h_n}} - \overline{\mathbf{H}}_{h_n}\rangle d\mathbf{x}dt \\
&+ \frac{\mathscr{J}_c}{p}\int_0^T\int_{\Omega_s} |\mathbf{curl}\,\overline{\phi_{l,h_n}}/\mathscr{J}_c|^p d\mathbf{x}dt - \frac{\mathscr{J}_c}{p}\int_0^T\int_{\Omega_s} |\mathbf{curl}\,\overline{\mathbf{H}}_{h_n}/\mathscr{J}_c|^p d\mathbf{x}dt \geqslant 0.
\end{aligned} \tag{4.31}$$

Noting a fact that there is a constant $C > 0$ such that $h_n \leqslant C/p$ by the condition (4.25) and (3.5), we see that

$$\begin{aligned}
\frac{\mathscr{J}_c}{p}\int_0^T\int_{\Omega_s} |\mathbf{curl}\,\overline{\phi_{l,h_n}}/\mathscr{J}_c|^p d\mathbf{x}dt &\leqslant \frac{\mathscr{J}_c T|\Omega_s|}{p}(Ch_n\|\nabla\,\mathbf{curl}\,\phi_l\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))} + 1)^p \\
&\leqslant \frac{\mathscr{J}_c T|\Omega_s|}{p}(C_l/p + 1)^p \to 0,
\end{aligned} \tag{4.32}$$

as $n \to +\infty$. Moreover, the bound (4.23) and corollary 2.1 (2) show that $\widehat{\mathbf{H}}(t) \in S$ for all $t \in [0,T]$.

Now by neglecting the last term in the left side of (4.31), noting (4.32) and passing $n \to +\infty$ and $l \to +\infty$ in (4.31) we obtain

$$\int_0^T \int_\Omega \mu\langle \partial_t\widehat{\mathbf{H}} + \partial_t\mathbf{H}_s, \phi - \widehat{\mathbf{H}}\rangle d\mathbf{x}dt \geqslant 0,$$

which is equivalent to $(\mathbf{P}^B\mathbf{1})$. Therefore, $\widehat{\mathbf{H}}$ is the solution of $(\mathbf{P}^B\mathbf{1})$. The uniqueness of $(\mathbf{P}^B\mathbf{1})$ assures the convergences as $h \searrow 0$ without extracting a subsequence. $\square$

Let us define the discrete functions $\widehat{(\psi|\nabla u)}_{h,\Delta t,p} \in C([0,T];H(\mathbf{curl};\Omega))$ and $\overline{(\psi|\nabla u)}_{h,\Delta t,p} \in L^\infty(0,T;H(\mathbf{curl};\Omega))$ made of the minimizers of the hybrid optimization problem $(\mathbf{P}^p_{h,\Delta t,p}\mathbf{2})$

in the same way as $\widehat{(\psi|\nabla u)}_{h,\Delta t,\varepsilon}$ and $\overline{(\psi|\nabla u)}_{h,\Delta t,\varepsilon}$. We see that $\widehat{(\psi|\nabla u)}_{h,\Delta t,p}(t)+\widehat{\mathbf{H}}_{s,h,\Delta t}(t)\in X_h^{(\mu)}(\Omega)$ and $\overline{(\psi|\nabla u)}_{h,\Delta t,p}(t)+\overline{\mathbf{H}}_{s,h,\Delta t}(t)\in X_h^{(\mu)}(\Omega)$ for all $t\in[0,T]$.

On the assumption (4.25) proposition 3.1 and theorem 4.2 yield

COROLLARY 4.2 The discrete approximations $\widehat{(\psi|\nabla u)}_{h,\Delta t(h),p(h)}$, $\overline{(\psi|\nabla u)}_{h,\Delta t(h),p(h)}$ converge to the unique solution $(\psi|\nabla u)$ of $(\mathbf{P}^B\mathbf{2})$ as $h\searrow 0$, $h\in\Lambda$ in the same sense as (4.26)-(4.30) for $\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)}=\widehat{(\psi|\nabla u)}_{h,\Delta t(h),p(h)}$, $\overline{\mathbf{H}}_{h,\Delta t(h),p(h)}=\overline{(\psi|\nabla u)}_{h,\Delta t(h),p(h)}$ and $\widehat{\mathbf{H}}=(\psi|\nabla u)$ as $h\searrow 0$, $h\in\Lambda$.

REMARK 4.2 If we fix $p\geqslant 2$ and assume the relations $\sup_{h\in\Lambda}\Delta t(h)<1$ and $\lim_{h\searrow 0}\Delta t(h)=0$, by using (3.7) we can similarly prove that the discrete solutions $\widehat{\mathbf{H}}_{h,\Delta t(h),p}$, $\partial_t\widehat{\mathbf{H}}_{h,\Delta t(h),p}$ and $\overline{\mathbf{H}}_{h,\Delta t(h),p}$ to the solution of $(\mathbf{P}_p^P\mathbf{1})$ in the same sense as (4.26)-(4.30) and

$$\mathbf{curl}\,\widehat{\mathbf{H}}_{h,\Delta t(h),p}\rightharpoonup\mathbf{curl}\,\widehat{\mathbf{H}}_p \text{ weakly}* \text{ in } L^\infty(0,T;L^p(\Omega;\mathbb{R}^3)), \tag{4.33}$$

$$\mathbf{curl}\,\overline{\mathbf{H}}_{h,\Delta t(h),p}\rightharpoonup\mathbf{curl}\,\widehat{\mathbf{H}}_p \text{ weakly}* \text{ in } L^\infty(0,T;L^p(\Omega;\mathbb{R}^3)). \tag{4.34}$$

The weak convergences (4.33),(4.34) are consequences of the bound (4.23).

5. Numerical results

In this section we present numerical results by computing the unconstrained optimization problems $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$, $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$. All the examples in this section are computed in the situation where Ω and Ω_s are parallelepipeds whose faces are either parallel or perpendicular to $x-y, y-z, z-x$ planes in (x,y,z) coordinate. We mesh the domain by tetrahedra in the way as we see in Figure 3.

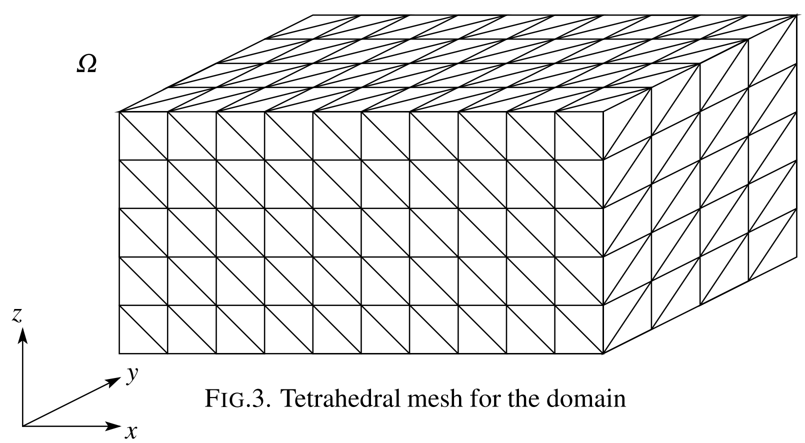

FIG.3. Tetrahedral mesh for the domain

We apply the external magnetic field $\mathbf{H}_s$ to be uniform in space and perpendicular to $x-y$ plane, so the boundary value $\mathbf{h}_s$ is given as $\mathbf{h}_s(t)=(0,0,\eta(t))$, where $\eta\in C^{1,1}([0,T])$ and $\eta(0)=0$. In this case the conditions (2.36), (2.37) are satisfied and the unique solution $\mathbf{H}_s$ of the system (2.7)-(2.10) is naturally given as $\mathbf{H}_s(t)=(0,0,\eta(t))$.

Let us note another equivalent characterization of the space $W_h(\Omega)$.

$$W_h(\Omega) = \{(\phi_h|\nabla u_h) \in L^2(\Omega;\mathbb{R}^3) \mid (\phi_h, u_h) \in U_h(\Omega_s)\times Z_h(\Omega_d),\\ \mathbf{n}\times\phi_h = \mathbf{n}\times\nabla u_h \text{ on } \partial\Omega_s,\ u_h|_{\partial\Omega} = 0\},$$

where $\mathbf{n}$ is the unit normal to $\partial\Omega_s$. Lemma 3.1 implies that the equality $\mathbf{n}\times\phi_h = \mathbf{n}\times\nabla u_h$ on $\partial\Omega_s$ holds if and only if

$$M_e(\phi_h - \nabla u_h) = 0, \tag{5.1}$$

for all edges e on $\partial\Omega_s$. The condition (5.1) is equivalent to the equality

$$M_e(\phi_h) = m_{v_2}(u_h) - m_{v_1}(u_h), \tag{5.2}$$

where v_1 and v_2 are the initial vertex and the terminal vertex of the edge e respectively. The relation (5.2) has to be always satisfied in the implementation of $W_h(\Omega)$ to fulfill the tangential continuity constraint $\mathbf{n}\times\phi_h = \mathbf{n}\times\nabla u_h$ on $\partial\Omega_s$.

5.1 *Definition of the penalized energy*

In order to search the minimizer of $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ by means of Newton method, we use C^2 class energy density so that we can calculate the Hessian of the energy functional. In our numerical simulation we employ the following regularized energy density g. For $0 < \alpha_1 < \alpha_2 < \alpha_3$ let $f_{\alpha_1,\alpha_2,\alpha_3} \in C^2(\mathbb{R})$ be a function satisfying that $f_{\alpha_1,\alpha_2,\alpha_3}(x) = 0$ for all $x \leqslant 0$,

$$f''_{\alpha_1,\alpha_2,\alpha_3}(x) = \begin{cases} x/\alpha_1 & \text{in } [0,\alpha_1],\\ 1 & \text{in } [\alpha_1,\alpha_2],\\ (-x+\alpha_3)/(\alpha_3-\alpha_2) & \text{in } [\alpha_2,\alpha_3],\\ 0 & \text{in } [\alpha_3,\infty), \end{cases}$$

Now $f_{\alpha_1,\alpha_2,\alpha_3}$ is a polynomial of degree 3 in $[0,\alpha_1]$, of degree 2 in $[\alpha_1,\alpha_2]$, of degree 3 in $[\alpha_2,\alpha_3]$ and of degree 1 in $[\alpha_3,\infty)$. Define $g(x) := f_{\alpha_1,\alpha_2,\alpha_3}(x^2 - \mathcal{J}_c^2)$, which is found to satisfy the required properties (2.14). This energy density $g(|\mathbf{v}|)/\varepsilon$ with $\varepsilon > 0$, $\mathbf{v}\in\mathbb{R}^3$ is a regularized version of the modified Bean model's energy density γ^{mB}_ε defined in (2.13) which is not continuously differentiable.

5.2 *Algebraic equation*

We rewrite the problems $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ and $(\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2})$ in a system of nonlinear algebraic equations. Here we use the general notation $\zeta(|\mathbf{p}|) := \gamma(\mathbf{p})$, with $\zeta \in C^2(\mathbb{R})$ to make the formulation below applicable for both $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ and $(\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2})$.

Let N be the degrees of freedom, N_d^i be the number of vertices in $\Omega\backslash\overline{\Omega_s}$, i.e, the number of vertices in the inside of Ω_d, N_d^b be the number of vertices on the boundary $\partial\Omega_s$, N_s^i be the number of edges in the inside of Ω_s, i.e, the number of edges which are not on the boundary $\partial\Omega_s$, N_s^b be the number of edges on the boundary $\partial\Omega_s$, and N_E be the number of elements contained in $\overline{\Omega_s}$.

Set $N_d := N_d^i + N_d^b$ and $N_s := N_s^i + N_s^b$. Then we observe

$$N = N_d + N_s^i = N_d^i + N_d^b + N_s^i.$$

We introduce a vector $\mathbf{H}\in\mathbb{R}^N$ defined by

$$\mathbf{H} = (\mathbf{H}_d^i, \mathbf{H}_d^b, \mathbf{H}_s^i) \in \mathbb{R}^{N_d^i}\times\mathbb{R}^{N_d^b}\times\mathbb{R}^{N_s^i},$$

where $\mathbf{H}_d^i$ and $\mathbf{H}_d^b$ are the nodal values of the magnetic scalar potential $u_{h,n}$ at the vertices in $\Omega\backslash\overline{\Omega_s}$ and on $\partial\Omega_s$ respectively, $\mathbf{H}_s^i$ is the values of line integral of the magnetic field $\psi_{h,n}$ along the edges in Ω_s. Our functionals $F_{h,\Delta t,\varepsilon}((\psi_{h,n}|\nabla u_{h,n}))$, $G_{h,\Delta t,p}((\psi_{h,n}|\nabla u_{h,n}))$ can be written as the following functional F on $\mathbb{R}^N$.

$$F(\mathbf{H}) := \frac{1}{2}\left(\begin{pmatrix}\mathbf{H}_d^i\\ \mathbf{H}_d^b\end{pmatrix} + 2\begin{pmatrix}\bar{\mathbf{H}}_d^i\\ \bar{\mathbf{H}}_d^b\end{pmatrix}\right)^T M_d \begin{pmatrix}\mathbf{H}_d^i\\ \mathbf{H}_d^b\end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix}\Xi(\mathbf{H}_d^b)\\ \mathbf{H}_s^i\end{pmatrix} + 2\begin{pmatrix}\Xi(\bar{\mathbf{H}}_d^b)\\ \bar{\mathbf{H}}_s^i\end{pmatrix}\right)^T M_s \begin{pmatrix}\Xi(\mathbf{H}_d^b)\\ \mathbf{H}_s^i\end{pmatrix}$$
$$+\sum_{k=1}^{N_E}|K_k|\zeta\left(\sqrt{\left(P_k\begin{pmatrix}\Xi(\mathbf{H}_d^b)\\ \mathbf{H}_s^i\end{pmatrix}\right)^T M_{s,k}P_k\begin{pmatrix}\Xi(\mathbf{H}_d^b)\\ \mathbf{H}_s^i\end{pmatrix}}\right), \tag{5.3}$$

where $(\bar{\mathbf{H}}_d^i, \bar{\mathbf{H}}_d^b, \bar{\mathbf{H}}_s^i) \in \mathbb{R}^{N_d^i} \times \mathbb{R}^{N_d^b} \times \mathbb{R}^{N_s^i}$ is the known vector associated with $-\widehat{\mathbf{H}}_{h,n-1} + \mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1}$, $M_d = (m_{q,r}^d)_{1\leqslant q,r\leqslant N_d}$ is a $N_d \times N_d$ matrix defined by

$$m_{q,r}^d = \frac{\mu_d}{\Delta t}\int_{\overline{\Omega_d}} \langle \nabla\phi_q, \nabla\phi_r\rangle d\mathbf{x},$$

ϕ_q $(q = 1, \cdots, N_d)$ are the basis functions in $Z_h(\Omega_d)$ corresponding to the N_d vertices in $\overline{\Omega_d}\backslash\partial\Omega$, $M_s = (m_{q,r}^s)_{1\leqslant q,r\leqslant N_s}$ is a $N_s \times N_s$ matrix defined by

$$m_{q,r}^s = \frac{\mu_s}{\Delta t}\int_{\overline{\Omega_s}} \langle \psi_q, \psi_r\rangle d\mathbf{x},$$

ψ_q $(q = 1, \cdots, N_s)$ are the basis functions in $U_h(\Omega_s)$ corresponding to the N_s edges in $\overline{\Omega_s}$, $\Xi : \mathbb{R}^{N_d^b} \to \mathbb{R}^{N_s^b}$ is a linear map, which expresses the connection (5.2) between the vertices and the edges on $\partial\Omega_s$, $P_k : \mathbb{R}^{N_s} \to \mathbb{R}^6$ $(k = 1, \cdots, N_E)$ are linear maps which project the edges in $\overline{\Omega_s}$ into 6 edges belonging to kth tetrahedron K_k and $M_{s,k} = (m_{q,r}^{s,k})_{1\leqslant q,r\leqslant 6}$ $(k = 1, \cdots, N_E)$ are 6×6 matrices defined by

$$m_{q,r}^{s,k} = \langle P_k^q \begin{pmatrix}\mathbf{curl}\,\psi_1\\ \vdots\\ \mathbf{curl}\,\psi_{N_s}\end{pmatrix}, P_k^r \begin{pmatrix}\mathbf{curl}\,\psi_1\\ \vdots\\ \mathbf{curl}\,\psi_{N_s}\end{pmatrix}\rangle,$$

where we write

$$P_k = \begin{pmatrix}P_k^1\\ \vdots\\ P_k^6\end{pmatrix},\ P_k^l \in \mathbb{R}^{N_s}(l = 1, \cdots, 6),$$

and $|K_k|$ is the volume of the tetrahedron K_k.

Thus, now we have the unconstrained optimization problem; find $\hat{\mathbf{H}} \in \mathbb{R}^N$ such that $F(\hat{\mathbf{H}}) = \min_{\mathbf{H}\in\mathbb{R}^N} F(\mathbf{H})$. This optimization problem gives the following nonlinear equations. Find $\hat{\mathbf{H}} \in \mathbb{R}^N$ such that

$$\nabla F(\hat{\mathbf{H}}) = \mathbf{0}_N,$$

which is

$$\begin{pmatrix} M_d\left(\begin{pmatrix}\hat{\mathbf{H}}_d^i\\ \hat{\mathbf{H}}_d^b\end{pmatrix} + \begin{pmatrix}\bar{\mathbf{H}}_d^i\\ \bar{\mathbf{H}}_d^b\end{pmatrix}\right)\\ \mathbf{0}_{N_s^i}\end{pmatrix} + \begin{pmatrix} \mathbf{0}_{N_d^i}\\ \begin{pmatrix}\Xi & \mathbf{0}_{N_s^b\times N_s^i}\\ \mathbf{0}_{N_s^i\times N_d^b} & I_{N_s^i}\end{pmatrix}^T M_s\left(\begin{pmatrix}\Xi(\hat{\mathbf{H}}_d^b)\\ \hat{\mathbf{H}}_s^i\end{pmatrix} + \begin{pmatrix}\Xi(\bar{\mathbf{H}}_d^b)\\ \bar{\mathbf{H}}_s^i\end{pmatrix}\right)\end{pmatrix}$$

$$+\sum_{k=1}^{N_E} \frac{|K_k|\zeta'(\sqrt{X_k})}{\sqrt{X_k}} \left(\begin{pmatrix} \mathbf{0}_{N_d^i} \\ \begin{pmatrix} \Xi & \mathbf{0}_{N_s^b \times N_s^i} \\ \mathbf{0}_{N_s^i \times N_d^b} & I_{N_s^i} \end{pmatrix}^T \end{pmatrix} P_k^T M_{s,k} P_k \begin{pmatrix} \Xi(\hat{\mathbf{H}}_d^b) \\ \hat{\mathbf{H}}_s^i \end{pmatrix} \right) = \mathbf{0}_N,$$

where $\mathbf{0}_l$ denotes zero vector in $\mathbb{R}^l$, $\mathbf{0}_{l\times m}$ is zero matrix of size $l \times m$, I_l is the identity matrix of size $l \times l$, and

$$X_k := \left(P_k \begin{pmatrix} \Xi(\hat{\mathbf{H}}_d^b) \\ \hat{\mathbf{H}}_s^i \end{pmatrix}\right)^T M_{s,k} P_k \begin{pmatrix} \Xi(\hat{\mathbf{H}}_d^b) \\ \hat{\mathbf{H}}_s^i \end{pmatrix}.$$

This nonlinear system is computed by Newton method coupled with conjugate gradient method. In order to compute the minimizer of $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ with a small $\varepsilon > 0$, successively performing Newton iteration for decreasing sequences of the coefficients $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n = \varepsilon$ and the parameters $\alpha_{i,1} > \alpha_{i,2} > \cdots > \alpha_{i,n} = \alpha_i$ $(i = 1-3)$ appearing in the penalty term at each time step is effective to reduce computation time. The code with which we obtain the numerical results was written in C.

5.3 *Distance between discrete solutions solving $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ and $(\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2})$*

Let us compute the distance between $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}$ and $\widehat{\mathbf{H}}_{h,\Delta t,p}$ for different parameters $h,\Delta t,\varepsilon,p$ in the situation where the uniform external magnetic field $\mathbf{H}_s(t) = (0,0,0.01t)$ is applied to the cubic superconductor with the diameter 1, $\mu_d = \mu_s = 1$, $\mathscr{J}_c = 1$ and the distance between $\partial\Omega_s$ and $\partial\Omega$ is 1. Moreover we assume the following relation

$$h = \Delta t = \varepsilon = \frac{1}{p} \leqslant \frac{1}{2}.$$

Note that this relation satisfies the sufficient conditions (4.13) and (4.25) for our convergence results. We set $D_\varepsilon(t) := \|\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t)\|_{L^2(\Omega;\mathbb{R}^3)} \times 1000$, $D_p(t) := \|\widehat{\mathbf{H}}_{h,\Delta t,p}(t)\|_{L^2(\Omega;\mathbb{R}^3)} \times 1000$, $D_{\varepsilon,p}(t) := \|\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) - \widehat{\mathbf{H}}_{h,\Delta t,p}(t)\|_{L^2(\Omega;\mathbb{R}^3)} \times 1000$, and DOF stands for the degrees of freedom in Table 1.

$t=1$		
$h \approx 1/5$	DOF= 3345	$D_\varepsilon(t) = 21.05460, D_p(t) = 18.44832, D_{\varepsilon,p}(t) = 5.243279$
$h \approx 1/10$	DOF= 29790	$D_\varepsilon(t) = 11.52809, D_p(t) = 11.42765, D_{\varepsilon,p}(t) = 0.466210$
$h \approx 1/15$	DOF= 104085	$D_\varepsilon(t) = 7.920608, D_p(t) = 7.904191, D_{\varepsilon,p}(t) = 0.133685$
$h \approx 1/20$	DOF= 250980	$D_\varepsilon(t) = 6.030922, D_p(t) = 6.025237, D_{\varepsilon,p}(t) = 0.062590$
$t=2$		
$h \approx 1/5$	DOF= 3345	$D_\varepsilon(t) = 42.10921, D_p(t) = 31.05171, D_{\varepsilon,p}(t) = 17.16281$
$h \approx 1/10$	DOF= 29790	$D_\varepsilon(t) = 23.01230, D_p(t) = 21.80453, D_{\varepsilon,p}(t) = 3.682135$
$h \approx 1/15$	DOF= 104085	$D_\varepsilon(t) = 15.79462, D_p(t) = 15.48331, D_{\varepsilon,p}(t) = 1.273215$
$h \approx 1/20$	DOF= 250980	$D_\varepsilon(t) = 12.03072, D_p(t) = 11.91770, D_{\varepsilon,p}(t) = 0.592443$
$t=3$		
$h \approx 1/5$	DOF= 3345	$D_\varepsilon(t) = 61.77371, D_p(t) = 37.90881, D_{\varepsilon,p}(t) = 30.65537$
$h \approx 1/10$	DOF= 29790	$D_\varepsilon(t) = 34.09663, D_p(t) = 31.50695, D_{\varepsilon,p}(t) = 7.247937$
$h \approx 1/15$	DOF= 104085	$D_\varepsilon(t) = 23.51332, D_p(t) = 22.66722, D_{\varepsilon,p}(t) = 3.059232$
$h \approx 1/20$	DOF= 250980	$D_\varepsilon(t) = 17.94297, D_p(t) = 17.55406, D_{\varepsilon,p}(t) = 1.610502$

TABLE 1.

We observe that the L^2 distance between $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t)$ and $\widehat{\mathbf{H}}_{h,\Delta t,p}(t)$ becomes smaller as the parameter h becomes smaller. This agrees with the fact that $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}$ and $\widehat{\mathbf{H}}_{h,\Delta t,p}$ converge to the same limit as $h \searrow 0$ in $C([0,T];L^2(\Omega;\mathbb{R}^3))$ satisfying the relation $h = \Delta t = \varepsilon = 1/p \leqslant 1/2$.

5.4 *The current density and the magnetic field*

We display some numerical results showing the behaviour of the electric current and the magnetic field, where $\mu_d = \mu_s = 1$, $\mathscr{J}_c = 1$, and $\mathbf{H}_s(t) = (0, 0, 0.01t)$ is applied.

5.4.1 *The current density for each* $\mathbf{E} - \mathbf{J}$ *relation* We assume that Ω_s is a parallelepiped of the size $x = 5/4, y = 1, z = 3/5$ and Ω is a parallelepiped of the size $x = 13/4, y = 3, z = 13/5$. The distance between each face of $\partial\Omega_s$ and the closest face of $\partial\Omega$ is 1. The computation involves 226497 degrees of freedom. In Figures 4-6 the current density $|\mathbf{J}|$ on the surface and the cross section of the superconductor Ω_s is displayed. No current is flowing in the blue region, the current $\mathbf{J}$ with $0 < |\mathbf{J}| < 1$ is flowing in the red region, and the critical current with $|\mathbf{J}| \approx 1$ is flowing in the yellow region.

5.4.2 *Motion of the subcritical region* We show the motion of the subcritical region where there is no current or the current $\mathbf{J}$ with $|\mathbf{J}| \leqslant 1/3$ is flowing in Figure 7 by solving $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$ with $p = 500$. We assume that Ω_s is a parallelepiped of the size $x = 5/4, y = 1, z = 3/4$ and Ω is a parallelepiped of the size $x = 13/4, y = 3, z = 11/4$. The distance between each face of $\partial\Omega_s$ and the closest face of $\partial\Omega$ to be 1. The computation involves 246555 degrees of freedom.

5.4.3 *The magnetic field* In the same situation as the section 5.4.2 we show the penetration of the magnetic flux $\mathbf{B} = \mu\widehat{\mathbf{H}} + \mu\mathbf{H}_s$ into the superconductor by solving $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$ with $p = 500$. In Figure 8, the cross section of Ω_s cut by a plane parallel to $x - z$ plane in the middle is displayed. The blue vector field stands for the smallest magnitude, the red vector field has the middle magnitude and the yellow vector has the largest magnitude.

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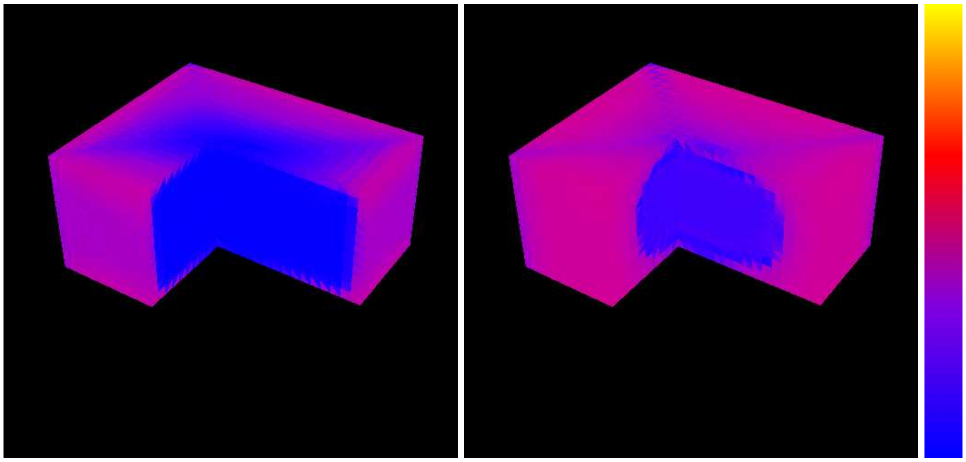

FIG.4. The current density $|\mathbf{J}|$ of the power law with $p = 10$ at $t = 1$ left, $t = 5$ right.

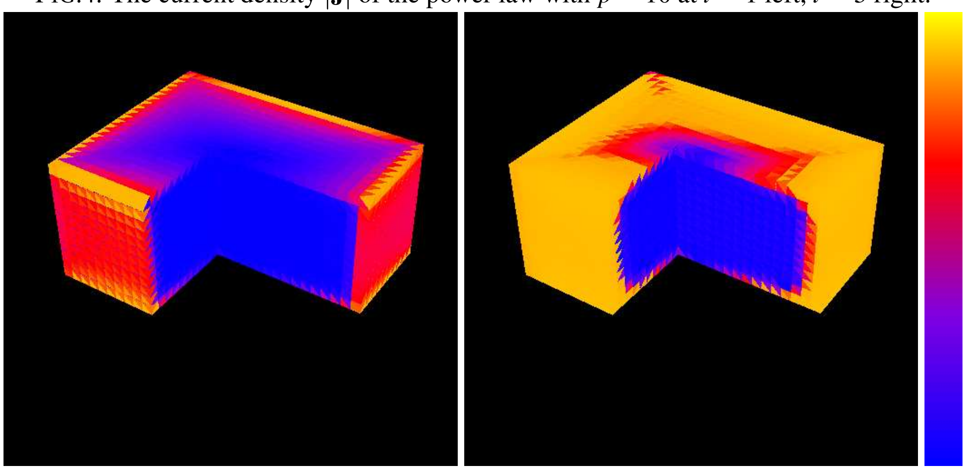

FIG.5. The current density $|\mathbf{J}|$ of the power law with $p = 100$ at $t = 1$ left, $t = 5$ right.

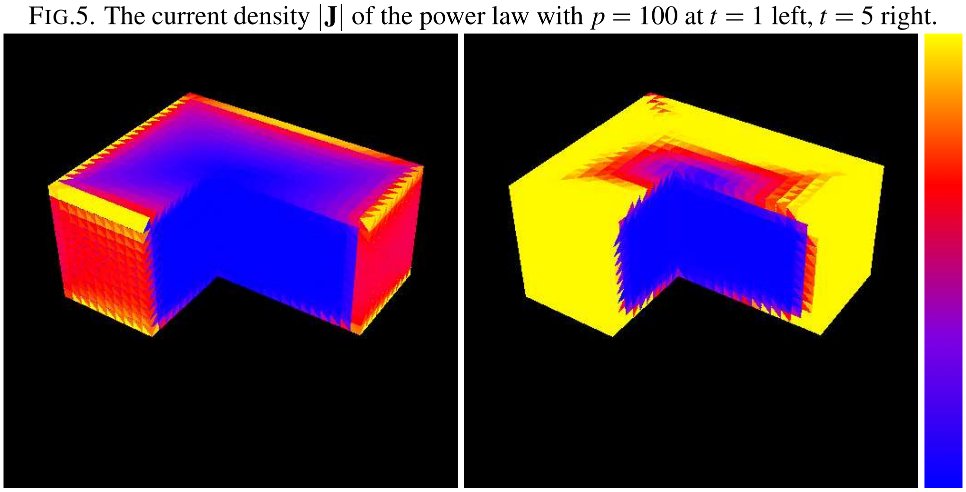

FIG.6. The current density $|\mathbf{J}|$ of the modified Bean model with $\varepsilon = 10^{-7}$ at $t = 1$ left, $t = 5$ right.

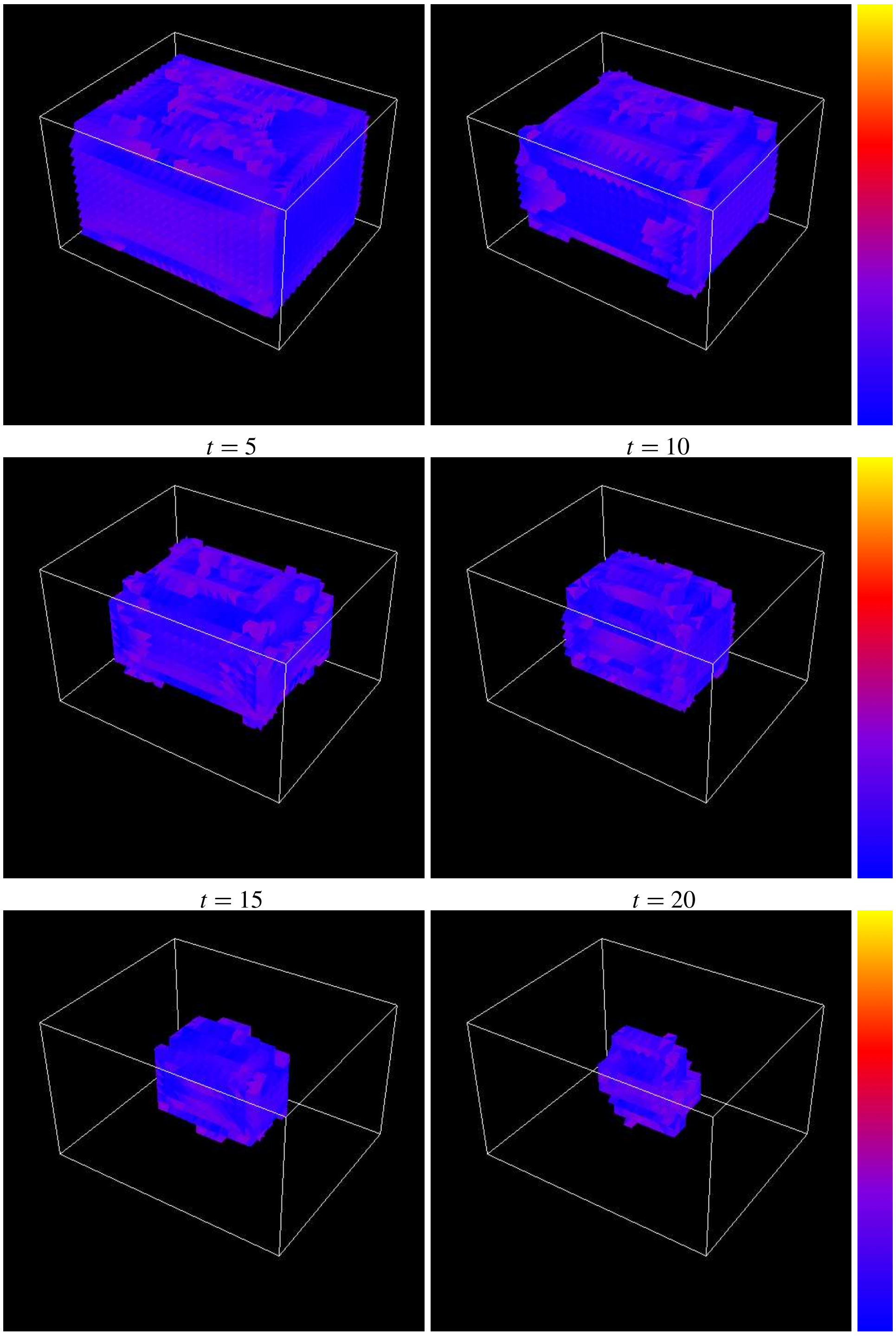

FIG.7. The subcritical region where $|\mathbf{J}| \leqslant 1/3$.

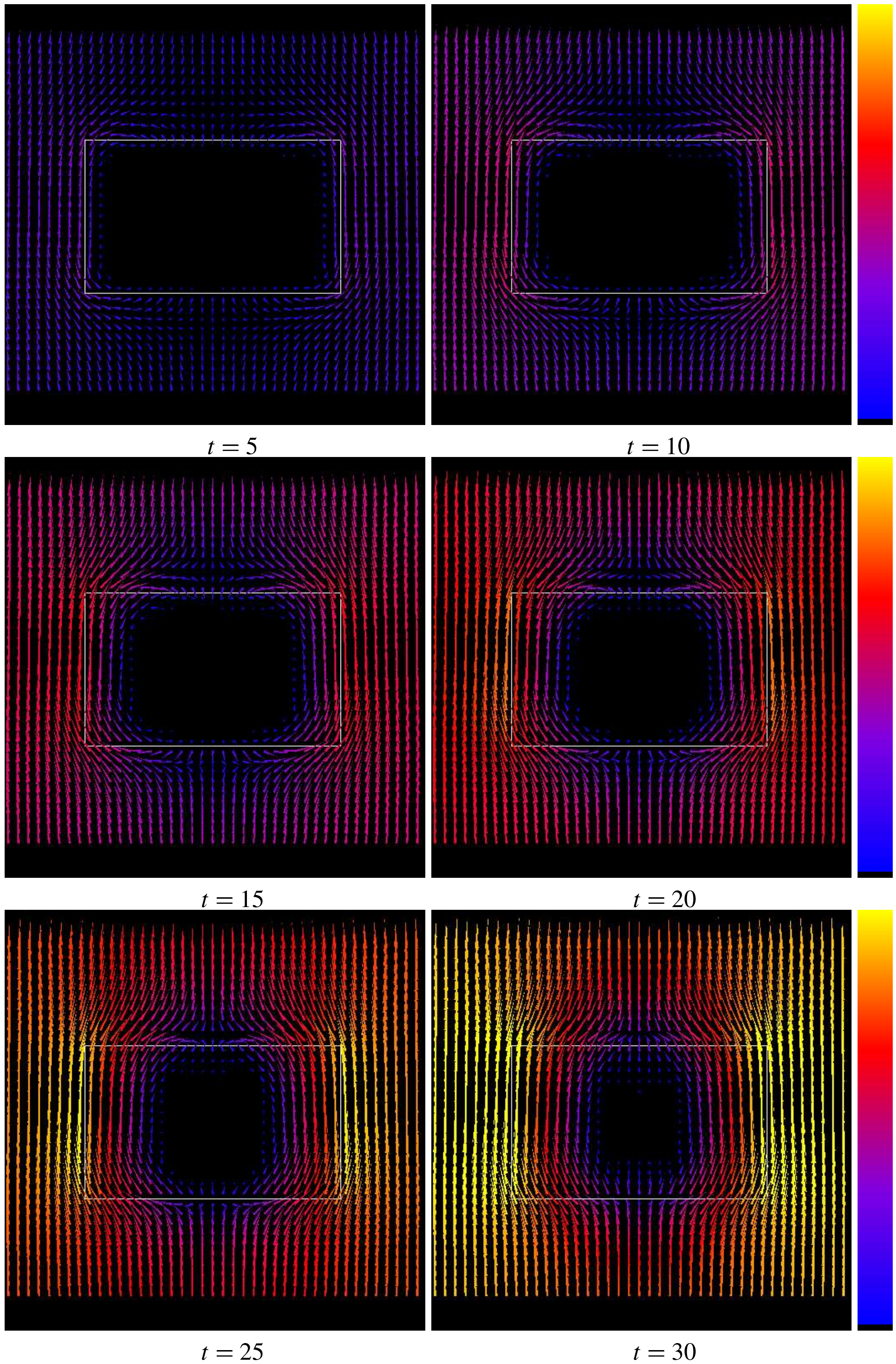

FIG.8. The penetration of the magnetic flux density $\mathbf{B} = \mu \widehat{\mathbf{H}} + \mu \mathbf{H}_s$.

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