Bounded recurrence of critical points and Jakobson's Theorem

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Abstract

We prove that the set of quadratic interval maps which satisfy a strong bounded recurrence condition on the orbit of the critical point has positive Lebesgue measure. This implies classical Theorems of Jakobson and Benedicks-Carleson.

1 Introduction

1.1 Statement of results

In this paper we will be concerned with the dynamics of maps belonging to the quadratic family

$$f_a(x) = x^2 - a$$

for $a \in \Omega_\varepsilon = [2-\varepsilon, 2]$, $\varepsilon > 0$ and $x \in I = [-2, 2]$. Let $c_0 = c_0(a) = f_a(0)$ denote the critical value of f_a and for $i \geq 0$, $c_i = c_i(a) = f^i(c_0)$. Notice that this differs slightly from the usual convention of letting c_1 denote the critical value.

Definition (Bounded recurrence of the critical point). For $\delta > 0, \alpha > 0$ and $a \in \mathbb{R}$ we say that f_a satisfies the bounded recurrence condition $(BR)_n = (BR)_n(\delta, \alpha)$ if for all $k \in [1, n]$:

$$\prod_{\substack{c_i \in (-\delta,\delta) \\ 0 \leq i \leq k}} |c_i| \geq e^{-\alpha k}. \qquad (BR)_n$$

We say that f satisfies (BR) if it satisfies $(BR)_n$ for all n and we let

$$\Omega^*_\varepsilon = \Omega^*_\varepsilon(\alpha, \delta) = \{a \in \Omega_\varepsilon : f_a \text{ satisfies } (BR)(\delta, \alpha)\}$$

Theorem. *For every $\alpha > 0$ there exists $\delta > 0$ such that*

$$\frac{|\Omega^*_\varepsilon|}{|\Omega_\varepsilon|} \to 1 \quad as \quad \varepsilon \to 0.$$

Here $|\cdot|$ denotes Lebesgue measure.

The proof also shows that the complement of Ω_ε^* in Ω_ε contains an open and dense set so that in particular Ω_ε^* is a nowhere dense set of positive Lebesgue measure. In section 4.1 we shall show that for small ε and any $\alpha < \log \sqrt{2}$, all parameters in Ω_ε^* have an exponentially growing derivative along the critical orbit. By standard results this implies that the corresponding maps f_a admit an absolutely continuous invariant probability measure μ. In fact quite minor modifications of the constructions and estimates given here can be used to obtain the same conclusion. In any case we have as a Corollary the following classical

Theorem (Jakobson, 1981). *The parameter value 2 is a (one-sided) full Lebesgue density point of the parameters for which the corresponding maps f_a admit an absolutely continuous invariant probability measure.*

This result is considered a milestone in the theory of dynamical systems, both for the intrinsic interest of the statement as well as for the ideas and techniques introduced. It has also motivated a large amount of deep and difficult research, although the full potential of the concepts and methods involved is still far from being fully grasped or realized. The main purpose of this paper is to take advantage of the substantial progress which has been made through the application and generalizations of the original ideas to more complicated examples, and to revisit the arguments in their original and simplest setting.

The formulation of the main result in terms of condition $(*)$ is new in this setting and is motivated by several factors. Amongst these are the fact that it appears naturally in the study of maps with infinite derivatives, where the same condition is imposed on the recurrence near the critical points and near the singularities; it is extremely simple to state and it is not hard to develop an intuitive feeling for it; a simple application of the Ergodic Theorem actually shows that it is satisfied quite generally, i.e. typical points satisfy $(*)$ in relation to their recurrence with respect to any other given typical point, for any maps admitting an absolutely continuous invariant measure with a density in $L^p, p > 1$.

The proof is divided into 5 sections, in which the different kinds of arguments required are developed: e.g. the combinatorial structure, the probabilistic argument, the distortion estimates. Throughout the proof I have tried to identify the exact assumptions used at each step and to bring these to the reader's attention. In connection with this I have also tried to show how most of the arguments apply in much more general situations than the one considered here and to give some indication as to how they have been or could be generalized.

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1.2 Idea of the proof

We fix $\varepsilon > 0$ and restrict our attention to the parameter interval $\Omega = \Omega_\varepsilon = [2-\varepsilon, 2]$ (from now on we shall usually omit the subscript ε). We begin with a combinatorial construction leading to the definition of the set Ω^*. The details are slightly technical but the basic idea is easy to explain. Fix a partition $\mathcal{I}$ of the critical neighbourhood $\Delta = (-\delta, \delta)$ which is fine enough so that two points which are close enough together and always fall in the same element of the partition can be said to have essentially the same recurrence. We then construct a sequence of partitions $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \ldots$ of Ω so that points in the same element of the partition of order n "stay together" up to time n; let $\Omega^{(n)}$ denote the union of elements which satisfy a certain condition which will be shown to imply $(BR)_n$.

It then remains to show that $\Omega^* = \cap_n \Omega^{(n)}$ has positive measure and $|\Omega^*|/\varepsilon \to 1$ as $\varepsilon \to 0$. For this it is sufficient to show (i) that the measure $|\Omega^{(n-1)} \setminus \Omega^{(n)}|$ of parameters which need to be excluded at each stage is exponentially small in n and (ii) that no exclusions have to be made before some time $N \to \infty$ as $\varepsilon \to 0$. The second claim follows immediately from the fact that the critical point is non-recurrent for the parameter value $a = 2$ while the first is non-trivial and constitutes the heart of the proof. In probabilistic language it corresponds to saying that the conditional probability of *not satisfying* $(BR)_n$ given that $(BR)_{n-1}$ *is satisfied*, is exponentially small in n.

Before going into the details let's consider a particularly simple example which illustrates some of the mechanisms involved. Let $T : I \to I$ be an expanding interval map with a Markov partition $\mathcal{P}$. For simplicity we assume that $T(\omega) = I$ for all elements $\omega \in \mathcal{P}$. Then by standard results there is a sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of refinements of $\mathcal{P}$ such that T^n maps each $\omega_n \in \mathcal{P}_n$ to I with uniformly bounded distortion. Now fix an arbitrary point $c \in I$ and consider the set $\Gamma^* = \cap_n \Gamma^{(n)}$ where

$$\Gamma^{(n)} = \{x \in I : |f^n(x) - c| \geq \delta e^{-\alpha n}\}$$

for some $\alpha > 0$ and $\delta > 0$. I claim that for arbitrarily small α and sufficiently small δ, $|\Gamma^*| > 0$. Indeed, it follows from the bounded distortion property that the proportion of each interval ω_n which falls into $\Delta_n = (c - \delta e^{-\alpha n}, c + \delta e^{-\alpha n})$ at time n is of the order of $\delta e^{-\alpha n}$. Thus the same is true of the entire interval and the total measure of points which fall into Δ_n at time n for some n, is bounded by some constant multiple of $\sum \delta e^{-\alpha n}$ which can be made arbitrarily small by taking δ small.

This simple argument can be interpreted in the following way. The distortion bounds (which in this case follow from the expansivity of the map) and the Markov property imply that small intervals ω_n grow sufficiently large sufficiently fast and in a sufficiently regular way that their distribution in the original interval I at time n is essentially random. Therefore the probability of falling into Δ_n is proportional to the size of Δ_n and the fact that this tends to zero exponentially fast in n then yields the conclusion. Notice that the construction shows that Γ^* must be nowhere dense.

Our condition $(BR)_n$ is very similar to the one given above although somewhat stronger since it takes into account the past orbit of points. Since we are interested in the recurrence of a single (critical) point for different parameters we shall consider the image in the dynamical interval I of the iterates of the critical point for different parameter values. It is easy to see that for any $n \geq 0$ the images $f_a^n(c)$ for an interval of parameter values in form an interval in I. To simplify the terminology we shall usually refer to these sets as *iterates* of an interval of parameter values. We shall show below that intervals of good parameters are growing in size at exponential speeds and that our construction of the partitions in parameter space yields uniform distortion estimates. The main difficulty here compared to the situation in the toy model described above is the absence of a Markov structure and even of a uniform lower bound on the images of partition elements. To overcome this problem we develop a statistical argument to show (and make precise the notion) that most partition elements have uniformly large images and therefore can be considered to have a sufficiently random distribution. More specifically we prove two key technical estimates which link the *size* and *number* of elements in $\mathcal{P}^{(n)}$ to their recurrence: elements with strong recurrence are *small* and *not too many*. Therefore the *average* recurrence of the critical points of maps in $\Omega^{(n-1)}$ up to time n is very low and it follows by a large deviation argument that for exponentially most parameters it will be sufficiently low so that they will belong to $\Omega^{(n)}$ (i.e. the ratio $|\Omega^{(n-1)} - \Omega^{(n)}|/|\Omega^{(n-1)}| \to 0$ exponentially fast).

In section 2 we give the complete combinatorial definition of the sets $\Omega^{(n)}$

and their partitions $\mathcal{P}^{(n)}$. In section 3 we assume two technical lemmas (the ones mentioned above) and carry out the large deviation argument. In section 4 we prove three fundamental lemmas which constitute the basis of the proof. These deal respectively with expansion estimates outside the critical neighbourhood, parameter dependence, and the notion of binding. In section 5 we prove the bounded distortion property for elements of $\mathcal{P}^{(n)}$ and show that all points of $\Omega^{(n)}$ actually satisfy condition $(BR)_n$. Finally, in section 6 we prove the technical lemmas used in the large deviations argument.

1.3 Technical remarks

Remarks about specific aspects of the proof will be interspersed throughout the text. I will limit myself here to some remarks of a more general nature.

Persistence of hyperbolicity. On one level, Jakobson's Theorem as well as the Theorem given here and the many other generalizations which will be mentioned below are about the general question of *persistence of hyperbolicity.* While uniform hyperbolicity is essentially an open condition, this is not the case for the weaker forms of hyperbolicity which occur in systems with critical points or (in higher dimensions) homoclinic and heteroclinic tangencies. However, these Theorems do say that in certain cases non-uniform hyperbolicity is persistent in a strong measure-theoretical sense. In the present setting the proof makes it quite clear that the uniform hyperbolicity in most of the phase space, in fact outside the critical neighbourhood Δ, which moreover persists under perturbations, is crucial. However the fact that these estimates do not hold everywhere in a persistent way means that parameter choices have to be made and is reflected in the fact that Ω^* contains no open sets. On the other hand the fact that we can take $|\Delta| \to 0$ without affecting the uniform expansion estimates is reflected in the fact that 2 is a full Lebesgue density point of Ω^*.

Connected to this point is that of characterizing dynamically the Lebesgue density points of all parameters for which the corresponding maps have absolutely continuous invariant probability measures (*acip*'s); an interesting statement would be something like: a^* is a (possibly one-sided) full Lebesgue density point of parameters for which an *acip* exists (such measures always reflect a certain degree of hyperbolicity, see next paragraph) if and only if it satisfies certain dynamical conditions, e.g. it has an *acip* itself. Some very partial results are known (see references below). Notice that maps without *acip*'s can be partial density points of maps with *acip*'s (i.e. the ratio of the measure of parameters for which an *acip* exists in an ε neighbourhood of such a map tend to a number strictly between 0 and 1 as ε tends to 0. For example it follows from the hyperbolicity of the renormalization operator that this is true for infinitely renormalizable maps in the quadratic family. There are no known examples of maps which are in the closure of the set of maps with *acip*'s but are *zero* Lebesgue density points of this set, or even for which such density is not defined. Such examples would be very interesting.

It is very likely that questions related to the degree of persistence of hyperbolicity should be naturally formulated and connected to questions related to the the very notion of hyperbolicity and to the degrees of hyperbolicity which can occur, even though this notion is not yet well deinfed ! All maps in the quadratic family (and even in much more general families) with an absolutely continuous invariant probability measure μ are *non-uniformly hyperbolic with respect to* μ: there exists $\lambda > 0$ such that $\log |(f^n)'(x)|^{1/n} \to \lambda$ for μ-almost every x, i.e. the derivative is growing asymptotically at a (uniform) exponential rate. However it is clear that not all such maps exhibit the same *degree of hyperbolicity*: for example the rate of decay of correlations, which reflects some fairly intrinsic properties of the map, can be exponential in certain cases and subexponential in others [NS98, BLvS99].

Construction of the invariant measure. The construction and the estimates given here can be used for parameters in Ω^* to prove the existence of an absolutely continuous invariant probability measure. Indeed they easily give rise to an induced map with very strong hyperbolicity properties which in fact implies exponential decay of correlations for Hölder continuous observables. Indeed, a version of Lemma 4.3 shows immediately that there exists a simple partition $\mathcal{P}$ of the phase space, a function $P : \mathcal{P} \to \mathbb{N}$ and an induced uniformly expanding map F with uniformly bounded distortion given by $F|\omega = f^{P(\omega)}|\omega$. This map however does not satisfy $\inf_{\omega\in\mathcal{P}} |P(\omega)| > 0$, i.e. elements of $\mathcal{P}$ do not grow to a fixed size under F. Nevertheless Proposition 3.1 implies that F can be iterated and $\mathcal{P}$ refined to give a new induced map $\hat{F}$, a refinement $\hat{\mathcal{P}}$ of $\mathcal{P}$ and a function $\hat{P} : \hat{\mathcal{P}} \to \mathbb{N}$ with $\hat{F}|(\hat{\omega}) = f^{\hat{P}}|(\omega)$ such that all $\hat{\omega} \in \hat{\mathcal{P}}$ grow to fixed size under $\hat{F}$. Moreover $\hat{P}$ is summable and has an exponentially decaying tail: $|\{x : \hat{P}(x) \geq n\}|$ is exponentially small in n. The existence of an *acip* μ for f then follows by classical arguments. By recent results of Lai-Sang Young [You98], the exponentially decaying tail of $\hat{N}$ immediately implies that if μ is mixing it also satisfies strong stochastic properties such as exponential decay of correlation and the central limit theorem. See [BLvS99] for a generalization to a larger class of maps satisfying some very weak conditions on the critical orbits.

Comparison with other approaches. All proofs and generalizations of Jakobson's Theorem must essentially rely on the construction of some sequence of partitions with some combinatorial information which keeps track of the certain aspects of the *itinerary* of partition elements. The partition needs to be constructed in such a way as to guarantee uniform distortion bounds and sufficiently strong analytic estimates on the size of partition elements in terms of their combinatorics. These ingredients are then all plugged into a statistical argument to obtain the necessary estimates on the conditional probabilities of satisfying the required conditions.

There are two main reasons for which this essential similarity in the various proofs is not always easily recognizable. One is that different approaches differ in the specifics and the order in which these constructions and estimates are carried out. In this paper for example, the partitions are constructed following [BC91] by fixing a partition in dynamical space near the critical point and essentially pulling it back via a special family of maps from parameter to dynamical space. In other approaches, such as recent notes of Yoccoz [Yoc], the partition is constructed topologically using preimages of a fixed point. Both constructions do the job but require different approaches to the distortion bounds and to the way the combinatorial information is defined. This can also lead to an (apparently) different structure to the overall argument as definitions and estimates need to be introduced at different stages.

Probably the most confusing aspect of trying to compare different proof is related to the choice of the (essentially geometric) conditions which are needed as an *intermediate step* in the construction. The existence of an *acip* is a relatively abstract notion and therefore a more concrete condition is introduced and is then shown on the one side to occur with positive probability in parameter space and on the other to imply the existence of an *acip*. Depending on what kind of condition is chosen, the proportion of work required to prove its persistence in parameter space and the fact that it implies the existence of an *acip* can vary considerably. In this paper a bounded recurrence condition on the critical orbit is used which I would say is about half way between the two: the proof of the existence of an *acip* given condition $(*)$ involves the construction of an induced Markov map and the estimates involved are almost identical to those required in the proof of the persistence of condition $(*)$ in parameter space. Yoccoz, on the other hand, defines a priori a set of parameters which by definition have induced Markov maps. The existence of an *acip* is then almost automatic and most of the work goes into proving the persistence of such structures in parameter space. Interestingly this involves also defining a condition on the critical orbits in terms of their itineraries with respect to the given partition, which is strongly reminiscent of our condition $(*)$.

1.4 Historical Remarks

The existence of quadratic maps with absolutely continuous invariant probabilities was first noticed by Ulam and von Neumann [UvN45] for the case

$a = 2$; they used the very special fact that f_2 is a Chebyshev polynomial smoothly conjugate to the angle-doubling map on the circle to obtain an explicit formula for the density of the invariant measure μ. The extensive work carried out during the 50's, 60's and 70's on uniformly expanding maps, see [Ren57, Wil79, LY73, Bow79] formed the basis for the more sophisticated techniques required to treat more general maps with critical points. The existence of an *acip* is now known to follow from some very weak conditions on the critical orbit, see [NvS88] and [BLvS99]. Jakobson [Jak81] was the first to show the existence of *acip*'s could be *typical* in some sense, even for systems with critical points. Benedicks and Carleson [BC85] gave an alternative proof based on controlling the recurrence of the critical point and other proofs include [Ryc88, BC91, Tsu93a, Yoc] and the one given in the present paper. The proof in [Tsu93a] is particularly remarkable for its conciseness as it takes full advantage of the special properties of maps close to the top quadratic map. It has recently been proved that in the special case of the quadratic family almost all parameters have either an attracting periodic orbit or admit an absolutely continuous invariant probability measure [GS97, LMN97].

There are many interesting generalizations of Jakobson's Theorem. These are usually limited to proving the persistence of some geometric and/or analytic condition (analogous to the bounded recurrence condition given above) but do not go into the details of the ergodic properties of systems satisfying such conditions. In general however, the methods used can be adapted to show that invariant measures with certain hyperbolicity properties exist. In [dMvS93] a generalization is proved to show that under a mild transversality condition, maps satisfying a *non-recurrence* condition on the critical orbit are full Lebesgue density points of maps with *acip*'s. In [TTY92] a sketch of the proof is given for one-parameter families of maps with finitely many critical points. In [Tsu93b] it is shown that under similar transversality assumptions as in [dMvS93] maps with a finite number of critical points and satisfying some exponential growth and bounded recurrence conditions along forward and backward critical orbits are Lebesgue density points of similar maps. Such transversality conditions have been shown to be satisfied for all maps with exponentially growing derivatives along the critical orbit in the quadratic family [Tsu] and for maps with non-recurrent critical points in rather general polynomial families [vS]. In [Ree86] a version of Jakobson's Theorem is proved in the context of rational maps; in [Rov93] "contracting Lorenz maps" are studied in which the map has a critical point coinciding with a discontinuity; in [Thu98]families of maps with completely degenerate ("flat") critical points are studied; in [LVb, LT] maps in which both singularities with infinite derivatives and critical points coexist and in [PRV98] a family of maps with an infinite number of critical points is analysed.

Of particular interest are generalizations to higher dimensional situations.

The ideas and techniques required for such a major breakthrough were first introduced in [BC91] in the context of the Hénon family of maps. These were generalized in [MV93, Via93] to quadratic-like maps in the context of homoclinic bifurcations. In [DRV96, Cos98] similar techniques were applied to study the dynamics near saddle-node bifurcation points. In [Via97] a class of systems with many expanding directions is shown to be persistent under perturbations. In [PR97] and [PRV] a class of flows with an equilibrium point with complex eigenvalues is considered and in [LVa] a generalized model of the Lorenz equations is developed to include parameters for which the return map contains folds as well as an unbounded derivative. These results do not study ergodic properties of the systems but rather concentrate on proving the persistence of certain geometrical/analytical structures which embody some degree of hyperbolicity. The ergodic properties in some cases have also been studied, see [BY93, BY98, BVb, BVa] for the Hénon maps, [Alv97, AV] for the systems studied in [Via97] and [HL] for generalized Lorenz-like attractors of [LVa].

2 The combinatorial structure

For the rest of the paper we fix constants

$$\log\sqrt{2} > \hat{\lambda} \gg \alpha \gg \delta \gg \iota \gg \varepsilon > 0$$

in the order given, where the symbol $\gg$ means that the constant on the right is chosen sufficiently small after the constant on the left has been fixed. To simplify the notation we shall also introduce the constants

$$\lambda = \hat{\lambda} - 2\alpha \quad \text{and} \quad \beta = \alpha/\lambda.$$

The constant $\hat{\lambda}$ gives the expansion rate outside a critical neighbourhood. It can be chosen arbitrarily close to $\log\sqrt{2}$ and we shall take advantage of this fact to simplify certain estimates. The constants α and δ are those used in the definition of condition $(*)$ and in the formulation of the results. Notice that taking α smaller *strengthens* the statement of the Theorem. The constant ι is an incidental constant used in the construction to define a critical neighbourhood $\Delta^+ = (-\delta^\iota, \delta^\iota)$ which strictly contains the critical neighbourhood $\Delta = (-\delta, \delta)$. The fact that ε is chosen last is no problem as our Main Theorem is an asymptotic statement about what happens as $\varepsilon \to 0$ once the other constants have been fixed.

2.1 Inductive assumptions

Let $\Omega^{(0)} = \Omega = \Omega_\varepsilon$ and $\mathcal{P}^{(0)} = \{\Omega^{(0)}\}$ denote the trivial partition of Ω. Given $n \geq 1$ suppose that for each $k \leq n-1$ there exists a set $\Omega^{(k)} \subseteq \Omega$ and a partition $\mathcal{P}^{(k)}$ of $\Omega^{(k)}$ into intervals such that each $\omega \in \mathcal{P}^{(k)}$ has an associated *itinerary* constituted by the following information (for the moment this is given as abstract data, the geometrical meaning of this data will become clear in the next section). A sequence

$$0 = \eta_0 < \eta_1 < \cdots < \eta_s \leq k, \qquad s = s(\omega) \geq 0$$

of *escape times*. Between any two escape times η_{i-1} and η_i (and between η_s and k) there is a sequence

$$\eta_{i-1} < \nu_1 < \cdots < \nu_t < \eta_i \qquad t = t(\omega, i) \geq 0$$

of *essential return times* (or *essential returns*) and between any two essential returns ν_{j-1} and ν_j (and between ν_t and η_i) there is a sequence

$$\nu_{j-1} < \mu_1 < \cdots < \mu_u < \nu_j \qquad u = u(\omega, i, j) \geq 0$$

of *inessential return times* (or *inessential returns*). Following every essential (resp. inessential) return there is a time interval

$$[\nu_j + 1, p_j] \ \ (\text{resp. } [\mu_j + 1, p_j]\ )$$

with $p_j > 0$ called the *binding period* associated to the return time ν_j (resp. μ_j). A binding period cannot contain any essential or inessential return times although it may contain a sequence of *bound return times* (or *bound returns*). Associated to each essential, inessential and bound return time is a positive integer r called the *return depth* (which is roughly the (norm of the) logarithm of the distance from the critical point) . We define the functions

$$\mathcal{R}^{(k)} : \Omega^{(k)} \to \mathbb{N} \quad \text{and} \quad \mathcal{E}^{(k)} : \Omega^{(k)} \to \mathbb{N}$$

which associate to each $a \in \Omega^{(k)}$ respectively the total sum of all return depths and the total sum of essential return depths only of the element $\omega \in \mathcal{P}^{(k)}$ containing a in its itinerary up to and including time k. By construction both functions are constant on elements of $\mathcal{P}^{(k)}$.

We also assume that the following two analytic properties are satisfied.

Bounded recurrence: Each $a \in \Omega^{(k)}$ satisfies $(BR)_k$.

Bounded distortion: For every $\omega \in \mathcal{P}^{(k)}$, the map

$$c_j : \omega \to \omega_j = \{c_j(a) : a \in \omega\}$$

is a diffeomorphism with uniformly bounded distortion $\forall\ j \leq \nu + p + 1$ where ν is the last essential or inessential return of ω before time k and

p is the binding period. More precisely, there exists a constant $D > 0$ independent of k and $\omega \in \mathcal{P}^{(k)}$ such that

$$\frac{|c_j'(a)|}{|c_j'(b)|} \leq D \qquad \forall\, a, b \in \omega,\ \forall\, j \leq \nu + p + 1$$

Moreover if $\nu+p+1 < k$ the same statement holds for all $\nu+p+1 < j \leq k+1$ replacing ω by any subinterval $\omega' \subseteq \omega$ which satisfies $\omega_j' \subseteq \Delta^+$.

2.2 Definition of $\Omega^{(n)}$ and $\mathcal{P}^{(n)}$

For $r \in \mathbb{N}$ let

$$I_r = [e^{-r}, e^{-r+1}) \quad \text{ and } I_{-r} = -I_r.$$

Then

$$\Delta^+ = \{0\} \cup \bigcup_{|r| \geq r_{\delta^+} - 1} I_r \quad \text{ and } \quad \Delta = \{0\} \cup \bigcup_{|r| \geq r_\delta - 1} I_r.$$

where

$$r_\delta = \log \delta^{-1} \quad \text{ and } \quad r_{\delta^+} = \iota \log \delta^{-1}$$

and we can suppose without loss of generality that $r_\delta, r_{\delta^+} \in \mathbb{N}$. Finally subdivide each I_r into r^2 subintervals of equal length. This defines partitions $\mathcal{I}, \mathcal{I}^+$ of Δ^+ with $\mathcal{I} = \mathcal{I}^+|_\Delta$. An interval belonging to either one of these partitions is of the form $I_{r,m}$ with $m \in [1, r^2]$ Let $I_{r,m}^\ell$ and $I_{r,m}^\rho$ denote the elements of $\mathcal{I}^+$ adjacent to $I_{r,m}$ and let $\hat{I}_{r,m} = I_{r,m}^\ell \cup I_{r,m} \cup I_{r,m}^\rho$. If $I_{r,m}$ happens to be one of the extreme subintervals of $\mathcal{I}^+$ then let $I_{r,m}^\ell$ or $I_{r,m}^\rho$, depending on whether $I_{r,m}$ is a left or right extreme, denote the intervals $(-2\delta^\iota, -\delta^\iota]$ or $[\delta^\iota, 2\delta^\iota)$ respectively.

Remark 1. The reason for the subdivision of each I_r into r^2 subintervals will become clear in the last part of the argument in the proof of the distortion estimates, see (12) and Sublemma 5.2.2. It is related to the fact that the distortion of a single iterate on each I_r is of the order of 1 (which is "not summable") whereas the distortion of a single iterate of $I_{r,m}$ is of the order of $1/r^2$ (which is "summable").

We now define a refinement $\hat{\mathcal{P}}^{(n)}$ of $\mathcal{P}^{(n-1)}$. Let $\omega \in \mathcal{P}^{(n-1)}$ and $\omega_n = \{c_n(a), a \in \omega\}$ denote its "image" at time n. We distinguish different cases.

Non-chopping times. We say that n is a non-chopping time for ω if one (or more) of the following situations occur:

1. $\omega_n \cap \Delta^+ = \emptyset$;
2. n belongs to the binding period associated to some (essential or inessential) return time $\nu < n$ of ω;
3. $\omega_n \cap \Delta_+ \neq \emptyset$ but ω_n does not intersect more than two elements of the partition $\mathcal{I}^+$.

In all three cases we let $\omega \in \hat{\mathcal{P}}^{(n)}$. If $\omega_n \cap (\Delta \cap I_{\pm r_\delta}) \neq \emptyset$ in cases b) or c) we say that n is a bound return time or an inessential return time respectively for $\omega \in \hat{\mathcal{P}}^{(n)}$ and we define the corresponding return depth by

$$r = \max\{|r| : \omega_n \cap I_r \neq \emptyset\}.$$

Notice that we are allowing the possibility of intervals having inessential or bound returns not actually intersecting Δ, but just the partition elements of $\mathcal{I}^+$ adjacent to (but outside) Δ. This is for minor technical reasons related to the estimates in Section 5.1.

Chopping times. In all remaining cases, i.e. if $\omega_n \cap \Delta^+ \neq \emptyset$ and ω_n intersects at least three elements of $\mathcal{I}^+$, we say that n is a chopping time for $\omega \in \mathcal{P}^{(n-1)}$. We define a natural subdivision

$$\omega = \omega^\ell \cup \bigcup_{(r,m)} \omega^{(r,m)} \cup \omega^\rho.$$

so that each $\omega_n^{(r,m)}$ fully contains a unique element of $\mathcal{I}_+$ (though possibly extending to intersect adjacent elements) and ω_n^ℓ and ω_n^ρ are components of $\omega_n \setminus (\Delta^+ \cup \omega_n)$ with $|\omega_n^\ell| \geq \delta^\iota$ and $|\omega_n^\rho| \geq \delta^\iota$. If the connected components of $\omega_n \setminus (\Delta^+ \cup \omega_n)$ fail to satisfy the above condition on their length we just glue them to the adjacent interval of the form $w_n^{(r,m)}$. By definition we let each of the resulting subintervals of ω be elements of $\hat{\mathcal{P}}^{(n)}$. The intervals ω^ℓ, ω^ρ and $\omega^{(r,m)}$ with $|r| < r_\delta$ are called *escape components* and are said to have an *escape* at time n. All other intervals are said to have an *essential return* at time n and the corresponding values of $|r|$ are the associated *essential return depths.* We make the same remark here as above: partition elements $I_{\pm r_\delta}$ do not belong to Δ but we still say that the associated intervals $\omega^{(\pm r_\delta, m)}$ have a return rather than an escape.

This defines the partition $\hat{\mathcal{P}}^{(n)}$ of $\Omega^{(n-1)}$. In particular the functions $\mathcal{E}^{(n)}$ and $\mathcal{R}^{(n)}$ are defined on $\Omega^{(n-1)}$ and constant on elements of $\hat{\mathcal{P}}^{(n)}$. Thus we can define

$$\Omega^{(n)} = \{\omega \in \hat{\mathcal{P}}^{(n)} : \mathcal{E}^{(n)}(\omega) \leq \alpha n/10\}$$

and

$$\mathcal{P}^{(n)} = \hat{\mathcal{P}}^{(n)} | \Omega^{(n)}.$$

Notice that an itinerary of escape times, return times, etc. is well defined for all elements of $\mathcal{P}^{(n)}$ up to time n. In Section 5 we shall prove that the two analytic assumptions on the recurrence and the distortion are also satisfied:

Proposition 2.1. *The inductive assumptions are satisfied for $\Omega^{(n)}$ and $\mathcal{P}^{(n)}$.*

In particular all parameters in $\Omega^{(n)}$ satisfy $(BR)_n$. Let

$$\Omega^* = \bigcap_{n \geq 0} \Omega^{(n)}.$$

2.3 Renormalization properties of the combinatorics

We give an alternative combinatorial description of the partition elements defined above. This description is crucial to the probabilistic argument which will be applied in the next section and highlights some remarkable renormalization properties of the construction. To each $\omega \in \hat{\mathcal{P}}^{(n)}$ is associated a sequence $0 = \eta_0 < \eta_1 < \cdots < \eta_s \leq n, \ s = s(\omega) \geq 0$ of escape times and a corresponding sequence of escaping components

$$\omega \subseteq \omega^{(\eta_s)} \subseteq \cdots \subseteq \omega^{(\eta_0)} \quad \text{with} \quad \omega^{(\eta_i)} \subseteq \Omega^{(\eta_i)} \ \text{ and } \ \omega^{(\eta_i)} \in \mathcal{P}^{(\eta_i)}.$$

Letting

$$\omega^{(\eta_i)} = \omega \quad \text{for all } \ s+1 \leq i \leq n$$

gives a well defined parameter interval $\omega^{(\eta_i)}$ associated to $\omega \in \hat{\mathcal{P}}^{(n)}$ for each $0 \leq i \leq n$. Notice that for two intervals $\omega, \tilde{\omega} \in \hat{\mathcal{P}}^{(n)}$ and any $0 \leq i \leq n$, the corresponding intervals $\omega^{(\eta_i)}$ and $\tilde{\omega}^{(\eta_i)}$ are either disjoint or coincide. Then we define

$$Q^{(i)} = \bigcup_{\omega \in \hat{\mathcal{P}}^{(n)}} \omega^{(\eta_i)}$$

and let

$$\mathcal{Q}^{(i)} = \{\omega^{(\eta_i)}\}$$

denote the natural partition of $Q^{(i)}$ into intervals of the form $\omega^{(\eta_i)}$. Notice that

$$\Omega^{(n-1)} = Q^{(n)} \subseteq \cdots \subseteq Q^{(0)} = \Omega^{(0)}$$

and

$$\mathcal{Q}^{(n)} = \hat{\mathcal{P}}^{(n)}$$

since the number s of escape times is always strictly less than n and therefore in particular $\omega^{(\eta_n)} = \omega$ for all $\omega \in \hat{\mathcal{P}}^{(n)}$. For a given $\omega = \omega^{(\eta_i)} \in \mathcal{Q}^{(i)}$, $0 \leq i \leq n-1$ we let

$$Q^{(i+1)}(\omega) = \{\omega' = \omega^{(\eta_{i+1})} \in \mathcal{Q}^{(i+1)} : \omega' \subseteq \omega\}$$

denote all the elements of $\mathcal{Q}^{(i+1)}$ which are contained in ω and $\mathcal{Q}^{(i+1)}(\omega)$ the corresponding partition. Then we define

$$\Delta\mathcal{E}^{(i)} : Q^{(i+1)}(\omega) \to \mathbb{N} \quad \text{where} \quad \Delta\mathcal{E}^{(i)}(a) = \mathcal{E}^{(\eta_{i+1})}(a) - \mathcal{E}^{(\eta_i)}(a)$$

gives the total sum of all essential return depths associated to the itinerary of the element $\omega' \in \mathcal{Q}^{(i+1)}(\omega)$ containing a, between the escape at time η_i and the escape at time η_{i+1}. Finally we let

$$\mathcal{Q}^{(i+1)}(\omega, R) = \{\omega' \in \mathcal{Q}^{(i+1)} : \omega' \subseteq \omega, \Delta\mathcal{E}^{(i)}(\omega') = R\}.$$

3 The probabilistic argument

Modulo the proof of Proposition 2.1 which will be given in section 5, it remains to prove that

$$|\Omega^*| > 0$$

and that 2 is a Lebesgue density point of Ω^*. As mentioned above our main strategy is to estimate the average value of the recurrence of the critical point for parameters in $\Omega^{(n-1)}$ taking into account the orbit up to time n. This is the heart of the proof, where the combinatorial structure and the analytical estimates really need to work together.

A relatively easy argument can be used to show that intervals in $\tilde{\mathcal{P}}^{(n)}$ are exponentially small in terms of their recurrence, i.e. in terms of the total sum of their return depths. However this in itself does not imply that the total measure of points belonging to intervals with strong recurrence is small, we also need to estimate the *cardinality* of the set of intervals in $\tilde{\mathcal{P}}^{(n)}$ with a given recurrence. The easiest way to do this is to use the sequence of return depths as a way of indexing the elements of $\tilde{\mathcal{P}}^{(n)}$ and to use standard combinatorial techniques to estimate the possible number of sequences whose terms add up to a given total. Unfortunately (unboundedly) many intervals may have the same sequence of returns depths and therefore we need to divide the combinatorial structure into *blocks* for which an indexing with bounded multiplicity can be obtained. This is precisely the purpose of the construction of section 2.3. These blocks have in an intrinsic interest as they contain certain scale-invariant properties of the construction. Recall that $\beta = \alpha/\lambda \ll 1$.

Proposition 3.1. *For all $i \le n-1$, $\omega \in \mathcal{Q}^{(i)}$ and $R \ge 0$ we have*

$$\sum_{\omega' \in \mathcal{Q}^{(i+1)}(\omega,R)} |\omega'| \le e^{(10\beta-1)R}|\omega|.$$

The statement in the this Proposition may be considered *the* key estimate in the entire paper. Starting with a "large" interval (the notion of escape time has the purpose precisely of fixing the concept of large in this context) we can construct a subdivision into subintervals which are again large after a certain history of returns to the critical neighbourhood, quantified through the total return depth function R, . The *tail* of R on one escaping component, i.e. the set of points belonging to intervals which have not yet achieved large scale after a total number of returns with return depths R, is exponentially small in R.

Proposition 3.1 will be proved in Section 6. For the moment we assume this estimate and complete the proof of our main Theorem. In section 3.1 we show how the estimate of Proposition 3.1 which applies to a single *block* leads to a bound for the overall average recurrence up to time n. In section 3.2 we complete the proof with the large deviations argument.

3.1 Average recurrence

Lemma 3.1.

$$\int_{\Omega^{(n-1)}} e^{\mathcal{E}^{(n)}/2} = \sum_{\omega \in \mathcal{Q}^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}|\omega| \le e^{3n/r_\delta}|\Omega|.$$

Proof. The equality follows immediately by the definitions. For the inequality, let $0 \le i \le n-1$, and $\omega \in \mathcal{Q}^{(i)}$ and write

$$\sum_{\omega' \in \mathcal{Q}^{(i+1)}(\omega)} e^{\Delta\mathcal{E}^{(i)}(\omega')/2}|\omega'| = \sum_{\omega' \in \mathcal{Q}^{(i+1)}(\omega,0)} |\omega'| + \sum_{R \ge r_\delta} e^{R/2} \sum_{\omega' \in \mathcal{Q}^{(i+1)}(\omega,R)} |\omega'|.$$

Proposition 3.1 implies

$$\sum_{\omega' \in \mathcal{Q}^{(i+1)}(\omega,0)} |\omega'| + \sum_{R \ge r_\delta} e^{R/2} \sum_{\omega' \in \mathcal{Q}^{(i+1)}(\omega,R)} |\omega'| \le \left(1 + \sum_{R \ge r_\delta} e^{(10\beta - \frac{1}{2})R}\right)|\omega|$$

and therefore

$$\sum_{\omega' \in \mathcal{Q}^{(i+1)}(\omega)} e^{\Delta\mathcal{E}^{(i)}(\omega')/2}|\omega'| \le (1 + e^{-r_\delta/3})|\omega| \le e^{3/r_\delta}|\omega|. \tag{1}$$

Since $\mathcal{E}^{(n)} = \Delta\mathcal{E}^{(0)} + \cdots + \Delta\mathcal{E}^{(n-1)}$ and $\Delta\mathcal{E}^{(i)}$ is constant on elements of $\mathcal{Q}^{(i)}$ we can write $\sum_{\omega\in\mathcal{Q}^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}|\omega|$ as

$$\sum_{\omega^{(1)}\in\mathcal{Q}^{(1)}(\omega^{(2)})} e^{\Delta\mathcal{E}^{(0)}(\omega^{(1)})/2} \sum_{\omega^{(2)}\in\mathcal{Q}^{(2)}(\omega^{(3)})} e^{\Delta\mathcal{E}^{(1)}(\omega^{(2)})/2} \cdots \sum_{\omega^{(n-1)}\in\mathcal{Q}^{(n-1)}(\omega^{(n)})} e^{\Delta\mathcal{E}^{(n-1)}(\omega^{(n-1)})/2} \sum_{\omega=\omega^{(n)}\in\mathcal{Q}^{(n)}} e^{\Delta\mathcal{E}^{(n-1)}(\omega^{(n)})/2}|\omega|.$$

Notice the *nested* nature of the expression. Applying (1) repeatedly gives

$$\sum_{\omega\in\mathcal{Q}^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}|\omega| \le e^{3n/r_\delta}|\Omega|.$$

$\square$

3.2 Large deviations

The definition of $\Omega^{(n)}$ gives

$$|\Omega^{(n-1)}| - |\Omega^{(n)}| = |\Omega^{(n-1)} \setminus \Omega^{(n)}| = |\{\omega \in \hat{\mathcal{P}}^{(n)} = \mathcal{Q}^{(n)} : e^{\mathcal{E}^{(n)}/2} \ge e^{\alpha n/20}\}|$$

and therefore using Chebyshev's inequality ($M\,|\{x\in I : f(x) \ge M\}| \le \int_I f$) and Lemma 3.1 we have

$$|\Omega^{(n-1)}| - |\Omega^{(n)}| \le e^{-\alpha n/20} \int_{\Omega^{(n-1)}} e^{\mathcal{E}^{(n)}/2} \le e^{\left(\frac{3}{r_\delta} - \frac{\alpha}{20}\right)n}|\Omega| \le e^{-\alpha n/30}|\Omega|$$

which implies

$$|\Omega^{(n)}| \ge |\Omega^{(n-1)}| - e^{-\alpha n/30}|\Omega|$$

Thus using the fact that for $\varepsilon > 0$ small there exists a $N > 0$ such that $\Omega^{(j)} = \Omega^{(j+1)}$ for all $j \le N$ (this follows easily from the fact that $c_j(2) \equiv -2, \forall j \ge 2$) we get

$$|\Omega^{(n)}| \ge \left(1 - \sum_{i=N}^{n} e^{-\alpha i/30}\right)|\Omega| \quad \text{and} \quad |\Omega^*| \ge \left(1 - \sum_{i=N}^{\infty} e^{-\alpha i/30}\right)|\Omega|.$$

Moreover it is easy to see that $N(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Therefore

$$|\Omega^*_\varepsilon|/|\Omega_\varepsilon| \to 1 \quad \text{as} \quad \varepsilon \to 0$$

which implies that 2 is a one sided Lebesgue density point of Ω^*.

4 Fundamental tools

We state and prove three preliminary Lemmas which are not difficult but whose importance cannot be overestimated. They provide the foundations for the estimates of Sections 5 and 6 and are also of independent interest. Generalizations of each of these results have been successfully developed and applied.

The first is an estimate on the expansion outside a given critical neighbourhood $\Delta = (-\delta, \delta)$. As a corollary we will show that all parameters in Ω_ε^* satisfy the Collet-Eckmann condition. The second is a basic result about the dependence of iterates of the critical points on the parameter. The third formalizes the intuitive idea that when an iterate of a critical point lands in Δ it then retraces part of its initial orbit. Therefore information on the past is useful to study the future.

4.1 Expansion outside Δ

Lemma 4.1. *For all $0 < \hat{\lambda} < \log 2$ and $\delta > 0$ small, there exist constants $\varepsilon > 0$ and $C_\delta > 0$ such that for any $a \in \Omega_\varepsilon$, $f = f_a$, $n \geq 1$ and $x \in I$ such that when $x, fx, \ldots, f^{n-1}x \notin \Delta$ we have*

$$|(f^n)'x| \geq C_\delta e^{\hat{\lambda} n}. \tag{UE1}$$

If moreover $f^n x \in \Delta^+$ we have

$$|(f^n)'x| \geq e^{\hat{\lambda} n} \tag{UE2}$$

Proof. Consider the map $h : [0,1] \to [-2,2]$ given by $h(\theta) = 2\cos\pi\theta$ and define a family of maps on $[0.1]$ by

$$g_a = h^{-1} \circ f_a \circ h.$$

Notice that for $a = 2$, h defines a conjugacy between $f_2 = x^2 - 2$ and the angle doubling map $g_2(\theta) = 2\theta \mod 1$. Indeed using the standard equality $cos^2\psi - 1 = cos2\psi$ we have

$$f_2(h(\theta)) = (2\cos\pi\theta)^2 - 2 = 2\cos 2\pi\theta = h(g_2(\theta)).$$

Now by the chain rule

$$\begin{aligned} |(f_a^n)'(x)| =& |h'(g_a^n(h^{-1}(x))) \cdot (g_a^n)'(h^{-1}(x)) \cdot (h^{-1})'(x)| \\ =& |(g_a^n)'(h^{-1}(x))| \cdot \left| \frac{h'(g_a^n(h^{-1}(x)))}{h'(h^{-1}(x))} \right|. \end{aligned}$$

Using the fact $h'(\theta) = \sin \pi\theta$ is close to zero only near the extreme points it is easy to see that

$$\left| \frac{h'(g_a^n(h^{-1}(x)))}{h'(h^{-1}(x))} \right| \geq \begin{cases} C_\delta & \text{if } f^{n-1}(x) \notin (-\delta, \delta) \\ 1 & \text{if } |f^n(x)| \leq |x| \end{cases}$$

for some constant $C_\delta > 0$ depending on δ. Since $|(g_2^n)'(\theta)| = 2^n$ for any θ this clearly implies the statement with $\hat{\lambda} = \log 2$ for $a = 2$. For $a < 2$ an explicit calculation shows that

$$|g_a'(\theta) - g_2'(\theta)| \leq C_\delta' |2 - a| \leq \varepsilon C_\delta'$$

for some (large) C_δ' and therefore

$$|(g_a^n)'(\theta)| \geq (2 - \varepsilon C_\delta')^n = e^{\log(2-\varepsilon C_\delta')} \geq e^{\hat{\lambda}}$$

for any given $\hat{\lambda} < \log 2$ and $\delta > 0$ as long as ε is small enough. $\square$

Remark 2. Estimates analogous to (UE1) with $\hat{\lambda}$ depending on δ hold in very great generality, e.g. for any C^2 unimodal map without periodic attractors [Mis81, Man85]. The remarkable aspects of the statement given here are 1) that the constant $\hat{\lambda}$ can be chosen independent of δ, and 2) that the constant C_δ can be dropped if $f^n(x)$ is closer to the criticalpoint than x is. The proof given here takes full advantage of the special nature of the parameter value $a = 2$, namely that it is smoothly conjugate to a uniformly expanding map. A similar result holds for perturbation of more general maps with non-recurrent critical orbits [dMvS93] or even for maps with certain exponential growth (Collet-Eckmann) conditions on the forward and backward orbits of the critical points [Tsu93b], however it is harder there to obtain good estimates on the dependence of constants such as C_δ, C_δ' and ε in terms of δ. Sharp estimates (or indeed any estimates at all) for $\hat{\lambda}$ and C_δ in terms of δ in more general situations would be extremely interesting. These issues are related to the questions mentioned above about the characterization of maps which are Lebesgue density points of maps with *acip*'s.

Recall that $\hat{\lambda}, \alpha$ and δ are assumed to have been fixed.

Corollary 4.1.1. *For all $\varepsilon > 0$ sufficiently small, $a \in \Omega_\varepsilon$ and $k \geq 1$ such that f_a satisfies $(BR)_k$ we have*

$$|(f^{k+1})'(c_0)| \geq e^{(\hat{\lambda}-2\alpha)(k+1)} = e^{\lambda(k+1)}.$$

Proof. Let N be a large constant such that $c_i(a) \notin \Delta$ and $(f_a^i)'(c_0(a)) \geq 3^i$ for all $i \leq N$ and all $a \in \Omega_\varepsilon$. N can be chosen arbitrarily large if ε is chosen small, this follows from the fact that for $a = 2$ the critical point is in the preimage of a fixed point with derivative equal to 4. Let $0 < N < \nu_1 < \cdots < \nu_s \leq k$ denote all the iterates such that $c_{\nu_j} \in \Delta$. By that chain rule we have, for any fixed $a \in \Omega_\varepsilon$ and $f = f_a$,

$$(f^{k+1})'(c_0) = (f^N)'(c_0) \cdot (f^{\nu_1 - N})'(c_N) \cdots (f^{\nu_s - \nu_{s-1}})'(c_{\nu_{s-1}}) \cdot (f^{k+1-\nu_s})'(c_{\nu_s}).$$

Moreover each term in the product, except for the first one, can be further split up into a piece of orbit outside Δ for which (UE2) holds, and the last iterate inside Δ. Thus rearranging the terms be separating iterates outside Δ and inside Δ gives

$$|(f^{k+1})'(c_0)| \geq 3^N e^{\hat{\lambda}(k+1-N-s)} \prod_{j=1}^{s} |f'(c_{\nu_j})|$$

and using the definition of $(BR)_k$ yields

$$|(f^{k+1})'(c_0)| \geq 3^N e^{\hat{\lambda}(k+1-N-s)} e^{-\alpha k}.$$

Finally notice that that whenever $c_{\nu_j} \in \Delta$ we have $\log |c_{\nu_j}|^{-1} \geq \log \delta^{-1}$ and therefore condition $(BR)_k$ also implies in particular a bound on the number of returns:

$$s \log \delta^{-1} \leq \sum_{\nu_i \leq n} \log |c_{\nu_i}|^{-1} \leq \alpha k.$$

Substituting this into the last display above completes the proof. □

4.2 Parameter dependence

Lemma 4.2. *For all $a \in \Omega_\varepsilon$ and all $k \geq 1$ such that f_a satisfies $(BR)_k$,*

$$\frac{1}{2} \leq \frac{|c_k'(a)|}{|(f_a^k)' c_0|} \leq 2.$$

Proof. Let $F : \Omega_\varepsilon \times I \to I$ be the function of two variables defined inductively by

$$F(a, x) = f_a(x) \quad \text{and} \quad F^k(a, x) = F(a, f_a^{k-1} x).$$

Then, for $x = c_0$,

$$c_k'(a) = \partial_a F^k(a, c_0) = \partial_a F(a, f_a^{k-1} c_0) = -1 + f_a'(c_{k-1}) c_{k-1}'(a).$$

Iterating this argument gives

$$-c_k'(a) = 1 + f_a'(c_{k-1}) + f_a'(c_{k-1})f_a'(c_{k-2}) + \dots$$
$$\dots + f_a'(c_{k-1})f_a'(c_{k-2})\dots f_a'(c_1)f_a'(c_0)$$

and dividing both sides by $(f^k)'(c_0) = f_a'(c_{k-1})f_a'(c_{k-2})\dots f_a'(c_1)f_a'(c_0)$ gives

$$-\frac{c_k'(a)}{(f_a^k)'(c_0)} = 1 + \sum_{i=1}^{k} \frac{1}{(f^i)'(c_0)}. \tag{2}$$

Now choose N so that

$$\sum_{i=N}^{\infty} e^{-\lambda i} \leq \frac{1}{10}$$

and $\varepsilon > 0$ sufficiently small so that for all $a \in \Omega_\varepsilon$,

$$(f_a^i)'(c_0) \leq -(3.5)^i \qquad \forall\, i \leq N. \tag{3}$$

Then, by (2),

$$\frac{|c_k'(a)|}{|f_a^k(c_0)|} \geq 1 - \sum_{i=1}^{N}(3.5)^{-i} - \sum_{N+1}^{\infty} e^{-\lambda i} \geq 1 - \frac{2}{5} - \frac{1}{10} \geq \frac{1}{2}.$$

The upper bound is obtained in the same way. □

Remark 3. A similar remark applies here as in the previous section. The proof is relatively straightforward in this case where we take advantage of (3) which follows from the fact that the critical point is preperiodic repelling. Moreover, in this case transversality of the family has a clear geometric meaning: a preperiodic repelling critical point is a codimension one condition in the space of maps and the statement in the Lemma is saying that the quadratic family crosses this subspace transversally and with positive speed. Similar statements are usually assumed in more general versions of Jakobson's Theorem; a geometric interpretation is not obvious in such cases but one can work with the analytic condition that the expression in (2) is bounded away from 0. Tsujii [Tsu] has recently shown that the quadratic family does indeed satisfy such a transversality condition at Collet-Eckmann parameters and van Strien [vS] has proved the same for rather more general polynomial families at Misiurewicz parameters (non-recurrent critical points).

The results just mentioned certainly have an intrinsic interest, although in this context it is not clear why any form of transversality at all should be required. Transversality conditions, in the theory of local bifurcations for example, are often imposed to guarantee that one moves *away from a bad situation* with positive speed; the opposite is true here: we begin with a map with an *acip* and want to show that this phenomenon *persists.* Probably some smoothness condition (maybe even a very weak one) on the dependence of the maps with respect to the parameter is all that is really needed. In fact we shall not use the statement of Lemma 4.2 directly but rather an immediate consequence stated in the following Corollary. The rest of the proof could most likely be adapted to work even if the constants 4 and 1/4 below were replaced by some other constants possibly even depending on ε, which would probably be necessary if the transversality conditions were not satisfied.

Corollary 4.2.1. *Let $\omega \subset \Omega$ and suppose that all $a \in \omega$ satisfy $(BR)_k$. Then for all $1 \leq i \leq j \leq k$ there exists $\xi \in \omega$ such that*

$$|(f_\xi^{j-i})'(\xi_i)|/4 \leq \frac{|\omega_l|}{|\omega_k|} \leq 4|(f_\xi^{j-1})'(\xi_i)|$$

Proof. Consider the map $\varphi : \omega_i \to \omega_j$ given by $\varphi(c_i(a)) = c_j \circ c_i^{-1}(a)$. By the mean value theorem there exists $\xi \in \omega$ such that $|\omega_j|/|\omega_i| = |\varphi'(\xi_i)|$. Then by the chain rule and Proposition 4.2 we get the desired statement. □

4.3 Binding

Lemma 4.3. *Let $0 \leq k \leq n-1$, $\omega \in \mathcal{P}^{(k)}$ and suppose that k is an essential return time for ω with return depth r. Then there exists an integer*

$$p \leq 2r$$

such that

$$|\omega_{k+p+1}| \geq |\omega_k|^{8\beta} \quad \textit{with } \beta = \alpha/\lambda < 1/8.$$

Proof. For $a \in \omega$ let $\gamma = \gamma(a)$ be the interval $[c_k(a), 0]$ (or $[0, c_k(a)]$) and for $j \geq 0$, let $\gamma_j = f_a^{j+1}(\gamma)$. We define the *binding period of* $c_k(a)$ as

$$p(a,k) = \sup\{m \in \mathbb{N} : |\gamma_j| \leq e^{-2\alpha j} \quad \forall\, j \leq m-1\}$$

and the *binding period of* ω as

$$p = p(\omega, k) = \min\{p(a,k) : a \in \omega\}$$

Intuitively $p(a,k)$ is essentially constant on ω but strictly speaking there can be some small variation. An additional argument is required at the end of the proof to deal with this problem.

For the next two sublemmas we fix a parameter $a \in \omega$ and let $\hat{p} = \min\{p-1, k\}$. We shall show below that $p \ll k$ but for the moment we want to make sure that our bounded recurrence condition holds during the time interval under consideration.

Sublemma 4.3.1 (Distortion during the binding period). *For all $y_0, z_0 \in \gamma_0$ and $0 \le i \le \hat{p}+1$ we have*

$$\left|\frac{(f^i)'(z_0)}{(f^i)'(y_0)}\right| \le e^{(1-e^{-\alpha})^{-2}} := D_\alpha$$

Proof. By the chain rule

$$\left|\frac{(f^i)'(z_0)}{(f^i)'(y_0)}\right| = \left|\prod_{j=0}^{i-1}\frac{f'(z_j)}{f'(y_j)}\right| = \left|\prod_{j=0}^{i-1}\left(1+\frac{f'(z_j)-f'(y_j)}{f'(y_j)}\right)\right|.$$

By the definition of f, $f'(z) = 2|z|\ \forall\ z \in I$ and $f''(z) \equiv 2$. Then by the mean value theorem, $|f'(z_j) - f'(y_j)| \le |\gamma_j| \sup_{z\in I} f''(z) = 2|\gamma_j|$. Then, using the fact that $\log(1+x) < x$,

$$\left|\frac{(f^i)'(z_0)}{(f^i)'(y_0)}\right| \le e^{\log\prod_{j=0}^{i-1}\left(1+\frac{|\gamma_j|}{|y_j|}\right)} = e^{\sum_{j=0}^{i-1}\log\left(1+\frac{|\gamma_j|}{|y_j|}\right)} \le e^{\sum_{j=0}^{i-1}\frac{|\gamma_j|}{|y_j|}}$$

and thus it is sufficient to show that

$$\sum_{j=0}^{i-1}\frac{|\gamma_j|}{|y_j|} \le (1-e^{-\alpha})^{-2}.$$

Since $|\gamma_j| \le e^{-2\alpha j}$ this follows immediately if we show that

$$|y_j| \ge (1-e^{-\alpha})e^{-\alpha j}$$

which in turn follows if $|c_j| \ge e^{\alpha j}$ since

$$|y_j| \ge |c_j| - |\gamma_j| \ge e^{-\alpha j} - e^{-2\alpha j} \ge e^{-\alpha j}(1-e^{-\alpha j}) \ge e^{-\alpha j}(1-e^{-\alpha}).$$

Let N be the smallest integer such that $e^{-\alpha N} \le \delta$ (i.e. N is the earliest time at which condition (BR) allows a return to Δ; notice that $N \sim \log\delta^{-1}/\alpha$). Then for $j \ge N$ we have clearly have $|c_j| \ge e^{-\alpha j}$ by condition $(BR)_j$ and for $j \le N$ we have the basic estimate $|c_j| \ge 1 - \varepsilon 4^j$ (since 4 is the maximum expansion rate). Therefore we just choose ε small enough (recall that it can be chosen arbitrarily small after fixing α and δ). □

Sublemma 4.3.2 (Expansion at the end of the binding period). *Letting $x = c_k(a)$ we have an estimate for the length of the binding period:*

$$\log|x|^{-1} \le p \le \frac{2}{\lambda}\log|x|^{-1} \ll k;$$

and two alternative estimates for the growth of the derivative at the end of the binding period:

$$|(f^{p+1})'(x)| \ge |x|^{5\beta-1} \qquad \text{and} \qquad |(f^{p+1})'(x)| \ge e^{\frac{\lambda(p+1)}{6}}.$$

Notice that the derivatives are calculated at $x = c_k(a)$, thus there is an overall expansion during the binding period which more than compensates the small derivative at the return.

Proof. By assumption $(BR)_{\hat{p}}$ is satisfied and therefore $|(f_a^j)'(c_0)| \geq e^{\lambda j}$ for all $j \leq \hat{p}$. By the mean value theorem, the definition of binding and the bounded distortion estimate above,

$$e^{-2\alpha\hat{p}} \geq |\gamma_{\hat{p}}| \geq |(f^{\hat{p}})'(c_0)||\gamma_0|/D_\alpha \geq e^{\lambda(\hat{p}-1)}|x|^2/D_\alpha.$$

Taking logarithms and rearranging yields

$$\begin{aligned} \hat{p} &\leq \frac{2\log|x|^{-1} + 2\log D_\alpha + \lambda}{\lambda + 2\alpha} \\ &\leq \frac{2}{\lambda}(\log|x|^{-1} + \log D_\alpha) + 1 \leq \frac{2\log|x|^{-1}}{\lambda} - 1 \end{aligned} \tag{4}$$

as long as $|x| \leq 2\delta^\iota$ is sufficiently small. Thus $\hat{p} < k$ and in particular $p = \hat{p}+1$. Similarly, using the fact that $|f'(z)| \leq 4,\ \forall z$ and that $|\gamma_p| \geq e^{-2\alpha p}$, we have

$$4^p|x|^2 D_\alpha \geq |(f^p)'(c_0)||\gamma_0|D_\alpha \geq |\gamma_p| \geq e^{-2\alpha p}$$

and therefore $p \geq (2\log|x|^{-1} - \log D_\alpha)/(2\alpha + \log 4) \geq \log|x|^{-1}$. This completes the estimates on the size of p. To get the derivative estimates, notice that by the mean value theorem there exists $\xi_0 \in \gamma_0$ such that $|(f^p)'\xi_0| = |\gamma_p|/|\gamma_0|$. Thus by Sublemma 4.3.1, the fact that $|f'(x)| = 2|x|$ and $|\gamma_0| = x^2$ and the chain rule, we have

$$|(f^{p+1})'(x)| = |f'(x)|\ |(f^p)'(x_0)| \geq \frac{1}{D_\alpha}\frac{|\gamma_p|}{|\gamma_0|} \geq \frac{2}{D_\alpha}\frac{e^{-2\alpha p}}{|x|}.$$

The two required estimates for the expansion are now easily obtained by substituting a bound for p in terms of x in one case and a bound for x in terms of p in the other. More precisely (4) gives $e^{-2\alpha p} \geq e^{-4\alpha\log|x|^{-1}/\lambda} \geq |x|^{4\alpha/\lambda}$ and therefore $|(f^{p+1})'(x)| \geq \frac{e^{-2\alpha p}}{/} D_\alpha|x| \geq |x|^{\frac{5\alpha}{\lambda}-1}$. Similarly (4) also gives $\log|x|^{-1} \geq \lambda p/4$ i.e. $|x|^{-1} \geq e^{\lambda p/4}$ and therefore $|(f^{p+1})'(x)| \geq e^{-2\alpha p}/D_\alpha|x| \geq e^{-2\alpha p}e^{\lambda p/4}/D_\alpha \geq e^{(\lambda-2\alpha)(p+1)/5}$. □

We can now complete the proof of the Lemma. Suppose first of all that $p(a)$ is constant on ω. Then by Sublemma 4.3.2 and using the fact that $|c_k(a)| \geq e^{-2r}$ by assumption, we have $|(f^{p+1})'(c_k(a))| \geq |c_k(a)|^{5\beta-1} \geq e^{(6\beta-1)r}$ and thus by Corollary 4.2.1 $|\omega_{k+p+1}| \geq e^{(7\beta-1)r}|\omega_k|$. The result then follows from the assumption that k is an essential return for ω implying that $e^{-r}/r^2 \leq |\omega_k| \leq 2e^{-r}$. If $p(a)$ is not constant on ω then for all $\tilde{a} \in \omega$ we have

$$|(f_{\tilde{a}}^p)'(c_{k+1}(\tilde{a}))| \sim |(f_{\tilde{a}}^p)'(c_0(\tilde{a}))| \sim |(f_a^p)'(c_0(a))| \sim |(f_a^p)'(c_{k+1}(a))|.$$

Here the symbol $\sim$ is used to indicate that the quantities on either side are the same up to a multiplicative constant which does not depend on a, ω or k. The middle $\sim$ follows from the inductive hypotheses and the other two from Sublemma 4.3.1. Therefore

$$|(f_{\tilde{a}}^{p+1})'(c_k(\tilde{a}))| \sim |(f_a^{p+1})'(c_k(a))|$$

and the rest of the argument proceeds as above. □

5 Distortion and recurrence bounds

In this section we prove the general inductive step in the definition of $\Omega^{(n)}$ and $\mathcal{P}^{(n)}$. Notice that the combinatorial properties follow by construction thus we only need to prove the two analytic conditions: bounded distortion and bounded recurrence. We begin with the latter.

5.1 Condition $(BR)_n$

Lemma 5.1. *All points in $\Omega^{(n)}$ satisfy $(BR)_n$.*

Recall that functions $\mathcal{R}^{(k)}$ and $\mathcal{E}^{(k)}$ are defined on $\Omega^{(n)}$ and constant on elements of $\mathcal{P}^{(n)}$ for all $k \leq n$ and most importantly we have $\mathcal{E}^{(k)}(a) \leq \lambda\alpha k/10$ for all $k \leq n$ and all $a \in \Omega^{(n)}$. We claim that it is sufficient to prove that for all $a \in \Omega^{(n)}$ and $0 \leq k \leq n$ we have

$$\mathcal{R}^{(k)}(a) \leq 5\mathcal{E}^{(k)}(a) \tag{5}$$

by virtue of the following

Sublemma 5.1.1. *Let $a \in \Omega^{(n)}$ and suppose that $\mathcal{R}^{(k)}(a) \leq \alpha k/2$ for all $k \leq n$. Then a satisfies $(BR)_n$.*

Proof. Notice first of all that by construction $c_\nu(a) \in \Delta$ only if a has some non-zero return depth associated to a return at time ν. In other words all the returns to Δ which contribute to the sum in the definition of condition (BR) are accounted for by considering all the return times and return depths associated to a. By construction, if $\nu \leq n$ is a return for ω and the associated return depth is r, then all points in ω satisfy $|c_\nu(a)| \geq e^{-2r}$ (in fact they satisfy $|\varphi^\nu a| \sim e^{-r}$ but the weaker estimate is sufficient for our purposes). Therefore $\log |c_\nu(a)|^{-1} \le 2r$ and

$$\sum_{\nu_j \le n} \log |c_{\nu_j}(a)|^{-1} \le \sum_{\nu_j \le n} 2r_j \le 2\mathcal{R}^{(n)}(\omega) \le \alpha n.$$

$\square$

Equation 5 says that the at least one fifth of the total sum of return depths associated to the orbit of the critical point of a given parameter a is formed by essential return depths. Thus all we need to do is prove that the inessential and bound returns occurring between two essential returns have a total sum of return depths which can be bounded in terms of the return depths of the preceding essential return. We divide the argument into two parts; first we consider the essential returns and then the bound returns.

Sublemma 5.1.2. *Let $\omega \in \mathcal{P}^{(\nu)}$, $\nu \in [1, n]$ and suppose that all points in ω satisfy $(BR)_\nu$ and that ν is an essential return for ω. Set $\nu := \mu_0$ and let $\nu < \mu_1 < \cdots < \mu_u$ be a maximal sequence of inessential returns occurring after time ν and before any subsequent chopping time. Let $r_0, r_1, \ldots, r_u$ be the corresponding inessential return depths. Then*

$$\sum_{r=0}^{u} r_i \le \frac{3}{2} r_0.$$

Proof. The idea of the proof is simple. Firstly, there are no chopping times between time μ_0 and time μ_u and the intervals are growing exponentially, in fact the growth can be estimates in terms of the return depths using the binding period estimates. Secondly, large intervals cannot fall too close to the critical point or else they would give rise to a chopping time. Thus we can bound the return depths at each return in terms of the size of the interval. More formally, let $p_0, \ldots, p_u$ be the lengths of the binding periods corresponding to the returns $\mu_0, \ldots, \mu_u$. By the bounded recurrence condition and Sublemma 4.3.2 we have for each $a \in \omega$ and $i = 0, \ldots, u$,

$$|(f_a^{p_i+1})'(c_{\mu_i}(a))| \ge |c_{\mu_i}(a)|^{\frac{5\alpha}{\lambda}-1} \ge e^{\left(1-\frac{6\alpha}{\lambda}\right)r}.$$

By Lemma 4.1 we also have

$$|(f_a^{\mu_{i+1}-(\mu_i+p_i+1)})'(c_{\mu_i+p_i+1}(a))| \ge 1$$

and therefore combining these two estimates and using the chain rule gives

$$|(f_a^{\mu_u+p_u+1-\mu_0})'(c_{\mu_0}(a))| \ge e^{\left(1-\frac{6\alpha}{\lambda}\right)\sum_{i=0}^{u} r_i}.$$

Thus by Corollary 4.2.1,

$$4 \geq 2|\omega_{\mu_u+p_u+1}| \geq |\omega_{\mu_0}| e^{\left(1-\frac{6\alpha}{\lambda}\right)\sum_{i=0}^{u} r_i}.$$

Now since μ_0 is an essential return we have $|\omega_{\mu_0}| \geq e^{-r_0}/r_0^2 \geq 8e^{-(1+\tilde{\alpha})r_0}$ for some $\tilde{\alpha} > 0$ which can be chosen small and so

$$e^{-(1+\tilde{\alpha})r_0} \leq |\omega_{\mu_0}| \leq 4e^{-\left(1-\frac{6\alpha}{\lambda}\right)\sum_{i=0}^{u} r_i}$$

which immediately implies

$$\sum_{i=0}^{u} r_i \leq \frac{1+\tilde{\alpha}}{1-\frac{6\alpha}{\lambda}} r_0 \leq \frac{3}{2} r_0.$$

□

Sublemma 5.1.3. *Let $\omega \in \mathcal{P}^{(\mu)}$, $\mu \in [1, n]$ and suppose that μ is an essential or inessential return for ω with return depth ρ. Let $p > 0$ be the length of the binding period following time μ and let $\mu < \zeta_1 < \cdots < \zeta_v \leq \mu + p$ be the bound return times of ω, i.e. those times for which $\omega_{\zeta_i} \cap \Delta \neq \emptyset$, and $\rho_1, \ldots, \rho_v$ the corresponding bound return depths: $\rho_i = \max\{|r| : \omega_{\zeta_i} \cap I_r \neq \emptyset\}$. Then*

$$\sum_{i=1}^{q} \rho_i \leq \rho.$$

Proof. For simplicity we fix $a \in \omega$ which has the deepest sequence of bound returns. By the definition of binding the points $c_{\mu+j}(a)$ is close to $c_{j-1}(a)$ for all $j \leq p$ and thus, intuitively, a bound return ζ_i for c_μ occurs exactly when $c_0(a)$ has a return at time $\zeta_i - \mu + 1$. We assume for the moment that this is indeed the case. The main idea is that by the definition of binding period c_{ζ_i} and $c_{\zeta_i-\mu+1}$ are very close (of the order of $e^{-2\alpha\zeta_i-\mu+1}$) and by condition $(BR)_{\zeta_i-\mu+1}$, $|c_{\zeta_i-\mu+1}|$ is of the order of $e^{-\alpha\zeta_i-\mu+1}$ and therefore the return depth ρ_i corresponding to the bound return at time ζ_i is almost the same as the associated return depth r_i corresponding to the return at time $\zeta_i - \mu + 1$. Thus the sum of the bound return depths is uniformly almost the same as the sum of all the return depths associated to $c_0(a)$ up to time p. By condition $(BR)_p$ and the estimates on the length of p this is of the order of $\alpha p \sim \alpha r$. The details are left to the reader.

It can happen however that $\zeta_i - \mu + 1$ is not a return time but $c_{\zeta_i-\mu+1}$ falls very close to Δ so that c_{ζ_i} falls inside Δ and gives rise to a bound return time. This situation has no real relevance to the dynamics and could to all effects and purposes be ignored by modifying slightly the definitions. To be formally correct however there are at least a couple of ways to deal with it. One way is to change slightly the definition of condition $(BR)_n$ so that not all returns to Δ are counted but those to a neighbourhood which shrinks slowly from Δ to Δ minus the two extreme subintervals of the partition. In this way we could guarantee that the above situation does not occur. Alternatively, notice that the bound return depths corresponding to such situations are necessarily very small, essentially r_δ. The techniques used for the binding estimates can be applied here to show that there is a binding period of length $\sim r_\delta$ following time $\zeta_i - \mu + 1$ during which this situation cannot happen, and is therefore very rare. Once again the details are straightforward and are left to the reader. $\square$

5.2 Bounded distortion

In this section we shall prove a strong version of the distortion bounds.

Lemma 5.2. *Let $\omega \in \mathcal{P}^{(n)}$. Then*

$$\frac{|c_k' a|}{|c_k' b|} \le e^{1/r_\delta} = D$$

for all $a, b \in \omega$ and all $k \le \nu_q + p_q + 1$ where ν_q is the last essential or inessential return of ω before time n and p_q is the length of the corresponding binding period. If $n > \nu_q + p_q + 1$ then the same statement holds for all $k \le n$ restricted to any subinterval $\bar\omega \subseteq \omega$ such that $|\bar\omega_k| \subseteq \Delta^+$.

Notice that the distortion bound *improves* as δ tends to zero. This is only true under the additional assumptions stated that $k \le \nu_q + p_q + 1$ or restricting to $\bar\omega \subseteq \omega$ such that $|\bar\omega_k| \subseteq \Delta^+$. Without any assumptions the result still holds for any $k \le n$ with a constant D_δ independent of n but depending on δ.

By Lemma 4.2 it is sufficient to show that

$$\frac{|(f_a^k)'(c_0)|}{|(f_b^k)'(c_0)|} \ll e^{1/r_\delta} = D, \tag{6}$$

i.e. critical orbits with the same combinatorics satisfy the same derivative estimates. By standard arguments (see also the proof of Lemma 4.3), (6) follows if we can show that

$$\sum_{j=0}^{k-1} D_j \ll 1/r_\delta \quad \text{where} \quad D_j = \sup_{a\in\omega} \frac{|\omega_j|}{|c_j(a)|}.$$

Let $0 < \nu_1 < \cdots < \nu_q < k$ be the essential and inessential returns of ω up to time k. By construction there is a unique element I_{ρ_i, m_i} in $\mathcal{I}^+$ associated to each ν_i. Let p_i be the length of the binding period associated to ν_i. For notational convenience define ν_0 and p_0 so that $\nu_0 + p_0 + 1 = 0$. We suppose first that $k \leq \nu_q + p_q + 1$. Then write

$$\sum_{j=0}^{\nu_q+p_q} D_j = \sum_{i=0}^{q-1} \sum_{\nu_i+p_i+1}^{\nu_{i+1}+p_{i+1}} D_j.$$

Sublemma 5.2.1. *For each $i = 0, \dots, q-1$ we have*

$$\sum_{\nu_i+p_i+1}^{\nu_{i+1}+p_{i+1}} D_j \leq D_\alpha^3 |\omega_{\nu_{i+1}}| e^{\rho_{i+1}}.$$

Proof. Write

$$\sum_{\nu_i+p_i+1}^{\nu_{i+1}+p_{i+1}} D_j = \sum_{\nu_i+p_i+1}^{\nu_{i+1}-1} D_j + D_{\nu_{i+1}} + \sum_{\nu_{i+1}+1}^{\nu_{i+1}+p_{i+1}} D_j. \tag{7}$$

We shall estimate each of the three terms on the right hand side in separate arguments.

For the first notice that since $\omega_{\nu_{i+1}} \subseteq \hat{I}_{\rho_i,m}$, condition (UE2) implies that $|(f^{\nu_{i+1}-j})' c_{\nu_{i+1}-j}(a)| \geq e^{\lambda(\nu_{i+1}-j)}$ and therefore by Corollary 4.2.1 we have $|\omega_j| \leq 4e^{-\lambda(\nu_{i+1}-j)} |\omega_{\nu_{i+1}}|$. Moreover $|c_{\nu_{i+1}-j}(a)| \geq \delta^\iota$ and so letting $\Lambda = \sum_{i\geq 0} e^{-\lambda i}$ we have

$$\sum_{j=\nu_i+p_i+1}^{\nu_{i+1}-1} D_j \leq 4 \sum_{j=\nu_i+p_i+1}^{\nu_{i+1}-1} e^{-\lambda j} |\omega_{\nu_{i+1}}| \delta^{-\iota} \leq 4\Lambda |\omega_{\nu_{i+1}}| e^{\rho_{i+1}}. \tag{8}$$

The last inequality follows from the fact that $\delta^{-\iota} < \delta^{-1} \leq e^{\rho_{i+1}}$. For the second term we immediately have

$$D_{\nu_{i+1}} \leq \frac{1}{2} |\omega_{\nu_{i+1}}| e^{\rho_{i+1}} \tag{9}$$

since $D_{\nu_{i+1}}$ is the supremum of $|\omega_{\nu_{i+1}}|/|c_{\nu_{i+1}}|$ over all points $c_{\nu_{i+1}}$ in $\omega_{\nu_{i+1}}$ and $|c_{\nu_{i+1}}| \geq 2e^{-\rho_{i+1}}$ by definition.

To estimate the third term we argue as follows. By Corollary 4.2.1 and the bounded distortion during binding periods we have for all $j \in [\nu_{i+1} + 1, \nu_{i+1} + p_{i+1}]$,

$$|\omega_j| \leq 8D_\alpha |\omega_{\nu_{i+1}}| \sup_{a\in\omega} |(f_a^{j-\nu_{i+1}})'(c_{\nu_{i+1}})(a)|. \tag{10}$$

Now fix $a \in \omega$ and let $\gamma_j, j \geq 0$ be as in section 4.3. By the definition of binding periods we have $|\gamma_j| \leq e^{-2\alpha(j-\nu_{i+1})}$ and $|\gamma_0| \geq e^{-2\rho_{i+1}}$ and therefore

$$|(f^{j-\nu_{i+1}-1})'(c_{\nu_{i+1}+1}(a))| \leq D_\alpha |\gamma_j|/|\gamma_0| \leq D_\alpha e^{-2\alpha(j-\nu_{i+1})} e^{-2\rho_{i+1}}.$$

Moreover $|f'_a(c_{\nu_{i+1}}(a))| \le 2e^{-\rho_{i+1}}$ and therefore

$$\sup_{a\in\omega} |(f_a^{j-\nu_{i+1}})'(c_{\nu_{i+1}})(a)| \le 2D_\alpha e^{-2\alpha(j-\nu_{i+1})}e^{-\rho_{i+1}}.$$

Substituting in (10) gives

$$\sum_{j=\nu_{i+1}+1}^{\nu_{i+1}+p_{i+1}} D_j \le 16D_\alpha|\omega_{\nu_{i+1}}|e^{\rho_{i+1}} \sum \frac{e^{-2\alpha(j-\nu_{i+1})}}{e^{-\alpha(j-\nu_{i+1})}} \le 16D_\alpha^2|\omega_{\nu_{i+1}}|e^{\rho_{i+1}}. \qquad (11)$$

Using the fact that Λ depends only on λ and D_α is large when α is small, equations (8)-(11) complete the proof. $\square$

Sublemma 5.2.1 gives immediately

$$\sum_{j=0}^{\nu_q+p_q} D_j = \sum_{i=0}^{q-1} \sum_{\nu_i+p_i+1}^{\nu_{i+1}+p_{i+1}} D_j \le D_\alpha^3 \sum_{i=0}^{q-1} |\omega_{\nu_i}|e^{\rho_i}. \qquad (12)$$

Since $|\omega_{\nu_i}| \lesssim e^{-\rho_i}/\rho_i^2$ we have almost obtained our estimate except for the fact that we have no control over the multiplicity with which each ρ_i occurs. To overcome this problem we show that essentially only the *last* return with a given return depth needs to be considered in the sum (12) since the lengths of the preceeding ones form a decreasing geometric sequence. First of all subdivide the sum on the right hand side into partial sums corresponding to return times with the same return depth:

$$\sum_{i=0}^{q-1} |\omega_{\nu_i}|e^{\rho_i} = \sum_{r\ge r_{\delta+}} e^r \sum_{i:\rho_i=r} |\omega_{\nu_i}|. \qquad (13)$$

Sublemma 5.2.2. *For any $r \ge r_{\delta^+}$,*

$$\sum_{i:\rho_i=r} |\omega_{\nu_i}| \le 10e^{-r}/r^2.$$

Proof. Let $\mu_j = \nu_{i_j}, j = 1,\dots,m$ be the subsequence of returns with return depths equal to r. By construction $|\omega_{\mu_m}| \le 5e^{-r}/r^2$. Using the binding period estimates and (UE2) we have for all $a \in \omega$ and $j = 1,\dots,m-1$,

$$|(f_a^{\mu_{i+1}-\mu_i})'c_{\mu_i}a| \ge |(f_a^{\mu_i+p_i+1})'c_{\mu_i}a| \ge e^{(1-5\beta)r} \ge e^{(1-5\beta)r_{\delta+}} = \delta^{-\iota(1-5\beta)}$$

Therefore by Corollary 4.2.1, $|\omega_{\mu_i}| \ge \delta^{\iota(1-5\beta)}|\omega_{\mu_{i+1}}|/4$ and

$$\sum_{j=1}^{m} |\omega_{\mu_i}| \le \sum_{j=0}^{m-1} \delta^{-\iota(1-5\beta)j}|\omega_{\mu_m}| \le 2|\omega_{\mu_m}|.$$

$\square$

Substituting the estimate of Sublemma 5.2.2 into (12) and (13) we get

$$\sum_{j=0}^{\nu_q+p_q} D_j \leq D_\alpha^3 \sum_{r\geq r_{\delta+}} e^r \sum_{i:\rho_i=r} |\omega_{\nu_i}| \leq D_\alpha^4 \sum_{r\geq r_{\delta+}} \frac{1}{r^2} \leq \frac{2D_\alpha^4}{r_{\delta+}^2} \ll 1/r_\delta.$$

This completes the proof of the Lemma for $k \leq \nu_q + p_q + 1$. If $k > \nu_q + p_q + 1$ we consider the additional terms D_j restricting ourselves to some subinterval $\bar{\omega} \subset \omega$ with $\bar{\omega}_k \subseteq \Delta^+$. Clearly the preceding estimates are unaffected by this restriction. Using (UE2) we have

$$|\bar{\omega}_j| \leq 4e^{-\hat{\lambda}(k-j)}|\bar{\omega}_k| \leq 2e^{-\hat{\lambda}(k-j)}\delta^\iota$$

and therefore using also the fact that $|f'(c_j(a))| \geq \delta^\iota$ since $\omega_j \cap \Delta^+ = \emptyset$ we get

$$\sum_{j=\nu_q+p_q+1}^{k-1} D_j \leq \sum_{j=\nu_q+p_q+1}^{k-1} e^{-\lambda(k-j)} \leq \sum_{i=0}^{\infty} e^{-\lambda i}.$$

6 Combinatorial and metric estimates

The aim of this section is to prove Proposition 3.1: for all $i \leq n-1$, $\omega \in \mathcal{Q}^{(i)}$ and $R \geq 0$ we have

$$\sum_{\omega' \in \mathcal{Q}^{(i+1)}(\omega,R)} |\omega'| \leq e^{(10\beta-1)R}|\omega|.$$

We use a counting argument to show that the cardinality of $\mathcal{Q}^{(i+1)}(\omega, R)$ is bounded above by $e^{\beta R}$ and an analytic argument to show that the size of each $\omega' \in \mathcal{Q}^{(i+1)}(\omega, R)$ is bounded above by $e^{(9\beta-1)R}|\omega|$.

6.1 The counting argument

Lemma 6.1. *For all $0 \leq i \leq n-1$, $\omega \in \mathcal{Q}^{(i)}$ and $R \geq 0$, we have*

$$\#\mathcal{Q}^{(i+1)}(\omega, R) \leq e^{\beta R}.$$

We begin with a standard abstract counting argument. Let $\mathcal{S}_R$ denote the the set of all sequences of pairs of integers $(\pm r_1, m_1), (\pm r_2, m_2), \ldots, (\pm r_s, m_s)$ with $s \geq 1$, $|r_1| + \cdots + |r_s| = R$, $|r_j| \geq r_\delta$ and $m_j \in [1, r_j^2]$ for all $j = 1, \ldots, s$.

Sublemma 6.1.1.

$$\#\mathcal{S}_R \leq e^{\beta R/2}.$$

Proof. We estimate first of all the cardinality of the set of possible sequences $r_1, \ldots, r_s$ of positive integers $r_j \geq r_\delta$ with $r_1 + \cdots + r_s = R$ for a fixed $s \geq 1$. Notice that the number of of possible sequences as prescribed corresponds exactly to the number of ways of partitioning R objects into s disjoint non-empty subsets and can be bounded above by the number of ways of choosing s balls from a row of $R + s$ balls since this will determine a partition of the remaining R balls into at most s disjoint subsets (the fact that each of these subsets must contain at least r_δ elements will be used in the calculation below). Using Stirling's approximation formula for factorials $k! \in [1, 1 + 1/4k]\sqrt{2\pi k}k^k e^{-k}$ we have

$$\binom{R+s}{s} \leq \frac{(R+s)!}{R!s!} \leq \frac{(R+s)^{R+s}}{R^R s^s} \leq \left(\frac{R+s}{R}\right)^R \left(\frac{R+s}{s}\right)^s.$$

To estimate the second term we use the fact that $sr_\delta \leq R$ and that $\log(1+x) < x$ for all $x > 0$ and we get

$$\left(\frac{R+s}{R}\right)^R \leq \left(\frac{R(1+\frac{1}{r_\delta})}{R}\right)^R \leq \left(1 + \frac{1}{r_\delta}\right)^R \leq e^{R\log(1+\frac{1}{r_\delta})} \leq e^{R/r_\delta}.$$

To estimate the first term we write first of all

$$\begin{aligned}
\left(\frac{R+s}{s}\right)^s &= \left[\left(\frac{s}{R+s}\right)^{-\frac{s}{R}}\right]^R \\
&\leq \left[\left(\frac{s}{R(1+\frac{1}{r_\delta})}\right)^{-\frac{s}{R}}\right]^R \leq \left[\left(\frac{s}{R}\right)^{-\frac{s}{R}} \left(1 + \frac{1}{r_\delta}\right)^{s/R}\right]^R
\end{aligned}$$

Since $s/R \leq 1/r_\delta$ and $x^{-x} \to 1$ as $x \to 0$ both terms inside the parenthesis tend to 1 as $\delta \to 0$ and it is sufficient to choose a sufficiently small δ to get

$$\binom{R+s}{s} \leq \left(\frac{R+s}{R}\right)^R \left(\frac{R+s}{s}\right)^s \leq e^{R/r_\delta} e^{\beta R/6} \leq e^{\beta R/5}$$

Next we need to allow each r_j to occur with multiplicity r_j^2 and this gives a bound

$$e^{\beta R/5} \sum_{j=1}^{s} r_j^2 \leq e^{\beta R/5} R^2 \leq e^{\beta R/5} e^{2\log R} \leq e^{\beta R/4}.$$

Also, each r_i can have a positive or negative sign. This gives an additional multiplicative factor of 2^s and thus a bound of

$$2^s e^{\beta R/4} \leq e^{s\log 2} e^{\beta R/4} \leq e^{(R\log 2)/r_\delta} e^{\beta R/4} \leq e^{\beta R/3}.$$

Finally, keeping in mind that for fixed R we have $s \le R/r_\delta$ we can sum over all possible sequence lengths s to get

$$\sum_{s \le R/r_\delta} e^{\beta R/3} = \frac{R}{r_\delta} e^{\beta R/3} = e^{\log(R/r_\delta)} e^{\beta R/3} \le e^{\beta R/2}.$$

$\square$

The statement in Lemma 6.1 now follows from the above and the following estimate about the actual number of partition elements which can share exactly the same sequence.

Sublemma 6.1.2. *For $\omega \in \mathcal{Q}^{(i-1)}$ and any sequence $(r_1, m_1), \dots, (r_s, m_s)$ with $s \ge 1, |r_j| \ge r_\delta,\ m_j \in [1, r_j^2]$ there exists at most r_δ^3 elements $\tilde{\omega} \in \mathcal{Q}^{(i)}(\omega)$ with a maximal sequence of essential returns $\eta_{i-1} < \nu_1 < \cdots < \nu_s \le \eta_i$ such that $\tilde{\omega}_{\nu_j} \cap I_{r_j, m_j} \ne \emptyset$ for $j = 1, \dots, s$.*

Proof. The proof is an immediate consequence of the construction described above. The first time, after time η_{i-1}, that ω intersects Δ^+ in a chopping time, every subinterval which arises out of the 'chopping' has either an escape time, in which case the sequence above is empty and the lemma does not apply or an essential return time in which case a unique pair of integers r_1 and m_1 are associated to it. Thus no two elements created up to this time can share the same sequence. Fixing one of these subintervals which has an essential return we consider higher iterates until the next time that it intersects Δ^+. At this time it is further subdivided into subintervals. Those which have essential returns at this time all have another uniquely defined pair of integers r_2 and m_2 associated to them. However multiplicity can occur for those which have escape times: all the subintervals which fall in $\Delta^+ \setminus \Delta$ have an escape at this time, and therefore belong to $\mathcal{Q}^{(i)}$ and we do not consider further iterates, but all share the same first (and only) term of the associated sequence of return depths. However the number of such subintervals can be estimated by the number of elements of the partition $\mathcal{I}^+|_{\Delta^+ \setminus \Delta}$ plus at most two elements which can escape by falling outside Δ^+. The number of such intervals is then $\le 2(r_{\delta^+} - r_\delta) r_\delta^2 + 2 \le r_\delta^3$. In the case of the intervals which have two or more returns we repeat the same reasoning to get the result. $\square$

6.2 The metric estimates

Lemma 6.2. *For all $0 \le i \le n-1$, $\omega \in \mathcal{Q}^{(i)}$, $R \ge 0$ and $\tilde{\omega} \in \mathcal{Q}^{(i+1)}(\omega, R)$ we have*

$$|\tilde{\omega}| \le e^{(9\beta - 1)R} |\omega|.$$

Proof. From the construction of $\tilde{\omega}$ there is a nested sequence of intervals

$$\tilde{\omega} \subseteq \omega^{(\nu_s)} \subseteq \cdots \subseteq \omega^{(\nu_1)} \subseteq \omega^{(\nu_0)} = \omega$$

such that each $\omega^{(\nu_j)}$ has an essential return at time ν_j (intuitively $\omega^{(\nu_j)}$ is created as a consequence of the intersection of $\omega^{(\nu_{j-1})}$ with Δ at time ν_j). Write

$$\frac{|\tilde{\omega}|}{|\omega|} = \frac{|\omega^{(\nu_1)}|}{|\omega^{(\nu_0)}|}\frac{|\omega^{(\nu_2)}|}{|\omega^{(\nu_1)}|}\cdots\frac{|\omega^{(\nu_s)}|}{|\omega^{(\nu_{s-1})}|}\frac{|\tilde{\omega}|}{|\omega^{(\nu_s)}|}. \tag{14}$$

Letting $r_0 = r_\delta$ we have

Sublemma 6.2.1. *For $j = 0, \ldots, s-1$ we have*

$$\frac{|\omega^{(\nu_{j+1})}|}{|\omega^{(\nu_j)}|} \leq e^{-r_{j+1}+9\beta r_j}$$

Proof. We consider first the cases $j = 1, \ldots, s-1$. The bounded distortion property holds up to time $\nu_j + p_j + 1$ and so

$$\frac{|\omega^{(\nu_{j+1})}|}{|\omega^{(\nu_j)}|} \leq D\frac{|\omega^{(\nu_{j+1})}_{\nu_j+p_j+1}|}{|\omega^{(\nu_j)}_{\nu_j+p_j+1}|}.$$

To estimate the numerator notice that by definition $\omega^{(\nu_{j+1})}_{\nu_{j+1}} \subset \Delta_+$ and therefore by (UE2) and Corollary 4.2.1 $|\omega^{(\nu_{j+1})}_{\nu_{j+1}}| \geq e^{\hat{\lambda}(\nu_{j+1}-\nu_j-p_j-1)}|\omega^{(\nu_{j+1})}_{\nu_{j+1}}|/4$ and therefore

$$|\omega^{(\nu_{j+1})}_{\nu_j+p_j+1}| \leq 4e^{-\lambda(\nu_{j+1}-\nu_j-p_j-1)}|\omega^{(\nu_{j+1})}_{\nu_{j+1}}| \leq |\omega^{(\nu_{j+1})}_{\nu_{j+1}}| \leq 5e^{-r_{j+1}}.$$

To estimate the denominator, observe that by construction $\omega^{(\nu_j)}$ has an essential return at time ν_j, and therefore by Lemma 4.3 $|\omega^{(\nu_j)}_{\nu_j+p_j+1}| \geq |\omega^{(\nu_j)}_{\nu_j}|^{8\beta} \geq e^{-8\beta r_j}/r_j^{16\beta}$.

For $j = 0$, recall that $\omega^{(\nu_0)} = \omega \in \mathcal{Q}^{(i-1)}$ which means that $\omega^{(\nu_0)}$ has an escape at time $\eta_{i-1} = \nu_0$. Suppose first that $\omega^{(\nu_0)}_{\nu_1} \subseteq \Delta^+$. The the bounded distortion holds up to time ν_1 and we have

$$\frac{|\omega^{(\nu_1)}|}{|\omega^{(\nu_0)}|} \leq D\frac{|\omega^{(\nu_1)}_{\nu_1}|}{|\omega^{(\nu_0)}_{\nu_1}|} \leq De^{-r_1}|\omega^{(\nu_0)}_{\nu_1}|^{-1}$$

and we just need to show that $|\omega^{(\nu_0)}_{\nu_1}| \geq De^{-9\beta r_\delta}$. There are two cases to consider according to the position of $\omega^{(\nu_0)}_{\nu_0}$. If $\omega^{(\nu_0)}_{\nu_0} \cap \Delta \neq \emptyset$ then $I_{r,m} \subseteq$

$\omega_{\nu_0}^{(\nu_0)} \subseteq \hat{I}_{r,m}$ for some $r_{\delta^+} \geq r \geq r_\delta$ and we can repeat the binding arguments of section 4.3 and together with (UE2) and Corollary 4.2.1 these give

$$|\omega_{\nu_0}^{(\nu_1)}| \geq |\omega_{\nu_0}^{(\nu_0+p_0+1)}|/4 \geq |\omega_{\nu_0}^{(\nu_0)}|^{8\beta}/4 \geq e^{-8\beta r}/4r^{16\beta} \geq De^{-9\beta r_\delta}.$$

On the other hand if $\omega_{\nu_0}^{(\nu_0)} \cap \Delta^+ = \emptyset$ then by construction $|\omega_{\nu_0}^{(\nu_0)}| \geq \delta^\iota$ and using (UE2) again and the fact that the critical point is quadratic

$$|\omega_{\nu_1}^{(\nu_0)}| \geq |\omega_{\nu_0+1}^{(\nu_0)}|/4 \geq \delta^{2\iota}/4 \geq \delta^{8\beta} \geq e^{-8\beta r_\delta}$$

as long as $\iota < 4\beta$. This completes the proof in the case $\omega_{\nu_0}^{(\nu_1)} \subseteq \Delta^+$.

If $\omega_{\nu_0}^{(\nu_1)}$ is not completely contained in Δ^+ we cannot apply the bounded distortion property to the whole of $\omega^{(\nu_0)}$ up to time ν_1. However, restricting ourselves to a maximal subinterval $\bar{\omega}^{(\nu_0)} \subset \omega^{(\nu_0)}$ with $\bar{\omega}_{\nu_1}^{(\nu_0)} \subseteq \Delta^+$ we do have

$$\frac{|\omega^{(\nu_1)}|}{|\omega^{(\nu_0)}|} \leq \frac{|\omega^{(\nu_1)}|}{|\bar{\omega}^{(\nu_0)}|} \leq D\frac{|\omega_n^{(\nu_1)} u_1|}{|\omega_{\nu_1}^{(\nu_0)}|} \leq De^{r_1}|\bar{\omega}_{\nu_1}^{(\nu_0)}|^{-1}.$$

Therefore it is sufficient to show that $|\bar{\omega}_{\nu_1}^{(\nu_0)}| \geq De^{-9\beta r_\delta}$. This is clearly true since by hypothesis $|\bar{\omega}_{\nu_1}^{(\nu_0)}|$ intersects Δ (since it contains $|\omega_{\nu_1}^{(\nu_1)}|$ which is contained in Δ) and also extends all the way to the boundary of Δ^+ (since we have assumed that $|\omega_{\nu_1}^{(\nu_0)}|$ is not properly contained in Δ^+). Therefore

$$|\bar{\omega}_{\nu_1}^{(\nu_0)}| \sim \delta^\iota \gg \delta^{8\beta} \geq e^{-8\beta r_\delta}.$$

□

The statement in the Lemma now follows immediately. Substituting into (14) and using the fact that $|\tilde{\omega}|/|\omega^{(\nu_s)}| \leq 1$ since $\tilde{\omega} \subseteq \omega^{(\nu_s)}$ gives

$$\frac{|\tilde{\omega}|}{|\omega|} \leq e^{-\sum_1^s r_j + 9\beta\sum_1^{s-1} r_j} e^{8\beta r_\delta} \leq e^{(8\beta-1)R} e^{-8\beta r_s + 8\beta r_\delta}.$$

The result follows since $r_s \geq r_\delta$. □

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