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EXTENDED GRADIENT SYSTEMS: DIMENSION ONE

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Abstract. We propose a general theory of formally gradient differential equations on unbounded one-dimensional domains, based on an energy-flow inequality, and on the study of the induced semiflow on the space of probability measures on the phase space. We prove that the ω-limit set of each point contains an equilibrium, and that the ω-limit set of μ-almost every point in the phase space consists of equilibria, where μ is any Borel probability measure invariant for spatial translations.

1. Introduction. The goal of this paper is to provide an abstract framework for studying the dynamical systems on unbounded domains, whose restriction to any bounded domain generates a gradient semiflow (i.e. a semiflow with a strict Lyapunov function). Such systems are called extended gradient semiflows in the following.

The asymptotic behaviour of gradient semiflows is well understood: their ω-limit set *consists* of stationary points. However, this result does not apply to extended gradient semiflows: typically one can not define a strict Lyapunov function on the phase space. We study instead a class of almost-Lyapunov functions H_n, $n \in \mathbf{N}$, and deduce the following general result: ω-limit set of each point *contains* a stationary point. Furthermore, given any translationally invariant measure μ, ω-limit set of μ-almost every point x *consists* of stationary points.

We expect that the theory developed here will be applicable to a wide class of parabolic and damped hyperbolic differential equations on unbounded domains. In particular, it is a contribution towards the goal to describe the global attractor of these systems (see [BV90], [Fei96], [MS95]).

In this paper we discuss properties of one-dimensional systems, namely the systems which generate a semiflow on a subset of the set of functions $u : \mathbf{R} \to \mathbf{R}^n$ (or $u : \mathbf{Z} \to \mathbf{R}^n$). Some generalisations and counterexamples in higher dimensions will be discussed in the forthcoming paper ([GS99]).

In section 2, we define extended gradient semiflows, and give some examples. The elementary properties, including non-existence of periodic orbits, are proved in section 3. In section 4, we give a sufficient condition for a point to be non-recurrent. The main result is in section 5, where we show that the average time spent before an arbitrary orbit returns to a small neighbourhood of a non-stationary point is $+\infty$. A corollary is that all the invariant measures of an extended gradient semiflow are supported on the set of stationary points.

In sections 6 and 7 we discuss spatially extended gradient systems, i.e. extended gradient semiflows invariant for spatial translations. We show that the induced

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semiflow on the space of translationally invariant Borel probability measures is gradient, and discuss various implications. Finally, we suggest applications to variational problems on unbounded domains, and Hamiltonian dynamical systems in particular.

In Appendices A and B we show that two typical examples satisfy all the given axioms of (spatially) extended gradient systems: the gradient dynamics of Frenkel-Kontorova models, and the real Ginzburg-Landau equation.

2. Axioms. Given a semiflow ϕ on a space $\mathcal{A}$, we use the following notation:

$$\begin{aligned} \phi^t(x) &:= \phi(x,t), \\ x^t &:= \phi(x,t), \\ x^{[0,T]} &:= \{\phi(x,t),\ t \in [0,T]\}. \end{aligned}$$

A stationary point of a semiflow ϕ is every point x such that for each $t \geq 0$, $x^t = x$. The set of all stationary points is denoted by $\mathcal{S}$.

Definition 2.1. *We say that a continuous semiflow ϕ on a topological space $\mathcal{A}$ is an extended gradient semiflow, if there exists a sequence of Borel-measurable functions $H_n, A_n : \mathcal{A} \to \mathbf{R}$ satisfying the following conditions:*

- **(E1): Lyapunov condition.** *There exists an open cover $\mathcal{F}$ of $\mathcal{A} - \mathcal{S}$, such that for each $\mathcal{U} \in \mathcal{F}$, there exists $\epsilon > 0$, and n_0 such that $x^{[0,T]} \subset \mathcal{U}$ and $n \geq n_0$ imply*
$$H_n(x^T) - H_n(x^0) + \int_0^T A_n(x^t)dt \leq -\epsilon T. \tag{2.1}$$
- **(E2): Boundedness.** *For each $n \geq 1$, H_n is bounded, and A_n bounded uniformly in n (i.e. there exists $M > 0$ such that for each $n \in \mathbf{N}$, $x \in \mathcal{A}$, $|A_n(x)| < M$) on $\mathcal{A}$.*
- **(E3): The flow inequality.** *There exists a strictly increasing function $E : \mathbf{R}^+ \to \mathbf{R}^+$, $E(0) = 0$, such that for each $x \in \mathcal{A}$, and $T > 0$,*
$$H_n(x^T) - H_n(x^0) + \int_0^T A_n(x^t)dt \leq -\sum_{k=1}^{n-1} \int_0^T E(|A_k(x^t)|)dt. \tag{2.2}$$

We say that a semiflow is a continuously extended gradient semiflow, if it is an extended gradient semiflow, and the functions H_n, A_n, $n \in \mathbf{N}$, are continuous.

A sequence of functions (H_n, A_n) satisfying (E1-3) is called the Lyapunov sequence.

In the examples, H_n will typically be the total energy of the system restricted to a finite domain of size $\sim n$, and A_n will be the energy current through the boundary of the finite domain. The one-dimensionality is implicitly assumed in (E2) (the uniform bound on A_n means that the size of the boundary does not grow with the size of the domain) and (E3) (see the examples below).

We use the following notation
$$\frac{\mathrm{d}H}{\mathrm{d}t}(x) = \frac{\mathrm{d}}{\mathrm{d}t} H(x^t)|_{t=0^+}, \tag{2.3}$$
and say that H is differentiable along trajectories, if (2.3) is well defined for each $x \in \mathcal{A}$. If for each $n \in \mathbf{N}$, H_n is differentiable along trajectories, (2.1) can be written as
$$\frac{\mathrm{d}}{\mathrm{d}t} H_n(x) + A_n(x) \leq -\epsilon \tag{2.4}$$

(where $x \in \mathcal{U}$ and $n \geq n_0$), and (2.2) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} H_n(x) + A_n(x) \leq -\sum_{k=1}^{n-1} E(|A_k(x)|). \tag{2.5}$$

Now we show that few typical examples satisfy the axioms.

Recall that we say that a semiflow ϕ on the phase space $\mathcal{A}$ is gradient, if there exists a function $H : \mathcal{A} \to \mathbf{R}$, decreasing along trajectories (the function H is called Lyapunov function). A semiflow is strictly gradient, if it is gradient and the Lyapunov function is strictly decreasing along all non-stationary trajectories.

Example 1: *Gradient flows.* Assume that ϕ is a strictly gradient semoflow on a compact $\mathcal{A}$, with a continuous Lyapunov function H. We then define $H_n \equiv H$, $A_n \equiv 0$. If H is differentiable along trajectories, and $\mathrm{d}H/\mathrm{d}t : \mathcal{A} \to \mathbf{R}$ is continuous, it is easy to check that the sequence H_n, A_n satisfies (E1-3).

Example 2: *Gradient dynamics of Frenkel-Kontorova models.* Given a C^2 function $V : \mathbf{R}^2 \to \mathbf{R}$, 1-periodic (i.e. $(V(x,y) = V(x+1,y+1))$ and satisfying the twist condition (there exists $\delta > 0$ such that $V_{12}(x,y) \leq -\delta < 0$), the Frenkel-Kontorova model is defined with the formal sum $V = \sum_{i \in \mathbf{Z}} V(u_{i-1}, u_i)$. Then the system

$$\dot{u} = -\nabla V(u) = -V_2(u_{i-1}, u_i) - V_1(u_i, u_{i+1}) \tag{2.6}$$

induces a continuously extended gradient semiflow ϕ on the set $\mathcal{B}_N$ of all $u \in \mathbf{R}^{\mathbf{Z}}$ such that $\sup_{i \in \mathbf{Z}} |u_i - u_{i-1}| \leq N$, for any $N \in \mathbf{N}$, in either product topology τ_p or sup-norm topology τ_∞ on $\mathcal{B}_N$. If we define the translation T on $\mathbf{R}^{\mathbf{Z}}$, $T(u)_i = u_i + 1$, then ϕ induces a continuously extended gradient semiflow on the quotient space $\mathcal{B}_N/T$ in either topologies. Precise definitions and the proof are given in Appendix A.

Example 3: *The real Ginzburg-Landau equation.* The real 1D Ginzburg-Landau equation is given by the equations

$$u_t(x,t) \quad = \quad u_{xx}(x,t) + u(x,t) - u^3(x,t), \tag{2.7}$$

where $u(.,t) : \mathbf{R} \to \mathbf{R}$. Then the equation (2.7) induces a continuous semiflow ϕ on the space $\mathcal{C}_{b,u}$ of all bounded, uniformly continuous functions, in either localised topology τ_{loc} or uniform topology τ_∞. The space $\mathcal{A}_C$ is the smallest ϕ-invariant, τ_∞-closed subset of $\mathcal{C}_{b,u}$ containing all functions u such that u, u_t, u_x, u_{tx}, u_{tt} are well-defined and uniformly bounded by C. Then (2.7) induces a continuously extended gradient semiflow on $\mathcal{A}_C$, for any $C > 0$. Precise definitions and an outline of the proof are given in Appendix B.

Given an extended gradient semiflow ϕ and a Lyapunov sequence (H_n, A_n), we say that $\mathcal{U}$ is an almost-Lyapunov set, if there exists $\epsilon > 0$ and n_0 such that for each $n \geq n_0$, $x^{[0,T]} \subset \mathcal{U}$ implies (2.1). Therefore (E1) can be reformulated as: there exists a cover $\mathcal{F}$ of $\mathcal{A} - \mathcal{S}$ consisting of open almost-Lyapunov sets.

3. Elementary properties. We now prove some elementary consequences of Definition 2.1. We assume in the following that ϕ is an extended gradient semiflow on $\mathcal{A}$. We first prove the following elementary lemma ($\mathbf{1}_A$ denotes the characteristic function of the set A).

Lemma 3.1. *If $A : [0,T] \to \mathbf{R}$ is a measurable bounded function such that for each $t \in [0,T]$, $0 \le A(t) \le M$ and $\int_0^T A(t) \ge 2\epsilon T$, then*

$$\int_0^T \mathbf{1}_{\{A(t)\ge\epsilon\}}(t)dt \ge \frac{\epsilon T}{M}.$$

Proof. We calculate:

$$\begin{aligned} 2\epsilon T &\le \int_0^T A(t)dt = \int_0^T \mathbf{1}_{\{A(t)\ge\epsilon\}}A(t)dt + \int_0^T \mathbf{1}_{\{A(t)<\epsilon\}}A(t)dt \\ &\le M\int_0^T \mathbf{1}_{\{A(t)\ge\epsilon\}}dt + \epsilon T, \end{aligned}$$

which implies the claim. □

Proposition 3.2. *(i) For each $x \in \mathcal{A} - \mathcal{S}$, $T > 0$ there exists an almost-Lyapunov neighbourhood $\mathcal{U}$ of $x^{[0,T]}$.*

(ii) If $x \in \mathcal{S}$, for each $n > 0$, $A_n(x) = 0$.

(iii) For each $x \in \mathcal{A} - \mathcal{S}$, and each $T > 0$ there exists n_0 such that

$$\text{for each } n > n_0, \quad H_n(x^T) < H_n(x^0). \tag{3.8}$$

Proof. (i) For each $t \in [0,T]$ we choose an almost-Lyapunov neighbourhood $\mathcal{U}_{x^t} \in \mathcal{F}$ of x_t, and n_{x^t}, ϵ_{x^t} such that (2.1) is satisfied on $\mathcal{U}_{x^t}$. We find a finite subcover $\mathcal{U}_{x_1}$,...,$\mathcal{U}_{x_n}$ of $x^{[0,T]}$, and define $\mathcal{U} = \cup_{k=1}^n \mathcal{U}_{x_k}$, $\epsilon = \min_{k=1,\dots,n}\epsilon_{x_k}$, $n_0 = \max_{k=1,\dots,n} n_{x_k}$.

(ii) Assume that for some k, $|A_k(x)| = \delta > 0$. Choose $n > k$ such that $(n-k)E(E(\delta)) > M$, where M is the bound on $|A_n|$ from (E2), and E the function from (E3). Since x is stationary, $H_n(x^T) - H_n(x) = 0$ for all $n \in \mathbf{N}$, and (E3) implies that for each $m > k$, $|A_m(x)| \ge E(\delta)$. Now (E3) implies that

$$|A_{n+2}(x)| \ge \sum_{i=1}^{n+1} E(|A_i(x)|) \ge (n-k)E(E(\delta)) > M,$$

which is in contradiction to M being an uniform bound on A_n.

(iii) Assume the contrary, and then there exists $x \notin \mathcal{S}$, and a strictly increasing sequence $(n_k)_{k\in\mathbf{N}}$ such that for each k, $H_{n_k}(x^T) \ge H_{n_k}(x)$. Then (i) implies that for some $\epsilon > 0$,

$$\left|\int_0^T A_{n_k}(x^t)dt\right| \ge 2\epsilon T + H_{n_k}(x^T) - H_{n_k}(x) \ge 2\epsilon T.$$

Denote by $S_k = \{t : \ |A_{n_k}(x^t)| \ge \epsilon\}$. Lemma 3.1 implies that

$$\int_0^T E(|A_{n_k}(x^t)|)dt \quad \ge \quad \int_0^T \mathbf{1}_{S_k} E(\epsilon)dt \ge \frac{\epsilon T E(\epsilon)}{M}. \tag{3.9}$$

Choose k large enough such that $k\epsilon E(\epsilon)/M > M$. Now (E3) and (3.9) imply that

$$\begin{aligned} H_{n_{k+1}}(x) - H_{n_{k+1}}(x^T) &\ge \sum_{i=1}^k \int_0^T E(|A_{n_i}(x^t)|)dt + \int_0^T A_{n_{k+1}}(x^t)dt \\ &\ge k\epsilon T E(\epsilon)/M - \int_0^T |A_{n_{k+1}}(x^t)|dt \\ &> MT - MT = 0, \end{aligned}$$

which is in contradiction to the assumption. □

The proposition above says that the local property (E1) can be extended arbitrarily far along a trajectory, and that the functions H_n indeed resemble Lyapunov functions of a semiflow. It is important that the property (iii) above can not in general be extended to a small neighbourhood of $x^{[0,T]}$ (one can easily check it for the semiflows in Examples 2 and 3, assuming τ_p and τ_{loc} topologies respectively).

Corollary 3.3. *Given an extended gradient semiflow ϕ, there exists no periodic orbits with a minimal period $T > 0$.*

Proof. Assume that x is on such an orbit with a period $T > 0$. Proposition 3.2 implies now that for n large enough, $H_n(x) < H_n(x^T) = H_n(x)$, which is a contradiction. □

Example 4: Corollary 3.3 implies non-existence of periodic solutions for the equation (2.6) on the spaces $\mathcal{B}_N$, $\mathcal{B}_N/T$ and for the equation (2.7) on the spaces $\mathcal{A}_C$.

It would be wrong to deduce at this moment that the dynamics of extended gradient systems is not interesting.

We conjecture that homoclinic orbits in an extended gradient system can exist. In particular, the family of "coarsening" orbits of the equation (2.7), constructed by Eckmann and Rougemont ([ER98]), is the family of orbits asymptotic to a homoclinic cycle (though a rigourous proof of that fact is still lacking). Further discussion of such questions will appear in [GS99].

However, an extended gradient system can never be chaotic, as will be shown in section 5.

4. Non-recurrent orbits. Now we give a sufficient condition for a point of a gradient extended system to be non-recurrent. Recall that a point x of a semiflow ϕ is recurrent, if for each neighbourhood U of x and each T there exists $t > T$ such that $x^t \in U$. It is easy to check that, given a continuous semiflow ϕ, if x is recurrent, so is x^t for each $t > 0$.

Proposition 4.1. *Given $x \in \mathcal{A}$, assume that there exists a neighbourhood $\mathcal{U}$ of x, a time T and a strictly increasing sequence of integers $(n_k)_{k\in\mathbf{N}}$ such that for each k,*

$$H_{n_k}(x^T) < \inf_{u\in\mathcal{U}} H_{n_k}(u). \tag{4.10}$$

Then for each $t \geq T$, $x^t \notin \mathcal{U}$, and x is not recurrent.

Proof. Assume that for some $t > T$, $x^t \in \mathcal{U}$. Then (4.10) implies that for each k, $H_{n_k}(x^t) > H_{n_k}(x^T)$, which is in contradiction to Proposition (3.2), (iii). □

Note that we actually proved more, i.e. that any point x satisfying (4.10) can not be in the ω-limit set of any point.

Example 5: It is easy to check, using the definitions of H_n and A_n in Appendices A and B and Proposition 4.1, that every point $u \in \mathcal{B}_N$ (assuming τ_∞ topology) satisfying

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=-k}^{k} \dot{u}_i^2 > 0 \tag{4.11}$$

is non-recurrent for the semiflow induced by (2.6), and that every point $u \in \mathcal{A}_C$ (assuming τ_∞ topology) satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \int_{-n}^{n} u_t^2 > 0. \tag{4.12}$$

is non-recurrent for the semiflow induced by (2.7).

Remark 1: *Topologically transitive extended gradient systems.* It is possible to construct a semiflow satisfying (E1-3) with a recurrent, and even a dense orbit. An example is an irrational flow on the two-torus, with a time change in a small neighbourhood of a point, which becomes a fixed point (the flow is constructed in [KH95], Exercise 4.1.9). A construction of a Lyapunov sequence (H_n, A_n) of functions for that semiflow is in [Sli99a].

We conjecture that in particular examples such as those in Appendices, one can give much stronger characterisation of non-recurrent orbits than the one in Proposition 4.1. This does not, however, follow from the extended-gradient structure of the semiflow alone, and other properties of the dynamics (order-preserving, invariance for spatial translations) must be taken into account.

5. Return times and invariant measures. We now prove the main result of the paper, and describe all the invariant measures of an extended gradient system. An important question is not only whether an orbit returns infinitely many times to a neighbourhood of x, but how often it happens *on average*.

We define the upper return time $r_{\mathcal{U}}(x)$ of an orbit of x to a set $\mathcal{U}$ by:

$$r_{\mathcal{U}}(x) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{U}}(x^t)dt,$$

where $\mathbf{1}_{\mathcal{U}}$ denotes the characteristic function of $\mathcal{U}$.

Theorem 5.1. *Given $y \in \mathcal{A}-\mathcal{S}$, there exists a neighbourhood $\mathcal{U}$ of y such that for each $x \in \mathcal{A}$, $r_{\mathcal{U}}(x) = 0$.*

Proof. Choose $y \in \mathcal{A}-\mathcal{S}$, $\mathcal{U} \in \mathcal{F}$ containing y, ϵ and n_0 satisfying (E1) on $\mathcal{U}$, and assume that $r_{\mathcal{U}}(x) = \delta > 0$. (E3) implies that for each $n \in \mathbf{N}$, $T > 0$,

$$H_n(x) - H_n(x^T) - \int_0^T A_n(x^t)dt \geq 0. \tag{5.13}$$

We can find a sequence of sets

$$R_k = [0, t_1] \cup [t_2, t_3] \cup \ldots \cup [t_{2k-2}, t_{2k-1}],$$

$0 = t_0 < t_1 < t_2 < \ldots < t_{2k-1} \leq t_{2k} = T$ such that $t \in R_k$ implies $x^t \in \mathcal{U}$ (i.e. $\mathbf{1}_{R_k}(t) \leq \mathbf{1}_{\mathcal{U}}(x^t)$), and such that $\mathbf{1}_{R_k}(t) \to \mathbf{1}_{\mathcal{U}}(x^t)$ as $k \to \infty$ a.e. with respect to the Lebesgue measure on $[0,T]$. We now get, applying (5.13) and (E1), that for all $n \geq n_0$

$$\begin{aligned} H_n(x) - H_n(x^T) - \int_0^T A_n(x^t)dt &= \sum_{i=0}^{2k-1}\left(H_n(x^{t_{i+1}}) - H_n(x^{t_i}) - \int_{t_i}^{t_{i+1}} A_n(x^t)dt\right) \\ &\geq \sum_{i=0}^{k-1}\left(H_n(x^{t_{2i+1}}) - H_n(x^{t_{2i}}) - \int_{t_{2i}}^{t_{2i+1}} A_n(x^t)dt\right) \\ &\geq \epsilon\sum_{i=0}^{k-1}(t_{2i+1} - t_{2i}) = \int_0^T \epsilon\mathbf{1}_{R_k}(t)dt. \end{aligned}$$

Lebesgue dominated convergence theorem now implies:

$$H_n(x) - H_n(x^T) - \int_0^T A_n(x^t)dt \geq \int_0^T \epsilon\mathbf{1}_{\mathcal{U}}(x^t)dt. \tag{5.14}$$

We find a strictly increasing sequence of times T_k such that

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{U}}(x^t)dt = \lim_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}\mathbf{1}_{\mathcal{U}}(x^t)dt = \delta.$$

Since H_n is bounded, we conclude that for each $n > 0$,

$$\lim_{k\to\infty}\frac{1}{T_k}(H_n(x) - H_n(x^{T_k})) = 0, \tag{5.15}$$

and then relation (5.14) implies that

$$\begin{aligned}\liminf_{k\to\infty}\frac{1}{T_k}\left|\int_0^{T_k} A_n(x^t)dt\right| &\geq \lim_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}\epsilon\mathbf{1}_{\mathcal{U}}(x^t)dt \\ &= \delta\epsilon. \end{aligned}\tag{5.16}$$

Now, denote by $S_n = \{t,\ |A_n(x^t)| \geq \delta\epsilon/2\}$. We calculate, applying (5.16):

$$\begin{aligned}\delta\epsilon &\leq \liminf_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}|A_n(x^t)|dt \\ &\leq \liminf_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}\mathbf{1}_{S_n}|A_n(x^t)|dt + \limsup_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}\mathbf{1}_{S_n^c}|A_n(x^t)|dt \\ &\leq M\liminf_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}\mathbf{1}_{S_n}dt + \delta\epsilon/2,\end{aligned}$$

and therefore for each $n \geq n_0$,

$$\liminf_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}\mathbf{1}_{S_n}dt \geq \delta\epsilon/2M, \tag{5.17}$$

where M is the constant from (E2).

Choose $n > n_0$ large enough such that $(n-n_0)E(\delta\epsilon/2)\delta\epsilon/2M > M$. Now we get, applying (5.17) and (E3):

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{T_k}(H_{n+1}(x) - H_{n+1}(x^T)) &\geq \liminf_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}A_n(x^t)dt \\ &\quad + \liminf_{k\to\infty}\frac{1}{T_k}\sum_{k=1}^{n}\int_0^{T_k}E(|A_n(x^t)|)dt \\ &\geq -M + \liminf_{k\to\infty}\frac{1}{T_k}\sum_{k=n_0}^{n}\int_0^{T_k}\mathbf{1}_{S_n}E(|A_n(x^t)|)dt \\ &\geq -M + E(\delta\epsilon/2)\sum_{k=n_0}^{n}\liminf_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}\mathbf{1}_{S_n}dt \\ &\geq -M + (n-n_0)E(\delta\epsilon/2)\delta\epsilon/2M > 0,\end{aligned}$$

which is in contradiction to (5.15). □

We conclude that a neighbourhood of each non-stationary point in $\mathcal{A}$ is visited never on average. A conclusion is that, while a gradient semiflow is asymptotically stationary (i.e. the limit set of each point consists of stationary points), a gradient extended system is asymptotically stationary on average. This is an example of a more general philosophy that a topological argument for systems on bounded domains should be replaced with a statistical (i.e. measure-theoretical) statement when discussing extended systems.

We clarify it in the following statements involving invariant measures. Recall that we say that a Borel probability measure is ϕ-invariant, if for each Borel-measurable A and $t \geq 0$, $\mu(A) = \mu(\phi^{-t}(A))$.

Corollary 5.2. *Assume that ϕ is an extended gradient semiflow on $\mathcal{A}$, and that ϕ is a Borel ϕ-invariant probability measure. If $\mathcal{U}$ is a countable union of elements of $\mathcal{F}$, then $\mu(\mathcal{U}) = 0$.*

Proof. It is enough to show that for each $\mathcal{U} \in \mathcal{F}$, $\mu(\mathcal{U}) = 0$, which follows immediately from the definition of $r_{\mathcal{U}}$, the fact that for each $y \in \mathcal{A}$, $r_{\mathcal{U}}(y) = 0$ (Theorem 5.1), and the Birkhoff Ergodic Theorem. □

Recall that a topological space satisfies the second axiom of countability, if it has a countable basis.

Corollary 5.3. *Assume that $\mathcal{A}$ is a second-countable space, and ϕ an extended gradient semiflow on $\mathcal{A}$. Then each Borel ϕ-invariant probability measure μ is supported on the set of stationary orbits, i.e. $\mu(\mathcal{S}) = 1$.*

Proof. Since $\mathcal{A}$ satisfies the second axiom of countability, we can find a countable subfamily of $\mathcal{F}$ covering $\mathcal{A} - \mathcal{S}$. Corollary 5.2 now implies that $\mu(\mathcal{A} - \mathcal{S}) = 0$. □

Corollary 5.4. *Assume that $\mathcal{A}$ is compact, metrisable, and that ϕ is an extended gradient semiflow. Then for each $x \in \mathcal{A}$, the ω-limit set $\omega(x)$ contains a stationary point.*

Proof. Since $\omega(x)$ is a closed ϕ-invariant subset of $\mathcal{A}$, and $\mathcal{A}$ is compact, $\omega(x)$ is compact and non-empty, and the Bogolyuboff-Kryloff Theorem (see e.g. [CFS82]) now implies that there exists a Borel ϕ-invariant probability measure μ supported on $\omega(x)$. Corollary 5.3 implies that μ is supported on $\mathcal{S}$, therefore $\mathcal{S} \cap \omega(x)$ is non-empty. □

Finally, we describe all compact minimal sets of an extended gradient system. Recall that, given a semiflow ϕ, a closed ϕ-invariant set is minimal if it does not contain proper closed ϕ-invariant sets.

Corollary 5.5. *Assume that $\mathcal{A}$ is metrisable, and that ϕ is an extended gradient semiflow. Then every compact minimal set consists of a single equilibrium.*

Proof. Given a compact minimal set $\mathcal{X}$, Corollary 5.4 implies that it contains a stationary point x. The definition of the minimal set implies that $\{x\}$ can not be a proper subset of $\mathcal{X}$, hence $\mathcal{X} = \{x\}$. □

An alternative proof of Corollary 5.5 is to use the argument by Furstenberg: every minimal set consists of uniformly recurrent orbits (see [Fur81], section 1.4 for definition of uniform recurrence, and Theorem 1.15 therein); Theorem 5.1 implies that the only uniformly recurrent points of an extended gradient system are stationary points.

Example 6: Corollary 5.3 implies that all the Borel ϕ-invariant probability measures of the semiflows ϕ induced by equations (2.6) and (2.7) on spaces $\mathcal{B}_N/T$, $\mathcal{A}_C$ with τ_p, τ_{loc} topologies respectively, are supported on the set of stationary points.

Remark 2: *On Golé's New proof of Aubry-Mather theorem.* An important step in Golé's proof is the claim that, for the dynamics (2.6), ω-limit set of each point contains a stationary point. The proof of that step in [Gol92] is incorrect (in [Gol98], it was replaced with a weaker statement). Corollary 5.4 recovers the original proof from [Gol92].

6. Spatially extended gradient systems. Now we add an additional feature to our discussion, and discuss the systems that are spatially extended. The spatial structure is represented with the action of a group G on the phase space. which will in the applications usually be the group of spatial translations, $G \cong \mathbf{Z}$ or $\mathbf{R}$.

Recall that we say that a group G acts on a set $\mathcal{A}$, if for each $a \in \mathcal{A}$, $ea = a$ (where $e \in G$ is identity), and for each $g_1, g_2 \in G$, $(g_1 g_2)a = g_1(g_2 a))$. We say that G acts continuously on the space $\mathcal{A}$, if $g : \mathcal{A} \to \mathcal{A}$ is continuous for each $g \in G$.

Definition 6.1. *Given a group G acting on $\mathcal{A}$, we say that a semiflow ϕ on $\mathcal{A}$ is a G-extended gradient semiflow, if ϕ is a continuously extended gradient semiflow satisfying the following:*

- **(S1):** *G acts continuously on $\mathcal{A}$, and commutes with the semiflow ϕ, i.e. for each $g \in G$, $t \geq 0$,*
$$g \circ \phi^t = \phi^t \circ g.$$
- **(S2):** *For each $n, k \in \mathbf{N}$, there exists $m \geq nk$, and group elements $g_1, g_2, ..., g_k \in G$ such that*
$$\begin{array}{rcl} H_m & = & \sum_{i=1}^k H_n \circ g_i, \\ A_m & = & \sum_{i=1}^k A_n \circ g_i. \end{array} \tag{6.18}$$

In applications, (S1) will mean that a system of equations is spatially homogeneous, and (S2) that the choice of H_n, A_n is compatible with the spatial structure.

Example 7: The equation (2.6) induces a $\mathbf{Z}$-extended gradient semiflow, where $\mathbf{Z}$ acts as the group of spatial translations S^n, $n \in \mathbf{Z}$, $S(u)_i = u_{i-1}$. Similarly, the equation (2.7) induces a $\mathbf{Z}$- (and $\mathbf{R}$-) extended gradient semiflow (see Appendices for details).

We denote by $M_G(\mathcal{A})$ the set of all Borel probability measures on the space $\mathcal{A}$, invariant for the group G, i.e. Borel probability measures μ, such that for each $g \in G$, and any measurable $A \subset \mathcal{A}$,
$$\mu(gA) = \mu(A).$$
Note that the set $M_G(\mathcal{A})$ depends on the topology on $\mathcal{A}$, and that it is not a-priori clear that $M_G(\mathcal{A})$ is non-empty. If $\mathcal{A}$ is compact and metrisable, and $G = \mathbf{Z}$ or $\mathbf{R}$, then the Bogolyuboff-Kryloff Theorem implies that $M_G(\mathcal{A})$ is non-empty (this will be the case in most applications). We implicitly assume that $M_G(\mathcal{A})$ is non-empty in the following.

Now we can define the induced semiflow ϕ^* on $M_G(\mathcal{A})$ as
$$\phi^{*t}\mu = \mu \circ \phi^{-t}. \tag{6.19}$$
The condition (S1) implies that $\mu \in M_G(\mathcal{A})$ implies $\phi^{*t}\mu \in M_G(\mathcal{A})$, hence the flow ϕ^{*t} is well-defined.

Denote by $M_G(\mathcal{S})$ the set of all Borel probability G-invariant measures supported on the set of stationary points $\mathcal{S}$, $M_G(\mathcal{S}) \subset M_G(\mathcal{A})$.

We assume in the following that ϕ is a G-extended gradient semiflow on $\mathcal{A}$ with a Lyapunov sequence (H_n, A_n).

Proposition 6.1. *Given any $\mu \in M_G(\mathcal{A})$, $T \geq 0$, and $n \in \mathbf{N}$. Then*
$$\int_{\mathcal{A}} \int_0^T A_n(x^t) dt d\mu(x) = 0.$$

Proof. We apply (S2), and for some $k > 0$ we find $m \geq nk$ and $g_1, g_2, ..., g_k$ such that (6.18) is satisfied. Now, applying G-invariance of μ, (S1) and (S2) respectively, we get

$$\begin{aligned}\int_{\mathcal{A}} \int_0^T A_n(x^t) dt d\mu(x) &= \frac{1}{k} \sum_{i=1}^k \int_{\mathcal{A}} \int_0^T A_n(\phi^t(g_i(x))) dt d\mu(x) \\ &= \frac{1}{k} \int_{\mathcal{A}} \int_0^T \left(\sum_{i=1}^k A_n \circ g_i(x^t)) \right) dt d\mu(x) \\ &= \frac{1}{k} \int_{\mathcal{A}} \int_0^T A_m(x^t) dt d\mu(x).\end{aligned}$$

If M is the uniform bound from (E2), we get:

$$\left| \int_{\mathcal{A}} \int_0^T A_n(x^t) dt d\mu(x) \right| \leq \frac{MT}{k}.$$

Since k can be arbitrarily large, the claim is proved. □

Theorem 6.2. *The semiflow ϕ^* on $M_G(\mathcal{A})$ defined in (6.19) is strictly gradient, and*

$$L_n(\mu) = \int_{\mathcal{A}} H_n d\mu$$

is a Lyapunov function for any $n \in \mathbf{N}$. The set of stationary points of ϕ^* is $M_G(\mathcal{S})$.

Proof. Choose $\mu \in M_G(\mathcal{A}) - M_G(\mathcal{S})$, $n \in \mathbf{N}$, $T > 0$. It is sufficient to show that $L_n(\mu^T) < L_n(\mu)$.

For a given $k \in \mathbf{N}$, denote by m_k the number $m_k \geq nk$ such that (S2) is satisfied. We define

$$D_k(x) = H_{m_k}(x^T) - H_{m_k}(x) + \int_0^T A_n(x^t) dt.$$

The condition (E3) implies that $D_k \leq 0$, and Proposition 3.2, (i), that for all $x \in \mathcal{A} - \mathcal{S}$, $\liminf_{k \to \infty} D_k(x) < 0$. Therefore we can find k large enough and $\epsilon > 0$ such that

$$\int_{\mathcal{A}} D_k(x) d\mu(x) \leq -\epsilon.$$

Now Proposition 6.1 implies that

$$\int_{\mathcal{A}} H_{m_k}(x^T) d\mu - \int_{\mathcal{A}} H_{m_k}(x) d\mu \leq -\epsilon.$$

Applying G-invariance of ϕ, (S1) and (S2), we deduce similarly as in the proof of Proposition (6.1):

$$\begin{aligned}L_n(\mu^T) - L_n(\mu) &= \int_{\mathcal{A}} H_n(x^T) d\mu - \int_{\mathcal{A}} H_n(x) \\ &= \frac{1}{k} \left(\int_{\mathcal{A}} H_{m_k}(x^T) d\mu - \int_{\mathcal{A}} H_{m_k}(x) \right) \\ &\leq -\frac{\epsilon}{k} < 0.\end{aligned}$$

□

In the following, we assume the weak*-topology on the set of measures $M_G(\mathcal{A})$. In the following, $\omega(\mu)$ denotes the ω-limit set of μ with respect to the semiflow ϕ^* and weak*-topology.

Corollary 6.3. *Assume that $\mathcal{A}$ is compact, metrisable. Then the semiflow ϕ^* defined in (6.19) is weak*-continuous, $M_G(\mathcal{A})$ is weak*-compact, for each $\mu \in M_G(\mathcal{A})$, $\omega(\mu)$ is non-empty, and $\omega(\mu) \subset M_G(\mathcal{S})$. Furthermore, $\omega(\mu)$ contains either one or infinitely many measures.*

Proof. Continuity of ϕ^* follows from the definition of weak*-topology. The set of all Borel probability measures on a compact metrisable set is weak*-compact and metrisable (see e.g. [Wal82], Theorems 6.4 and 6.5), and since $M_G(\mathcal{A})$ is weak*-closed, it is weak*-compact. Other claims follow from Theorem 6.2 and the well-known properties of gradient semiflows on compact sets (see e.g. [KH95], Prop. 1.6.3 and 1.6.4). □

7. Limit sets of G-extended gradient semiflow. Here we show that if a semiflow ϕ is a spatially extended gradient semiflow in the sense of Definition 6.1, then we can show that the asymptotic behaviour of ϕ is "almost everywhere" analogous to the behaviour of a gradient semiflow.

Theorem 7.1. *Assume that ϕ is a G-extended gradient semiflow on $\mathcal{A}$, and that $\mu \in M_G(\mathcal{A})$. Then $\omega(x) \subset \mathcal{S}$ for μ-almost all $x \in \mathcal{A}$.*

Proof. Denote by $\mathcal{X}$ the set of all $x \in \mathcal{A}$ such that $\omega(x)$ is not a subset of $\mathcal{S}$.

Step 1. We first describe $\mathcal{X}$. Choose $x \in \mathcal{X}$, and find $y \in \omega(x) \cap (\mathcal{A} - \mathcal{S})$, a neighbourhood $\mathcal{U} \in \mathcal{F}$ of y, ϵ and n_0 such that (E1) is satisfied. We can find a neighbourhood $V \subset U$ of y and $\delta > 0$ such that $z \in V$ implies $z^{[0,\delta]} \subset U$. Now we define

$$D_n^{m,\delta} = H_n(x^{m\delta}) - H_n(x^{(m+1)\delta}) - \int_{m\delta}^{(m+1)\delta} A_n(x^t)dt.$$

Now (E3) implies that $D_n^{m,\delta} \geq 0$, and the choice of V and (E1) imply the following: if $x^{[m\delta,(m+1)\delta]} \cap V \neq \emptyset$, then for all $n \geq n_0$,

$$D_n^{m,\delta} \geq \delta\epsilon/2 \quad \text{or} \quad D_n^{m+1,\delta} \geq \delta\epsilon/2.$$

Since $y \in \omega(x)$, there exists a sequence m_i such that $D_n^{m_i,\delta} \geq \delta\epsilon/2$. Let $\mathcal{X}_{\delta,\epsilon,n}$ be the set of all $x \in \mathcal{A}$, such that there exists a sequence $m_i \to \infty$ such that $D_n^{m_i,\delta} \geq \delta\epsilon/2$. We conclude that

$$\mathcal{X} \subset \bigcup_{\substack{\delta,\epsilon>0,\\ n\in\mathbf{N}}} \mathcal{X}_{\delta,\epsilon,n}. \tag{7.20}$$

Step 2. We now show that $\mu(\mathcal{X}_{\delta,\epsilon,n}) = 0$. Proposition 6.1 implies that

$$a_m := \int_{\mathcal{A}} D_n^{m,\delta}(x)d\mu(x) = \int_{\mathcal{A}} \left(H_n(x^{m\delta}) - H_n(x^{(m+1)\delta})\right) d\mu(x).$$

Now Theorem 6.2 implies that $a_m \geq 0$ for all $m \in \mathbf{N}$, and (E2) (i.e. the fact that H_n is bounded) implies that the series $\sum_{m\in\mathbf{Z}} a_m$ is convergent. Therefore for each $\xi > 0$, we can find $m_0 \in \mathbf{N}$ such that

$$\sum_{m\geq m_0} a_m \leq \xi.$$

A simple calculation implies easily that

$$\mu(D_n^{m,\delta} \geq \delta\epsilon/2) \leq \frac{2}{\delta\epsilon}\int_{\mathcal{A}} D_n^{m,\delta}(x)d\mu(x) = \frac{2}{\delta\epsilon}a_m.$$

Therefore

$$\mu(\mathcal{X}_{\delta,\epsilon,n}) \leq \mu(D_n^{m,\delta} \geq \delta\epsilon/2 \text{ for some } m > m_0) \leq \frac{2}{\delta\epsilon} \sum_{m \geq m_0} a_m \leq \frac{2}{\delta\epsilon}\xi.$$

Since ξ was arbitrarily small, $\mu(\mathcal{X}_{\delta,\epsilon,n}) = 0$.

Since $\mathcal{X}_{\delta,\epsilon,n}$ is monotone in δ, ϵ, (7.20) implies that $\mu(\mathcal{X}) = 0$. □

Example 8: *Eckmann - Rougemont coarsening orbits.* Let ϕ be the G-extended gradient semiflow generated by (2.7) on $\mathcal{A}_C$ for some C large enough (G is the group of spatial translations T_y, $y \in \mathbf{Z}$). Let μ_B be the Bernoulli measure on the space Σ of all sequences $(\eta_i)_{i \in \mathbf{Z}}$, $\eta_i \in \{-1, 1\}$, and $\pi : \Sigma \to \mathcal{C}_{u,b}$ the map such that $\pi(\eta)$ is a piecewise-linear function $u(x)$, $u(i) = \eta_i$. We assume the product topology on Σ and τ_{loc} topology on $\mathcal{C}_{u,b}$, and then π is continuous. We set $\mu = \mu_B \circ \pi^{-1}$, and then μ is G-invariant. Regularity properties of (2.7) imply that for each $t > 0$, μ^t is supported on $\mathcal{A}_C$ for C large enough. Now Theorem 7.1 implies that for μ-a.e. $x \in \mathcal{A}$, $\omega(x) \subset \mathcal{S}$.

Eckmann and Rougemont in [ER98] showed that for certain initial conditions x of the type $\pi(\eta)$, both the solutions $u \equiv -1$ and $u \equiv 1$ are in the τ_{loc} ω-limit set of x (the "coarsening" dynamics), and that $\omega(x)$ is not a subset of $\mathcal{S}$. We showed that the set of such initial conditions has μ-measure 0.

8. Discussion. We expect that the results proved in this paper apply to a large class of parabolic and damped hyperbolic differential equations. We apply the ideas to particular examples (simplifying and modifying some arguments), and suggest generalisations and counterexamples in higher dimensions in the forthcoming paper [GS99].

We now briefly discuss some other possible applications of the results.

Variational problems. A standard difficulty when solving variational problems on unbounded domains is that the functional in question is not well defined, i.e. it is divergent. A novel approach to variational problems on unbounded domains which we suggest is to minimise the induced functional (well-defined!) on the subset of the space of all measures invariant for spatial translations, invariant for the induced semiflow ϕ^*. An example where the idea could be applied is the following.

Hamiltonian dynamics. We can study the gradient dynamics of the action functional of a Lagrangian system (a flow or a map); the gradient dynamics of Frenkel-Kontorova models (see Appendix A) as action functionals for twist maps is an example. Since the time is one-dimensional, the theory developed in this paper can be applied to that dynamics, where the Lagrangian system can have arbitrarily many degrees of freedom. Corollaries 5.4 and 6.3 are powerful new tools for construction of orbits and invariant measures respectively of Lagrangian flows and maps.

A new proof of existence of Mather's shadowing orbits of twist maps ([Mat91b]) using the dynamics (2.6) is given in [Sli99b]. Corollary 5.4 is a step towards generalisation of the result to Lagrangian systems with more than one degree of freedom.

In [Sli99a], the author gives a simple construction of action-minimising measures of Lagrangian maps (the analogue of Mather's results in [Mat91a] for Lagrangian flows) based on Corollary 6.3, and discusses related open problems.

Appendix A. Gradient dynamics of Frenkel-Kontorova models

The Frenkel-Kontorova model is defined with a C^2 generating function $V : \mathbf{R}^2 \to \mathbf{R}$ satisfying the following conditions:

(i) **Periodicity**: for each $x, y \in \mathbf{R}^2$, $V(x+1, y+1) = V(x, y)$;
(ii) **The twist condition**: there exists $\delta > 0$, such that $V_{12}(x, y) \le -\delta$,

where indices denote partial derivatives. The Frenkel-Kontorova model is defined with the following formal sum $V(u) = \sum_{i \in \mathbf{Z}} V(u_{i-1}, u_i)$, where $u \in \mathbf{R}^{\mathbf{Z}}$. The Frenkel-Kontorova model is studied in a number of physical applications ([FM96]), as well as the action functional for twist maps, in particular in Aubry-Mather theory ([MF94]).

The dynamics is defined with the following system of equations (the Cauchy problem):

$$\begin{aligned} \dot{u}_i &= \nabla V(u) = -V_2(u_{i-1}, u_i) - V_1(u_i, u_{i+1}), \qquad (8.21) \\ u_0 &= u^0, \end{aligned}$$

$u^0 \in \mathbf{R}^{\mathbf{Z}}$ (a Cauchy problem). We study the system in the space

$$\mathcal{B}_N = \{u \in \mathbf{R}^{\mathbf{Z}}, \ : \ \sup_{i \in \mathbf{Z}} |u_i - u_{i+1}| \le N\},$$

where $N \in \mathbf{N}$. We are interested in two topologies on $\mathcal{B}_N$, namely τ_∞-topology, generated by the norm $\sup_{i \in \mathbf{Z}} |u_i|$ (in this topology, $\mathcal{B}_N$ is a Banach manifold); and the product topology τ_p induced by the product topology on $\mathbf{R}^{\mathbf{Z}}$.

One can show that the conditions (i), (ii) imply that the equation (8.21) generates a semiflow ϕ on $\mathcal{B}_N$ for each $N \ge \mathbf{N}$, and that ϕ is τ_∞- and τ_p-continuous (see [Sli99a] for details).

Given the translation $T : \mathbf{R}^{\mathbf{Z}} \to \mathbf{R}^{\mathbf{Z}}$, $T(u)_i = u_i + 1$, periodicity of V implies that ϕ commutes with T, hence we can study the induced semiflow on the quotient space $\mathcal{B}_N/T$ (we here abuse the notation and denote by T the group of translations generated by T). Tychonoff theorem implies that the space $\mathcal{B}_N/T$ is τ_p-compact. We set $\mathcal{A} = \mathcal{B}_N/T$, and define:

$$\begin{aligned} H_n(u) &= \sum_{k=-n}^{n-1} V(u_k, u_{k+1}), \qquad (8.22) \\ A_n(u) &= -V_1(u_{-n}, u_{-n+1})\dot{u}_{-n} + V_1(u_n, u_{n+1})\dot{u}_n \\ &= V_1(u_{-n}, u_{-n+1})(V_2(u_{-n-1}, u_{-n}) + V_1(u_{-n}, u_{-n+1})) \qquad (8.23) \\ &\quad -V_1(u_n, u_{n+1})(V_2(u_{n-1}, u_n) + V_1(u_n, u_{n+1}), \qquad (8.24) \end{aligned}$$

$H_n, A_n : \mathcal{A} \to \mathbf{R}$. Clearly, H_n, A_n are τ_p- and τ_∞ continuous, and τ_p-compactness of $\mathcal{A}$ implies (E2). Differentiating $\mathrm{d}H_n/\mathrm{d}t$ and substituting (8.21), we get

$$\frac{\mathrm{d}H_n(u^t)}{\mathrm{d}t} + A_n(u) \le -\sum_{k=-n+1}^{n} \dot{u}_k^2. \qquad (8.25)$$

Now, given $u \notin \mathcal{S}$, we can find n_0 large enough such that

$$\sum_{k=-n_0+1}^{n_0} \dot{u}_k^2 \ge 2\epsilon > 0,$$

and a τ_p (or τ_∞) neighbourhood $\mathcal{U}_u$ of u such that for all $v \in \mathcal{U}_u$,

$$\sum_{k=-n_0+1}^{n_0} \dot{v}_k^2 \geq \epsilon.$$

Now if we define $\mathcal{F} = \{\mathcal{U}_u, \; : \; u \in \mathcal{A} - \mathcal{S}\}$, (2.4) and then (E1) are satisfied.

We can uniformly bound $|V_1(x,y)|$ on the set $\{|x-y| \leq N\}$ by a constant $C > 0$. Now the relation

$$A_n^2 = (-V_1(-u_{-n}, u_{-n+1})\dot{u}_{-n} + V_1(u_n, u_{n+1})\dot{u}_n)^2 \leq 2C(\dot{u}_n^2 + \dot{u}_{-n}^2)$$

and (8.25) imply (2.5) and (E3), with $E(x) = \frac{1}{2C}x^2$. We conclude:

Proposition 8.1. *The equation (8.21) induces a continuously extended gradient semiflow on spaces $\mathcal{B}_N$, $\mathcal{B}_N/T$, where $N \in \mathbf{N}$, in either τ_p or τ_∞ topology.*

We note that $\mathcal{B}_N/T$ (with τ_p topology) is homeomorphic to $[-N,N]^{\mathbf{N}} \times \mathbf{S}^1 \times [-N,N]^{\mathbf{N}}$, and therefore τ_p-compact and metrisable.

The definition of H_n and A_n implies immediately that the dynamics 8.21 satisfies (S1), (S2), and induces a G-extended gradient semiflow, where G is the group of translations S^n, $n \in \mathbf{Z}$; $S(u)_i = u_{i-1}$.

Appendix B. The real Ginzburg-Landau equation

The real 1D Ginzburg-Landau equation is given by the equations

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + u(x,t) - u^3(x,t) && (8.26)\\ u(x,0) &= u^0, && (8.27) \end{aligned}$$

where indices as usual denote the partial derivatives, $u : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$, and $u^0 : \mathbf{R} \to \mathbf{R}$ is the initial condition ((8.26), (8.27) is a Cauchy problem). We here outline the proof that (8.26) induces a gradient extended semiflow; detailed discussion will appear in [GS99].

Denote by $\mathcal{C}_{b,u}$ the space of all bounded, uniformly continuous functions on $\mathbf{R}$. Denote by τ_∞ the topology on $\mathcal{C}_{b,u}$ induced by the norm

$$||f||_\infty = \sup_{x \in \mathbf{R}} |f(x)|.$$

Let $s : \mathbf{R} \times (0,1]$ be an integrable function, say $s(x) = (1+x^2)^{-1}$. Denote by τ_{loc} the topology on $\mathcal{C}_{b,u}$, induced by the norm

$$||f||_s = \sup_{x \in \mathbf{R}} s(x)|f(x)|.$$

It is well-known that (8.26) induces a τ_∞- and τ_{loc}-continuous semiflow ϕ on $\mathcal{C}_{b,u}$ (see [Col94] and references therein).

Denote by $\tilde{\mathcal{A}}_C$, the subset of $\mathcal{C}_{b,u}$, such that u, u_t, u_x, u_{tt}, u_{tx} are well-defined and bounded in $||.||_\infty$ by C. Denote by $\mathcal{A}_C$ the smallest ϕ-invariant τ_∞-closed set containing $\tilde{\mathcal{A}}_C$. It is easy to show (e.g. using the integral representation of the solution of (8.26), (8.27)) that there exists $D > 0$ such that $\mathcal{A}_C \subset \tilde{\mathcal{A}}_D$ (i.e. that D is the uniform bound on u, u_t, u_x, u_{tt}, u_{tx}), and that $\mathcal{A}_C$ is τ_{loc}-closed.

For a given $C > 0$, we define $H_n, A_n : \mathcal{A}_C \to \mathbf{R}$,

$$\begin{aligned} H_n(u) &= \int_{-n}^{n} \left(\frac{u_x^2}{2} - \frac{u^2}{2} + \frac{u^4}{4} \right) dx, && (8.28)\\ A_n(u) &= -u_x(-n)u_t(-n) + u_x(n)u_t(n). && (8.29) \end{aligned}$$

The definition of the set $\mathcal{A}_C$ implies (E2). Now, partially integrating, we get:

$$\frac{\mathrm{d}H_n(u)}{\mathrm{d}t} + A_n(u) = -\int_{-n}^{n} u_t^2 dx. \tag{8.30}$$

Analogously as in Appendix A, we deduce that A_n, H_n are τ_∞- and τ_{loc}- continuous, and that (E1) is satisfied.

Recall that, if $u \in \mathcal{A}_C$, that D is the uniform bound on $|u_x|$, $|u_t|$, $|u_{tx}|$. Then for each $y > x$, the mean value theorem implies that $u_t^2(y) \geq u_t^2(x) - D^2(y-x)$. A simple calculation implies now that:

$$\int_{n-1}^{n} u_t^2 dx \geq \tilde{E}(|u_t(n)|), \tag{8.31}$$

where $\tilde{E} : \mathbf{R}^+ \to \mathbf{R}^+$ is defined with:

$$\tilde{E}(x) := \begin{cases} \frac{x^4}{2D^2}, & \text{for } x \leq D^2, \\ x^2 - \frac{D^2}{2} & \text{for } x \geq D^2. \end{cases}$$

It is easy to check that $\tilde{E}$ is strictly increasing, convex, and $\tilde{E}(0) = 0$. Using (8.31), convexity and monotonicity of $\tilde{E}$, we get:

$$\begin{aligned} \int_{-n}^{-n+1} u_t^2 dx + \int_{n-1}^{n} u_t^2 dx &\geq \tilde{E}(|u_t(-n)|) + \tilde{E}(|u_t(n)|) \\ &\geq 2\tilde{E}(\frac{1}{2}(|u_t(-n)| + |u_t(n)|)) \\ &\geq 2\tilde{E}(\frac{1}{2D}| - u_x(-n)u_t(-n) + u_x(n)u_t(n)|) \\ &= 2\tilde{E}(\frac{1}{2D}|A_n(u)|). \end{aligned} \tag{8.32}$$

We define $E(x) = 2\tilde{E}(\frac{1}{2D}x)$, and substituting (8.32) into (8.30) we prove (E3).

We conclude:

Proposition 8.2. *For each $C > 0$, the equation (8.26) induces a continuously extended gradient semiflow on $\mathcal{A}_C$, in either τ_∞ or τ_{loc} topology.*

We note that $\mathcal{A}_C$ is τ_{loc}-compact, metric space.

We denote by T_y the translation $T_y(u)(x) = u(x-y)$. The definition of H_n, A_n implies that (8.26) induces a G-extended gradient semiflow, where G is the group of translations T_y, where either $y \in \mathbf{Z}$ or $y \in \mathbf{R}$.

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