Holomorphic motions and puzzles (following M. Shishikura)

Pascale Rœsch*

Abstract

We prove that the Mandelbrot set is locally connected at Yoccoz parameters, following an idea of Shishikura.

The aim of this article is to present a short proof, due to M. Shishikura (unpublished), of the following theorem first obtained by J.-C. Yoccoz:

Main Theorem (Yoccoz). *The Mandelbrot set M is locally connected at parameters c for which $f_c(z) = z^2 + c$ is a non-renormalizable map whose fixed points are repelling and whose critical point is recurrent.*

For parameters c satisfying the above conditions, another theorem of Yoccoz (see the paper by J. Milnor in this volume) asserts that the Julia set J_c of f_c is locally connected, and in particular exhibits a fundamental system of neighborhoods $P_n(c)$ of c—in the dynamical plane of f_c—whose closures have connected intersections with J_c. We derive the Main Theorem from this dynamical result in the following way. Given a prescribed parameter $c_0 \in M$, we use the Douady-Hubbard conformal representation of $\mathbb{C} \setminus M$ to construct neighborhoods $\mathcal{P}_n$ of c_0—in the parameter plane—whose closures have connected intersections with M. Then, to show that $\{c_0\} = \bigcap_n \mathcal{P}_n$, we consider the annuli $\mathcal{A}_n = \mathcal{P}_n \setminus \overline{\mathcal{P}_{n+1}}$ and prove that the sum of their moduli is infinite. To get this estimate, we compare the modulus of $\mathcal{A}_n$ with the modulus of the "corresponding" annulus $A_n(c_0) = P_n(c_0) \setminus \overline{P_{n+1}(c_0)}$. In M. Shishikura's proof, this key comparison relies on quasiconformality properties of holomorphic motions and the observation that, for $c \in \mathcal{P}_n$, the dynamical piece $P_n(c)$ moves holomorphically.

Section 1 briefly recalls the construction of Yoccoz puzzles and some pertinent results which will be needed later (for proofs, the reader is referred to Milnor's article [M2]). Section 2 deals with parallel constructions in the parameter plane; it provides a para-puzzle "corresponding" to the dynamical Yoccoz puzzle (see also [B, H] for similar constructions). Section 3 completes the proof of the Main Theorem, and in particular explains how to compare the modulus of a parameter annulus with the modulus of a critical annulus in the dynamical plane.

*Research partially supported by the ESF.

I am very grateful to M. Shishikura for explaining this proof that I adapted in [R] to study local connectivity in the parameter plane of cubic Newton maps.

1 Puzzles in the dynamical planes

1.1 External rays

We consider the family of quadratic polynomials $f_c(z) = z^2 + c$ parameterized by $c \in \mathbb{C}$. By definition, the filled Julia set K_c of f_c is the complement of the open set

$$B_c(\infty) = \left\{ z \in \mathbb{C} \mid f_c^n(z) \xrightarrow[n \to \infty]{} \infty \right\}$$

which is the basin of infinity. The potential function

$$G_c\colon \mathbb{C} \longrightarrow \mathbb{R}_+, \quad z \longmapsto \lim_{n\to\infty} 2^{-n} \log^+ \left| f_c^n(z) \right|,$$

(where $\log^+ x = \sup\{0, \log x\}$) is a proper harmonic function whose zero set is K_c and whose critical values are bounded by $G_c(0)$. Hence, the open set $U_c = \{z \mid G_c(z) > G_c(0)\}$ is a punctured disc and the classical Böttcher Theorem provides a unique conformal representation

$$\phi_c\colon U_c \longrightarrow \left\{ z \mid |z| > r_c \right\}, \qquad \text{where} \quad \log r_c = G_c(0),$$

which conjugates f_c to $z \mapsto z^2$ and satisfies $G_c = \log|\phi_c|$.

The level curve $\{z \in \mathbb{C} \mid G_c(z) = v\}$, $v > 0$, is called the *equipotential* of level v and is denoted by $E_c(v)$. The *external ray* of angle $t \in \mathbb{R}/\mathbb{Z}$, denoted by $R_c(t)$, is the gradient line of G_c stemming from infinity with the angle t (measured via the Böttcher coordinate ϕ_c). When K_c is connected, *i.e.*, when c belongs to the Mandelbrot set, ϕ_c is a conformal representation from $B_c(\infty)$ to $\mathbb{C} \setminus \overline{\mathbb{D}}$ and external rays are just the inverse images by ϕ_c of the rays in $\mathbb{C} \setminus \overline{\mathbb{D}}$.

An external ray is an analytic curve that either hits a critical point of G_c or accumulates on the Julia set J_c. Moreover, if t is rational and $R_c(t)$ hits no critical point, then $R_c(t)$ lands at (*i.e.* converges to) some point of J_c that is a repelling or parabolic (eventually) periodic point [DH1]. Conversely, for $c \in M$, each repelling periodic point is the landing point of at least one (and at most finitely many) external ray(s) (see [M1]).

1.2 Wakes of the Mandelbrot set

We now briefly describe the regions in the parameter plane where the fixed points of f_c are both repelling and one of them is the landing point of a prescribed cycle of external rays. These conditions define parameters for which the Yoccoz puzzle can be constructed.

Theorem 1.1 [DH1]. *The map*

$$\Phi_M\colon \begin{cases} \mathbb{C}\setminus M \longrightarrow \mathbb{C}\setminus\overline{\mathbb{D}} \\ \quad c \quad\ \longmapsto \Phi_M(c)=\phi_c(c) \end{cases}$$

is a conformal representation—where ϕ_c is the Böttcher coordinate on $U_c \ni c$.

Thus, the conformal homeomorphisms ϕ_c and ϕ_M have the same image, $\mathbb{C}\setminus\overline{\mathbb{D}}$, and they map c—with different meanings—to the same point. In particular, every point $\omega\in\mathbb{C}\setminus\overline{\mathbb{D}}$ has the form $\omega=\phi_c(c)$ for $c=\phi_M^{-1}(\omega)$.

As in the dynamical plane, the *external ray* of angle $t\in\mathbb{R}/\mathbb{Z}$ in the parameter plane is the set

$$\mathcal{R}(t)=\Phi_M^{-1}\Big(\big\{r\,e^{2i\pi t},\ r>1\big\}\Big),$$

and the *equipotential* of level $v>0$ is the closed curve

$$\mathcal{E}(v)=\big\{c\in\mathbb{C}\mid \log\big|\Phi_M(c)\big|=v\big\}\,.$$

One easily checks that the ray $\mathcal{R}(0)$ is exactly the half-line $]1/4,+\infty[$. For $c\in\mathbb{C}\setminus\overline{\mathcal{R}}(0)$, the ray $R_c(0)$ lands at a repelling fixed point of f_c usually denoted by β_c. The second fixed point α_c of f_c is attracting if and only if the parameter c lies inside the open region $\mathcal{H}\subset M$ bounded by the central cardioid. The multiplier $f_c'(\alpha_c)$ defines a conformal representation from $\mathcal{H}$ to the unit disc $\mathbb{D}$ which extends to a homeomorphism between the boundary curves. We denote by $\gamma\colon\mathbb{R}/\mathbb{Z}\simeq\partial\mathbb{D}\to\partial\mathcal{H}$ the inverse of the latter homeomorphism.

The following two theorems (see [DH1, M2]) describe the regions of $\mathbb{C}\setminus\mathcal{H}$ over which the external rays landing at α_c have fixed angles.

Theorem 1.2 [DH1].
a) *Every external ray $\mathcal{R}(t)$ of rational angle t converges to some parameter in M.*
b) *A parameter $c\in M$ is the landing point of some external ray $\mathcal{R}(p/q)$ with $(p,q)=1$ and q even if and only if the critical point of f_c is eventually periodic. Moreover, for such a parameter c, the external ray $\mathcal{R}(r/s)$ converges to c if and only if, in the dynamical plane of f_c, the external ray $R_c(r/s)$ lands at the critical value c.*
c) *For any rational p/q with $(p,q)=1$ and $q>1$, the parameter $\gamma(p/q)\in\partial\mathcal{H}$ is the landing point of exactly two external rays $\mathcal{R}(t^-)$ and $\mathcal{R}(t^+)$, where the angles $t^\pm=t^\pm(p/q)$ belong to the same orbit under multiplication by two and satisfy $0<t^-<t^+<1$.*

The proper arc $\mathcal{R}(t^-)\cup\mathcal{R}(t^+)\cup\gamma(p/q)$ divides the plane into two connected components. We denote by $\mathcal{W}_{p/q}$ the component which does not contain the domain $\mathcal{H}$.

Theorem 1.3 [GM]. *For any parameter c in $\mathcal{W}_{p/q}$, there are exactly q external rays that land at the fixed point α_c, namely the rays $R_c(2^i t^+)$, where $t^+ = t^+(p/q)$ and $i = 0, \ldots, q-1$. Moreover, if we cut out the plane along these rays, the critical value c lies in the region bounded by $R_c(t^-)$ and $R_c(t^+)$.*

1.3 The Yoccoz puzzle

Finally, we define the Yoccoz puzzle (see [M2]) for every parameter c belonging to a wake $\mathcal{W}_{p/q}$ with $(p,q)=1$ and $q>1$ (every point of $M \setminus \overline{\mathcal{H}}$ belongs to such a $\mathcal{W}_{p/q}$).

Definition 1.4. Let $c \in \mathcal{W}_{p/q}$, $t^+ = t^+(p/q)$ and $X^c = \{z \in \mathbb{C} \mid G_c(z) < 1\}$. The *Yoccoz puzzle* of f_c is given by the following sequence of graphs:

$$I_0^c = \partial X^c \cup \Big(X^c \cap \bigcup_{i\ge 0} \overline{R}_c(2^i t^+)\Big),$$
$$I_n^c = f_c^{-n}(I_0^c) \quad \text{for all } n \ge 1.$$

The *puzzle pieces of depth* 0 are the q connected components of $X^c \setminus I_0^c$. Each of them contains exactly one point $f_c^i(0)$ with $0 \le i \le q-1$ and will be denoted by $P_0^c\big(f_c^i(0)\big)$ accordingly. The *puzzle pieces of depth* $n \ge 1$ are the connected components of $f_c^{-n}(X^c) \setminus I_n^c = f_c^{-n}(X^c \setminus I_0^c)$. The puzzle piece containing a given point z will be denoted by $P_n^c(z)$. The puzzle pieces containing the critical value c will be simply denoted by $P_0^c, \ldots, P_n^c, \ldots$.

For $c \in \mathcal{W}_{p/q}$, a theorem of Yoccoz [M2] asserts that, if f_c is non-renormalizable and combinatorially recurrent, then its Julia set is locally connected, and in particular $\bigcap_{n=1}^{\infty} P_n^c = \{c\}$.

The map f_c is *combinatorially recurrent* if, for every n, there exists $m>0$ such that $f_c^m(0)$ belongs to $P_n^c(0)$. If the critical point 0 is recurrent then f_c is obviously combinatorially recurrent. Conversely, if f_c is non-renormalizable and combinatorially recurrent then it follows from the theorem of Yoccoz that the point 0 is recurrent.

Milnor's paper [M2] gives further information about the puzzle pieces P_n^c that will be needed in Section 3. In particular, it provides a sequence of annuli $A_{n_i}^c$ having the following properties:

- $A_{n_i}^c = P_{n_i}^c \setminus \overline{P_{n_i+1}^c}$ surrounds the critical value c of f_c and is *non-degenerate*—meaning that $P_{n_i}^c \supset \overline{P_{n_i+1}^c}$;

- $f_c^{n_i-n_0}$ induces an unramified covering map from $A^c_{n_i}$ onto $A^c_{n_0}$; and
- the sum $\sum_{i\geq 0} \operatorname{mod} A^c_{n_i}$ is infinite, where $\operatorname{mod} A^c_{n_i}$ denotes the modulus of $A^c_{n_i}$.

2 Puzzles in the parameter plane

As in the previous section, we will use the following conventions: objects in the parameter plane will be denoted by calligraphic capital letters (such as $\mathcal{P}$, $\mathcal{R}$), while those in dynamical planes will be denoted by roman capitals (such as P, R). Symbols associated with objects from the dynamical plane of f_c will also carry a subscript or superscript c.

2.1 Yoccoz para-puzzles

Since the combinatorics of the Yoccoz puzzle of f_c depends on the wake $\mathcal{W}_{p/q}$ that contains c, we will construct one para-puzzle for each $\mathcal{W}_{p/q}$, using the same angles as in the dynamical plane and replacing the Böttcher coordinate by the Douady-Hubbard conformal representation. Fix two relatively prime integers p, q with $q > 1$.

Definition 2.1. Let $t^+ = t^+(p/q)$ and $\mathcal{X}_n = \{c \in \mathcal{W}_{p/q} \mid \log|\phi_M(c)| < 2^{-n}\}$, $n \geq 0$. The *Yoccoz para-puzzle* is given by the following sequence of graphs defined in the closure of $\mathcal{W}_{p/q}$:

$$\mathcal{I}_n = \partial\mathcal{X}_n \cup \Big(\bigcup_{t\in J_n} \overline{\mathcal{R}}(t) \cap \mathcal{X}_n\Big)$$

where $J_n = \Big\{t \in \mathbb{R}/\mathbb{Z} \mid 2^n t \in \{t^+, \ldots, 2^{q-1}t^+\}\Big\}$. The *para-puzzle pieces of depth n* are the connected components of the $\mathcal{X}_n \setminus \mathcal{I}_n$. A general para-puzzle piece of depth n will be denoted by $\mathcal{P}_n$ and the one containing a given parameter c by $\mathcal{P}_n(c)$.

Remark 2.2. For any parameter c in $\mathcal{W}_{p/q}$, Theorem 1.1 shows that I^c_0 consists of converging rays and equipotentials. The graph I^c_1 has the same property provided the critical value c does not lie on I^c_0. By induction, if c does not belong to $I^c_{n-1} \cup \cdots \cup I^c_0$ then $I^c_n \cup \cdots \cup I^c_0$ is a union of converging rays and closed equipotentials. We will now characterize the set of parameters c for which this occurs.

Lemma 2.3. *A parameter $c \in \overline{\mathcal{W}}_{p/q}$ belongs to $\mathcal{I}_n$ if and only if the critical value c of f_c lies on I^c_n. As a consequence, if c belongs to a para-puzzle piece $\mathcal{P}_n$ of depth n, then c is outside of $I^c_n \cup \cdots \cup I^c_0$.*

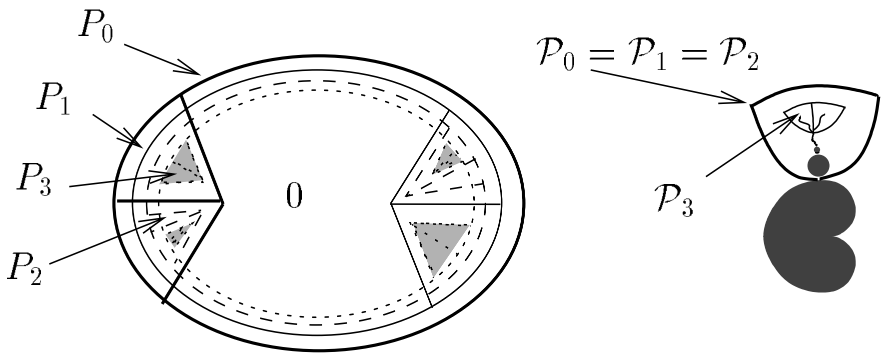

Figure 1: Shape of dynamical and parameter pieces of depth ≤ 3 for $\mathcal{W}_{1/3}$.

Proof. This is clear for $n = 0$. Now recall that if c does not belong to K_c—or equivalently if c does not belong to M—the Böttcher coordinate ϕ_c is defined on a disc U_c containing the critical value. For parameters c in $\overline{\mathcal{W}_{p/q}} \setminus M$, the equality $\Phi_M(c) = \phi_c(c)$ implies that the argument of $\phi_c(c)$ belongs to $[t^-, t^+]$ (Theorem 1.3). Hence, by definition of graphs and para-graphs, a parameter $c \in \overline{\mathcal{W}_{p/q}}$ is in $\mathcal{I}_n \setminus M$ if and only if the critical value c is in $I_n^c \setminus K_c$. Then Theorem1.2-b) completes the proof for $c \in M$. Indeed, the rays of $\mathcal{I}_n \setminus \mathcal{I}_0$ land at parameters c such that $f_c^n(c) = \alpha_c$, and each of these parameters c is the landing point of q external rays corresponding to the q dynamical rays landing at the critical value c (c is a preimage of α_c). The second part of the statement follows readily. □

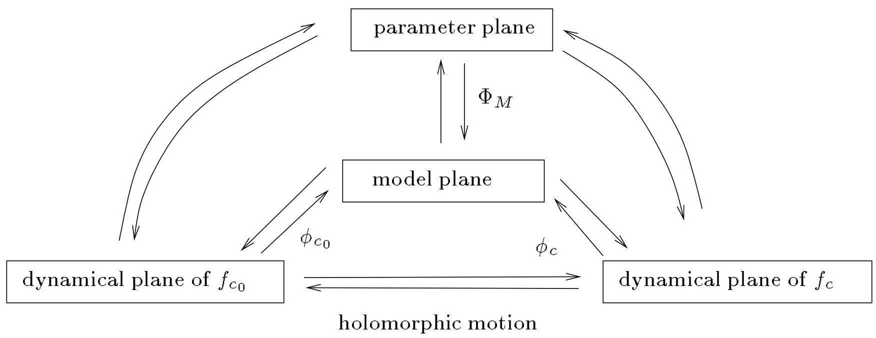

Figure 2: Relations between dynamical, parameter and reference planes.

Remarks 2.4.

a) Let n be the minimal index such that I_n^c contains the critical value. Then the graphs $I_0^c, \ldots, I_n^c$ do not contain the critical point, and therefore consist

of rays and equipotentials. This holds in particular when the parameter c belongs to $\mathcal{P}_{n-1}$.
b) All para-puzzle pieces are simply-connected since each graph $\mathcal{I}_n$ is made connected by the equipotential it involves.
c) The unique para-puzzle piece $\mathcal{P}_0$ of depth 0 is bounded by $\mathcal{R}(t^+)$, $\mathcal{R}(t^-)$ and $\mathcal{E}(1)$. In the dynamical plane of f_c for $c \in \mathcal{P}_0$, the curves $R_c(t^+)$, $R_c(t^-)$ and $E_c(1)$ are exactly the rays and equipotential that bound the piece P_0^c containing the critical value (Theorem 1.3). Thus, using the canonical parameterizations of rays and equipotentials, we get a natural homeomorphism from $\partial\mathcal{P}_0$ to ∂P_0^c for any $c \in \mathcal{P}_0$.

2.2 Relations between puzzles and para-puzzles

We want now to generalize Remark 2.4-c) to deeper graphs, *i.e.*, to construct a "canonical" homeomorphism from $\partial\mathcal{P}_n$ to $\partial P_n^{c_0}$ for any $n \geq 0$ and any parameter $c_0 \in \mathcal{P}_n$. Canonical here means the homeomorphism extends to a quasiconformal map from $\mathcal{P}_n$ to $P_n^{c_0}$. The construction proceeds by induction and relies on the fact that, when c moves within a para-puzzle piece $\mathcal{P}_n$, the graph I_{n+1}^c moves holomorphically.

Recall that a *holomorphic motion* of a given subset $Z \subset \widehat{\mathbb{C}}$ parameterized by a complex manifold $\mathcal{C}$ is a map $\mathcal{C}\times Z \to \mathcal{C}\times\widehat{\mathbb{C}}$ of the form $(c,z) \mapsto \big(c, h^c(z)\big)$ which is holomorphic on each slice $\mathcal{C} \times \{z\}$, $z \in Z$, injective on each slice $\{c\} \times Z$, $c \in \mathcal{C}$, and satisfies $h^{c_0}(z) = z$ for every $z \in Z$ and a given $c_0 \in \mathcal{C}$.

Lemma 2.5. *Let c_0 be a parameter in a para-puzzle piece $\mathcal{P}_n$ of depth $n \geq 0$. There exists a holomorphic motion*

$$h_n\colon \begin{cases} \mathcal{P}_n \times I_{n+1}^{c_0} \longrightarrow \mathcal{P}_n \times \widehat{\mathbb{C}} \\ \quad (c,z) \quad \longmapsto h_n(c,z) = \big(c, h_n^c(z)\big) \end{cases}$$

such that $I_{n+1}^c = h_n^c(I_{n+1}^{c_0})$ for every $c \in \mathcal{P}_n$.

Moreover, if $n \geq 1$, then h_n coincides with h_{n-1} on $\mathcal{P}_n \times (I_n^{c_0} \cap I_{n+1}^{c_0})$ (where $\mathcal{P}_{n-1} = \mathcal{P}_{n-1}(c_0) \supset \mathcal{P}_n$) and for every $c \in \mathcal{P}_n$ the diagram

$$\begin{array}{ccc} I_{n+1}^{c_0} & \xrightarrow{h_n^c} & I_{n+1}^c \\ {\scriptstyle f_{c_0}}\downarrow & & \downarrow{\scriptstyle f_c} \\ I_n^{c_0} & \xrightarrow[h_{n-1}^c]{} & I_n^c \end{array}$$

is commutative.

Proof. Let $V_n = \{z \in \mathbb{C} \mid \log^+|z| \geq 2^{-(n+1)}\}$. Since $\phi_c^{-1}(z)$ is holomorphic in the two variables (c, z) and is defined for $(c, z) \in \mathcal{P}_n \times V_n$, the map

$$h_n : \begin{cases} \mathcal{P}_n \times \phi_{c_0}^{-1}(V_n) \longrightarrow \mathcal{P}_n \times \widehat{\mathbb{C}} \\ \qquad (c, z) \qquad \longmapsto \big(c, \phi_c^{-1} \circ \phi_{c_0}(z)\big) \end{cases}$$

is a holomorphic motion of $\phi_{c_0}^{-1}(V_n)$ (note that this set contains the equipotential of $I_{n+1}^{c_0}$). Moreover, for $c \in \mathcal{P}_n$, the critical point of f_c remains outside of I_{n+1}^c (Lemma 2.3), so that f_c admits local inverses on I_{n+1}^c. Therefore, h_n extends uniquely over $\mathcal{P}_n \times (I_{n+1}^{c_0} \setminus K_{c_0})$ to a holomorphic motion—still denoted by h_n—satisfying $f_c \circ h_n^c = h_n^c \circ f_{c_0}$. One easily checks that $h_n^c(I_{n+1}^{c_0} \setminus K_{c_0}) = I_{n+1}^c \setminus K_c$. The final extension

$$h_n : \mathcal{P}_n \times \big(\phi_{c_0}^{-1}(V_n) \cup I_{n+1}^{c_0}\big) \longrightarrow \mathcal{P}_n \times \widehat{\mathbb{C}}$$

is given by the Λ-Lemma of Mañé-Sad-Sullivan (see [D]) which shows that any holomorphic motion of a given subset in $\widehat{\mathbb{C}}$ extends to a holomorphic motion of the closure. Since the extended map h_n^c is continuous, $h_n^c(I_{n+1}^{c_0}) = I_{n+1}^c$. The others desired properties are satisfied by construction. □

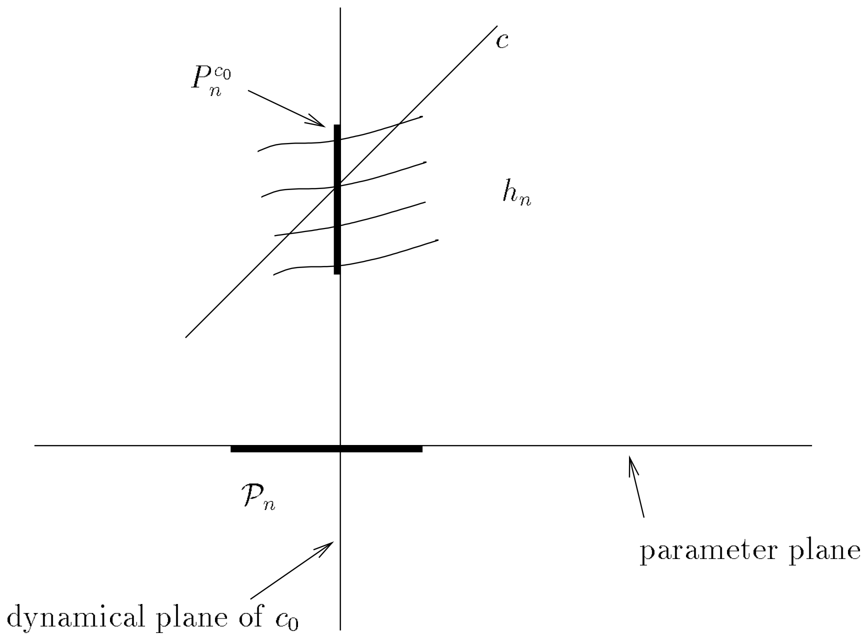

Figure 3: Holomorphic motion of $I_{n+1}^{c_0}$ over $\mathcal{P}_n$.

Notations 1. From now on, we fix a parameter $c_0 \in M$ and we suppose that c_0 lies in some para-puzzle piece $\mathcal{P}_n$ at each depth n. For every c in $\mathcal{P}_n$, we denote by D^c_{n+1} the puzzle piece of depth $n+1$ in the dynamical plane of f_c which is bounded by $h^c_n(\partial P^{c_0}_{n+1})$, where h_n is the holomorphic motion given by Lemma 2.5.

Lemma 2.6. *There exists a unique homeomorphism*

$$H\colon \bigcup_{n\geq 0}(\overline{\mathcal{P}_n}\cap\mathcal{I}_{n+1}) \longrightarrow \bigcup_{n\geq 0}(\overline{P^{c_0}_n}\cap I^{c_0}_{n+1})$$

that coincides with $(\phi_{c_0})^{-1}\circ\Phi_M$ outside of M. Moreover, H maps $\partial\mathcal{P}_n$ onto $\partial P^{c_0}_n$ and is given, on $\mathcal{P}_n\cap\mathcal{I}_{n+1}$, by the inverse holomorphic motion of the critical value: $H(c) = (h^c_n)^{-1}(c)$.

Proof. We will prove the following assertion $\mathfrak{P}(k)$ by induction on $k\geq 0$:

$\mathfrak{P}(k)$: *The map $(\phi_{c_0})^{-1}\circ\Phi_M$, restricted to*

$$\big(\partial\mathcal{P}_0\cup(\overline{\mathcal{P}_0}\cap\mathcal{I}_1)\cup\cdots\cup(\overline{\mathcal{P}_{k-1}}\cap\mathcal{I}_k)\big)\setminus M,$$

extends to a homeomorphism H from $\partial\mathcal{P}_0\cup(\overline{\mathcal{P}_0}\cap\mathcal{I}_1)\cup\cdots\cup(\overline{\mathcal{P}_{k-1}}\cap\mathcal{I}_k)$ to $\partial P^{c_0}_0\cup(\overline{P^{c_0}_0}\cap I^{c_0}_1)\cup\cdots\cup(\overline{P^{c_0}_{k-1}}\cap I^{c_0}_k)$ which takes $\partial\mathcal{P}_k$ onto $\partial P^{c_0}_k$.

The step $k=0$ follows from remark 2.4-c) and we now assume that $\mathfrak{P}(k)$ is satisfied.

We first show that $\phi^{-1}_{c_0}\circ\Phi_M$ induces a homeomorphism from $(\overline{\mathcal{P}}_k\cap\mathcal{I}_{k+1})\setminus M$ onto $(\overline{P^{c_0}_k}\cap I^{c_0}_{k+1})\setminus J_{c_0}$. In fact, $\phi^{-1}_{c_0}\circ\Phi_M\colon\mathbb{C}\setminus M\to\mathbb{C}\setminus J_{c_0}$ maps the point whose Douady-Hubbard coordinate is $re^{2i\pi t}$ to the point whose Böttcher coordinate is the same. Moreover, by induction, the homeomorphism H takes $\partial\mathcal{P}_k$ onto $\partial P^{c_0}_k$. Therefore, the image of $(\overline{\mathcal{P}_k}\cap\mathcal{I}_{k+1})\setminus M$ under $\phi^{-1}_{c_0}\circ\Phi_M$ is exactly $(\overline{P^{c_0}_k}\cap I^{c_0}_{k+1})\setminus J_{c_0}$.

Note that on $(\mathcal{P}_k\cap\mathcal{I}_{k+1})\setminus M$, the map H satisfies

$$H(c) = \phi^{-1}_{c_0}\circ\Phi_M(c) = \phi^{-1}_{c_0}\big(\phi_c(c)\big) = (h^c_k)^{-1}(c).$$

Thus we extend H by setting $H(c) = (h^c_k)^{-1}(c)$ for $c\in\mathcal{P}_k\cap\mathcal{I}_{k+1}$. To see that this extension is still injective, we have to show that, if two distinct external rays $\mathcal{R}(t_1),\mathcal{R}(t_2)\subset\mathcal{I}_{k+1}$ land at points $c_1,c_2\in\mathcal{P}_k$ with the same image under H, then $c_1=c_2$. Since c_1,c_2 belong to $\mathcal{P}_k$ with $k\geq 1$, they are Misiurewicz parameters (*i.e.*, parameters for which the critical point is eventually periodic). This follows readily from Theorem 1.2-b) and the fact that t_1,t_2 have even denominators since they both differ from $t^{\pm}$. Then Theorem 1.2-b) implies that the two dynamical rays $R_{c_1}(t_1)$, $R_{c_2}(t_2)$ land at c_1 and c_2 respectively. Since $h^{c_j}_k\big(R_{c_0}(t_j)\big) = R_{c_j}(t_j)$ for $j\in\{1,2\}$, the rays

$R_{c_0}(t_1)$ and $R_{c_0}(t_2)$ both land at $H(c_1) = H(c_2)$. In turn, this implies that $R_{c_1}(t_1)$ and $R_{c_1}(t_2)$ land at c_1. By Theorem 1.2-b), $\mathcal{R}(t_2)$ lands at c_1, and therefore $c_1 = c_2$.

Finally, we have to show that H maps $\partial\mathcal{P}_{k+1}$ onto $\partial P^{c_0}_{k+1}$. Since H is a homeomorphism from $\partial\mathcal{P}_k \cup (\overline{\mathcal{P}_k} \cap \mathcal{I}_{k+1})$ to $\partial P^{c_0}_k \cup (\overline{P^{c_0}_k} \cap I^{c_0}_{k+1})$, it maps $\partial\mathcal{P}_{k+1}$ to ∂Q_{k+1} for some piece Q_{k+1} of depth $k+1$. On the other hand, for $c \in \mathcal{P}_{k+1}$, the critical value c of f_c is in D^c_{k+1} because it moves continuously with c, belongs to $D^{c_0}_{k+1}$ for $c = c_0$ and does not meet I^c_{k+1} (Lemma 2.3). Hence, if $c \in \partial\mathcal{P}_{k+1}$ lies inside some equipotential segment of $\partial\mathcal{P}_{k+1}$, then the critical value c belongs to ∂D^c_{k+1}. Therefore, $H(c) = (h^c_k)^{-1}(c) \in \partial P^{c_0}_{k+1}$ and lies inside some equipotential segment of $\partial P^{c_0}_{k+1}$. Since $P^{c_0}_{k+1}$ is the only piece of depth $k+1$ whose boundary contains this point, $Q_{k+1} = P^{c_0}_{k+1}$. □

Corollary 2.7. *The para-puzzle piece $\mathcal{P}_{k+1}$ containing c_0 is the set of parameters c for which the two puzzle pieces P^c_{k+1} and D^c_{k+1} coincide. Thus c belongs to $\mathcal{P}_k \setminus \overline{\mathcal{P}_{k+1}}$ if and only if the critical value c lies in $D^c_k \setminus \overline{D^c_{k+1}}$.*

Proof. As in Lemma 2.6, for $c \in \mathcal{P}_{k+1}$, the critical value c of f_c is in D^c_{k+1}.

Conversely, suppose c belongs to $\mathcal{P}_k \setminus \overline{\mathcal{P}}_{k+1}$. Let c_t, $t \in [0,1]$, be a continuous path in $\mathcal{P}_k$ joining c_0 to $c_1 = c$, crossing $\partial\mathcal{P}_{k+1}$ at exactly one point c_{t_0}, and avoiding $\mathcal{I}_{k+1} \cap M$ (such a path can be constructed using equipotentials and rays). Then for $t < t_0$, the critical value $f_{c_t}(0) = c_t$ belongs to $D^{c_t}_{k+1}$. Moreover, since the parameter path c_t crosses $\mathcal{I}_{k+1}$ at $t = t_0$, the critical value $f_{c_t}(0)$ gets out of $D^{c_t}_{k+1}$ when t passes over t_0. Then Lemma 2.6 insures that, for $t > t_0$, the critical value does not cross $\partial D^{c_t}_{k+1}$ again (because the path does not cross $\mathcal{P}_{k+1}$). Hence $c_1 = c$ lies outside of D^c_{k+1}. □

Lemma 2.8. *If $\overline{D^{c_0}_{n+1}} \subset D^{c_0}_n$ then $\overline{\mathcal{P}_{n+1}} \subset \mathcal{P}_n$.*

Proof. The corollary 2.7 shows that the homeomorphism H of Lemma 2.6 maps $\partial\mathcal{P}_{n+1} \cap \mathcal{P}_n$ into $\partial D^{c_0}_{n+1}$. If $\overline{D^{c_0}_{n+1}} \subset D^{c_0}_n$, then $H_n^{-1}(\partial D^{c_0}_{n+1})$ is compact, so $\partial\mathcal{P}_{n+1} \cap \mathcal{P}_n$ is also compact (since it is closed in $\mathcal{P}_n$). On the other hand, $\partial\mathcal{P}_{n+1}$ is connected, and it is not entirely included in $\partial\mathcal{P}_n$ because of the piece of equipotential it contains. Therefore $\overline{\mathcal{P}_{n+1}} \subset \mathcal{P}_n$. □

3 Proof of the Main Theorem

3.1 Extending the holomorphic motions

Let $c_0 \in M$ be a parameter satisfying the hypotheses of the Main Theorem *i.e.*, for which $f_{c_0}(z) = z^2 + c_0$ is a non-renormalizable map whose fixed points are repelling and whose critical point is recurrent. Since c_0 does not belong to the central cardioid, a well-known result (see for example [GM]) asserts that c_0 lies in some wake $\mathcal{W}_{p/q}$ with $(p,q) = 1$ and $q > 1$; moreover, c_0 is not a Misiurewicz parameter. Thus c_0 belongs to a complete sequence of pieces $\mathcal{P}_n$, $n \geq 0$, of the para-puzzle constructed in Section 2. As in the dynamical planes, we consider the sequence of (possibly degenerate) annuli $\mathcal{A}_n = \mathcal{P}_n \backslash \overline{\mathcal{P}_{n+1}}$. We also denote by $A_n^{c_0}$ the annulus $P_n^{c_0} \backslash \overline{P_{n+1}^{c_0}}$ which surrounds the critical value of f_{c_0}.

The key ingredient for constructing a quasiconformal homeomorphism $\mathcal{A}_n \to A_n^{c_0}$ is the following theorem of Z. Słodkowski [S, D]:

Theorem 3.1 (Słodkowski). *Every holomorphic motion H of $Z \subset \widehat{\mathbb{C}}$ parameterized by the open unit disc $\mathbb{D}$ and inducing the inclusion map on $\{0\} \times Z$ extends to a holomorphic motion $\widetilde{h}\colon \mathbb{D} \times \widehat{\mathbb{C}} \to \mathbb{D} \times \widehat{\mathbb{C}}$. Moreover, for every parameter $c \in \mathbb{D}$, the map $\widetilde{h}^c\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal homeomorphism whose dilatation constant K_c is bounded by $\frac{1+|c|}{1-|c|}$.*

Applying this theorem, we can extend each h_n to a holomorphic motion $\widetilde{h}_n$ of $\widehat{\mathbb{C}}$ parameterized by $\mathcal{P}_n$. Then, as in Lemma 2.6, we consider the map $\mathcal{A}_n \to A_n^{c_0}$ which assigns to each parameter c the point $(\widetilde{h}_n^c)^{-1}(c)$. This map is a quasiconformal homeomorphism (see the proof of Proposition 3.2 below). However, without further care, its dilatation will depend on n. We will solve this problem by restricting to the sequence of non-degenerate annuli $A_{n_i}^{c_0}$ described at the end of Section 1, whose modulus sum is infinite and for which $f^{n_i - n_0}$ induces an unramified covering map from $A_{n_i}^{c_0}$ to $A_{n_0}^{c_0}$.

3.2 Comparing the moduli

The Main Theorem is now a straightforward consequence of the following:

Proposition 3.2. *There exists a constant $K > 1$ such that, for every integer $n \in \{n_i,\ i \geq 0\}$,*

$$\frac{1}{K} \operatorname{mod} A_n^{c_0} \leq \operatorname{mod} \mathcal{A}_n \leq K \operatorname{mod} A_n^{c_0}\,.$$

Proof. Let $n \in \{n_i,\ i \geq 0\}$ and d_n be the degree of the covering map $f_{c_0}^{n-n_0}\colon A_n^{c_0} \to A_{n_0}^{c_0}$. To get the required inequality, it suffices to construct a quasiconformal homeomorphism $\widetilde{H}_n\colon \mathcal{A}_n \to A_n^{c_0}$ whose dilatation is bounded independent of n.

For every $c \in \mathcal{P}_n$ and $m \in \{n_0, n\}$, let A_m^c denote the annulus $D_m^c \setminus \overline{D_{m+1}^c}$. Then the map $f_c^{n-n_0}$ induces an unramified covering map of degree d_n from A_n^c to $A_{n_0}^c$. Indeed, Lemma 2.5 and the definition of D_m^c, D_{m+1}^c show that the diagram

$$\begin{array}{ccc} \partial A_n^{c_0} & \xrightarrow{h_n^c} & \partial A_n^c \\ {\scriptstyle f_{c_0}^{n-n_0}}\big\downarrow & & \big\downarrow{\scriptstyle f_c^{n-n_0}} \\ \partial A_{n_0}^{c_0} & \xrightarrow[h_{n_0}^c]{} & \partial A_{n_0}^c \end{array}$$

is commutative, so $f_c^{n-n_0}$ maps A_n^c onto $A_{n_0}^c$. Lemma 2.3 ensures that no critical point of $f_c^{n-n_0}$ can enter A_n^c while c remains in $\mathcal{P}_n$, so the map $f_c^{n-n_0}\mid_{A_n^c}$ has no ramification points and its degree is d_n.

Now Słodkowski's Theorem gives a holomorphic motion

$$\widetilde{h}_{n_0}\colon \mathcal{P}_{n_0}\times\widehat{\mathbb{C}} \longrightarrow \mathcal{P}_{n_0}\times\widehat{\mathbb{C}}$$

which extends h_{n_0} and consists of K_c-quasiconformal homeomorphisms $\widetilde{h}_{n_0}^c$. For every $c\in\mathcal{P}_n$ the homeomorphism $\widetilde{h}_{n_0}^c\colon \overline{A_{n_0}^{c_0}}\to\overline{A_{n_0}^c}$ lifts—via the holomorphic covering maps $f_{c_0}^{n-n_0}$ and $f_c^{n-n_0}$—to a homeomorphism

$$\widetilde{h}_n^c\colon \overline{A_n^{c_0}} \longrightarrow \overline{A_n^c}$$

which is again K_c-quasiconformal, coincides with h_n^c on $\partial P_{n+1}^{c_0}$, and depends holomorphically on c—because $f_c^{n-n_0}\circ\widetilde{h}_n^c = \widetilde{h}_{n_0}^c\circ f_{c_0}^{n-n_0}$. Hence the map

$$\widetilde{h}_n\colon \mathcal{P}_n\times\overline{A_n^{c_0}} \longrightarrow \mathcal{P}_n\times\widehat{\mathbb{C}},\quad (c,z)\longmapsto \big(c,\widetilde{h}_n^c(z)\big),$$

is a holomorphic motion.

We now recall that, according to Corollary 2.7 and Lemma 2.6, a parameter c is in $\mathcal{A}_n\cup\partial\mathcal{P}_{n+1}$ if and only if the critical value c belongs to $A_n^c\cup\partial D_{n+1}^c$, or equivalently if and only if $(\widetilde{h}_n^c)^{-1}(c)$ lies in $A_n^{c_0}\cup\partial P_{n+1}^{c_0}$. Let $\widetilde{H}_n$ be the map

$$\widetilde{H}_n\colon \begin{cases} \mathcal{A}_n\cup\partial\mathcal{P}_{n+1} \longrightarrow A_n^{c_0}\cup\partial P_{n+1}^{c_0}, \\ \qquad c \qquad\quad \longmapsto \widetilde{H}_n(c) = (\widetilde{h}_n^c)^{-1}(c)\,. \end{cases}$$

On $\partial\mathcal{P}_{n+1}$, the map $\widetilde{H}_n$ coincides with H and hence is a homeomorphism onto $\partial P_{n+1}^{c_0}$ (Lemma 2.6).

Assertion. *The map $\widetilde{H}_n$ is a quasiconformal homeomorphism with dilatation*

$$K_n = \sup\big\{K_c,\ c\in\mathcal{P}_n\big\} \le \sup\big\{K_c,\ c\in\overline{\mathcal{P}_{n_0+1}}\big\} = K < +\infty\,.$$

We prove this assertion below assuming that all maps are sufficiently smooth. The general case can be derived by approximation arguments, using compactness of quasiconformal homeomorphisms (see [DH2, Lemma IV.3] and [LV, Theorem 2.2 in Section VI.2]). Note that the proof of Theorem 3.1

(in [D]) yields an explicit approximation of class $\mathcal{C}^\infty$ for the extended holomorphic motion.

Consider the Beltrami coefficient $\mu = \overline{\partial}\widetilde{H}_n/\partial\widetilde{H}_n$. Differentiating the relation $\widetilde{h}_n^c \circ \widetilde{H}_n(c) = c$, we get the identity

$$\overline{\partial}\widetilde{h}_n^c(\widetilde{H}_n(c))\,\overline{\partial\widetilde{H}_n(c)} + \partial\widetilde{h}_n^c(\widetilde{H}_n(c))\,\overline{\partial}\widetilde{H}_n(c) = 0$$

which shows that

$$|\mu(c)| = \left|\frac{\overline{\partial}\widetilde{h}_n^c(\widetilde{H}_n(c))}{\partial\widetilde{h}_n^c(\widetilde{H}_n(c))}\right| \leq \frac{K_n - 1}{K_n + 1} < 1\,.$$

The Ahlfors-Bers Theorem then yields a K_n-quasiconformal homeomorphism $\phi\colon \mathcal{A}_n \to \phi(\mathcal{A}_n)$ satisfying $\overline{\partial}\phi = \mu\,\partial\phi$. By a straightforward calculation of $\overline{\partial}(\widetilde{H}_n \circ \phi^{-1})$, the map $\Phi_n = \widetilde{H}_n \circ \phi^{-1}$ is holomorphic. One can easily see that $\widetilde{H}_n$ is a proper map, so Φ_n is a holomorphic ramified covering map from $\phi(\mathcal{A}_n)$ to $A_{n_0}^{c_0}$. By Lemma 2.8 and the Riemann-Hurwitz formula, Φ_n has no ramification points, so its degree is given by the degree of H on the inner boundary component of $\mathcal{A}_n$. Using the fact that $\widetilde{H}_n$ induces a homeomorphism from $\partial\mathcal{P}_{n+1}$ to $\partial P_{n+1}^{c_0}$, we conclude that Φ_n is a conformal homeomorphism. The result follows. □

In fact the above proof gives the following more precise result:

Corollary 3.3. *For each $n \in \{n_i,\ i \geq 0\}$, there exists a number $K_n > 1$ such that*

$$\frac{1}{K_n}\operatorname{mod} A_n^{c_0} \leq \operatorname{mod}\mathcal{A}_n \leq K_n \operatorname{mod} A_n^{c_0}$$

and $K_n \longrightarrow 1$ as n tends to infinity.

Proof. Since the sum $\sum_{i\geq 0}\operatorname{mod} A_{n_i}^{c_0}$ is infinite, Proposition 3.2 shows that $\sum_{i\geq 0}\operatorname{mod}\mathcal{A}_{n_i}$ is also infinite, so that the intersection of the para-puzzle pieces $\mathcal{P}_n$ reduces to $\{c_0\}$. On the other hand, the preceding proof bounds the dilatation of $\widetilde{H}_n$ by $K_n = \sup\{K_c,\ c \in \mathcal{P}_n\}$. To see that the sequence K_n tends to 1, choose a conformal representation $\chi\colon \mathcal{P}_{n_0} \to \mathbb{D}$ which maps c_0 to 0. Then Słodkowski's Theorem 3.1 asserts that $K_c \leq \frac{1+|\chi(c)|}{1-|\chi(c)|}$. This implies the result since the pieces $\mathcal{P}_n$ shrink to c_0. □

To conclude this paper, we characterize the parameters rays $\mathcal{R}(t)$ which converge to a given Yoccoz parameter c.

Corollary 3.4. *Let $c \in M$ be a parameter satisfying the hypotheses of the Main Theorem. The angles $t \in \mathbb{R}/\mathbb{Z}$ for which the parameter ray $\mathcal{R}(t)$ converges to c are exactly those for which the dynamical ray $R_c(t)$ converges to the critical value c of f_c.*

Proof. Observe that if a ray $\mathcal{R}(t)$ or $R_c(t)$ enters some (para-) puzzle piece of depth n then all its points of potential less than 2^{-n} lie in that piece. Now Lemma 2.6 shows that $\mathcal{R}(t)$ enters $\mathcal{P}_n(c)$ if and only if $R_c(t)$ enters the piece P_n^c containing the critical value of f_c. Since $\bigcap \mathcal{P}_n(c) = \bigcap P_n^c = \{c\}$, the result follows. $\square$

References

[A] Ahlfors L. — *Lectures on Quasiconformal Mappings*, Wadsworth & Brook/Cole, Advanced Books & Software, Monterey 1987.

[B] Branner B. — *Puzzles and para-puzzles of quadratic and cubic polynomials*, Proc. of Symp. in Appl. Math. **49** (1994), 31–69.

[D] Douady A. — *Prolongement de mouvements holomorphes* [*d'après Słodkowski et autres*], Séminaire Bourbaki n° 775, Astérisque **227** (1995), 7–20.

[DH1] Douady A., Hubbard J. H. — *Étude dynamique des polynômes complexes I & II*, Publ. math. d'Orsay 1984.

[DH2] Douady A., Hubbard J. H. — *On the dynamics of polynomial-like mappings*, Ann. Sci. Éc. Norm. Sup. **18** (1985), 287–343.

[GM] Goldberg L. R., Milnor J. — *Fixed points of polynomial maps, part II: fixed point portraits*, Ann. Sci. Éc. Norm. Sup. **26** (1993), 51–98.

[H] Hubbard J. H. — *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*, in *Topological Methods in Modern Mathematics*, 467–511, Goldberg and Phillips eds, Publish or Perish 1993.

[LV] Lehto O., Virtanen K. I. — *Quasiconformal Mappings in the Plane*, Springer-Verlag, 1973.

[M1] Milnor J. — *Dynamics in One Complex Variable: Introductory Lectures*, IMS Preprint, SUNY Stony Brook 1990.

[M2] Milnor J. — *Local connectivity of Julia Sets*, in this volume.

[R] Roesch P. — *Topologie locale des méthodes de Newton cubiques*, thèse, ENS de Lyon 1997.

[S] Słodkowski Z. — *Extensions of holomorphic motions*, Ann. Scuola Norm. Sup. Cl. Sci. **22** (1995), 185–210.

Pascale ROESCH
Université des Sciences et Technologies de Lille
UFR de Mathématiques
59655 Villeneuve d'Ascq CEDEX, FRANCE
email:roesch@agat.univ-lille1.fr