Stability of the spectrum for transfer operators

Gerhard Keller, Universität Erlangen *
Carlangelo Liverani, Universitá di Roma, Tor Vergata †

December 15, 1998

Abstract

We prove stability of the isolated eigenvalues of transfer operators satisfying a Lasota-Yorke type inequality under a broad class of random and nonrandom perturbations including Ulam-type discretizations. The results are formulated in an abstract framework.

Mathematics Subject Classification Numbers: 28A65, 47A55.
Key words: Expanding Maps, Stectral Stability, Transfer Operators.

1 Introduction

Let $(B, \|\cdot\|)$ be a Banach space which is equipped with a second norm $|\cdot| \leq \|\cdot\|$ with respect to which B is typically non-complete. For any bounded linear operator $Q : B \to B$ let

$$|||Q||| := \sup\{|Qf| : f \in B, \|f\| \leq 1\} \ . \tag{1}$$

We consider a family $(P_\epsilon)_{\epsilon \geq 0}$ of bounded linear operators on $(B, \|\cdot\|)$ with the following properties: There are $C_1, M > 0$ such that for all $\epsilon \geq 0$

$$|P_\epsilon^n| \leq C_1 M^n \quad \forall n \in \mathbb{N} \ ; \tag{2}$$

there are $C_2, C_3 > 0$ and $\alpha \in (0,1)$, $\alpha < M$, such that for all $\epsilon \geq 0$

$$\|P_\epsilon^n f\| \leq C_2 \alpha^n \|f\| + C_3 M^n |f| \quad \forall n \in \mathbb{N} \ \forall f \in B \ ; \tag{3}$$

also (but see Remarks 1,2,6 for alternative conditions),

$$\text{if } z \in \sigma(P_\epsilon),\ |z| > \alpha, \text{ then } z \text{ is not in the residual spectrum of } P_\epsilon \ ; \tag{4}$$

and there is a monotone upper-semicontinuous function $\tau : [0,\infty) \to [0,\infty)$ such that $\tau_\epsilon > 0$ if $\epsilon > 0$ and

$$|||P_0 - P_\epsilon||| \leq \tau_\epsilon \to 0 \quad \text{as } \epsilon \to 0 \ . \tag{5}$$

For families of operators satisfying (2) – (5) we derive uniform bounds on the resolvents with respect to the norm (1), $|||(z - P_\epsilon)^{-1}|||$, when z is uniformly bounded away from the spectrum $\sigma(P_0)$ of P_0, and we show that for such z the difference $|||(z-P_\epsilon)^{-1} - (z-P_0)^{-1}||| = O(\tau_\epsilon^\eta)$ for a suitable $\eta > 0$ (Theorem 1). An immediate corollary to these estimates is the stability of isolated eigenvalues λ of P_0 with $|\lambda| > \alpha$; stability in the sense that if $\delta > 0$ is such that $B_\delta(\lambda) \cap \sigma(P_0) = \{\lambda\}$ then $\lim_{\epsilon\to 0} |||\Pi_\epsilon - \Pi_0||| = 0$. Here $B_\delta(\lambda)$ denotes the open unit ball of radius δ around λ and $\Pi_\epsilon := \frac{1}{2\pi i}\int_{\partial B_\delta(\lambda)} (z - P_\epsilon)^{-1}\, dz$ is the total spectral projection of P_ϵ associated with $B_\delta(\lambda) \cap \sigma(P_\epsilon)$ (Corollary 1). More precise statements and further corollaries are deferred to the next section.

*Partially supported by the Deutsche Forschungsgemeinschaft (DFG).
†Acknowledges the support of the Deutsche Forschungsgemeinschaft (DFG), the GNFM-CNR and the ESF programme PRODYN.

Remark 1 a) If inequality (3) is satisfied for some n_0 such that $C_2^{1/n_0}\alpha =: \alpha_0 < 1$, then it holds for all n with α replaced by α_0.

b) It is a simple consequence of (3) that for $C_4 = C_2 + C_3$ holds

$$\|P_\epsilon^n\| \leq C_4 M^n \quad \forall \epsilon \geq 0 \, \forall n \in \mathbb{N} . \tag{6}$$

c) In the mathematical literature there are many examples of single operators P_0 satisfying (2) and (3). In nearly all cases the two norms involved have the additional property that

$$\text{the closed unit ball of } (B, \|\cdot\|) \text{ is } |\cdot|\text{-compact,} \tag{7}$$

and in all of these examples this property is the key to proving that the essential spectral radius of P_ϵ is bounded by α which implies in particular assumption (4), see also Remark 3.

In the rest of this introduction we discuss a number of situations where assumptions (2), (3) and (7) are satisfied. This may serve to illustrate the broad applicability of the results proved in this note.

The basic result for this setting is the Theorem of Ionescu Tulcea and Marinsecu [15]: If the constant M in (2) and (3) is equal to 1, then $P_0 : B \to B$ has at most finitely many eigenvalues of modulus 1. All these eigenvalues have finite multiplicity, and the rest of the spectrum is contained in a disk around the origin of radius less than 1. In other words: P_0 is quasicompact.

The following more concrete setting historically motivated this theorem: B is the space of complex-valued Lipschitz functions on a compact metric space, $\|\cdot\|$ is the Lipschitz norm, and $|\cdot|$ is the supremum norm. P_0 is a Markov transition operator of Doeblin-Fortet type, i.e. satisfying (3). For a while this model played a prominent role in mathematical learning theory, see the monographs [16, Theorem 2.1.40] and [27, Theorem 3.2.1] for a comprehensive treatment and further references. Later, the same analytic setting was used to study Ruelle transfer operators for subshifts of finite type, see e.g. [28] for more information.

Independently of these developments another concrete setting in which the three assumptions (2), (3) and (7) are met emerged in 1973, when Lasota and Yorke [24] studied Perron-Frobenius operators P_0 of piecewise C^2 and piecewise expanding maps: Now B is the space of functions of bounded variation on an interval, $\|\cdot\|$ is the variation norm, and $|\cdot|$ the usual L^1-norm. Inequalities (2) and (3) with $M = 1$ were derived in [24], and the applicability of the Theorem of Ionecu Tulcea and Marinescu was noticed later in [18, 19], where [18] in fact deals with an extension of the Lasota-Yorke result to piecewise expanding maps of the unit square. Soon after, Rychlik [30] showed how to bypass the Ionescu Tulcea-Marinescu theorem and proved quasicompactness of P_0 more directly, an approach that was exploited in [20] to show that the constant α from (3) is an upper bound for the essential spectral radius of P_0. Later, in [12], it was shown how to derive this estimate directly from the Ionescu Tulcea-Marinescu theorem. A good reference to these and related results for Perron-Frobenius operators of one-dimensional maps is the monograph [8]. Variants of this theory for nonexpanding or higher-dimensional maps can be found in [23, 7, 22].

Passing from a single operator P_0 to a family $(P_\epsilon)_{\epsilon \geq 0}$ satisfying conditions (2)–(5) can have various interpretations. We mention the following ones: P_0 is the Perron-Frobenius operator of a piecewise expanding map T as discussed above.

a) The P_ϵ are Perron-Frobenius operators of maps T_ϵ which are "close" to T. For maps of the interval $I = [0, 1]$ a suitable notion of closeness is

$$\begin{aligned} d(T, T_\epsilon) := \inf\{\kappa > 0 \mid \exists\, A \subseteq I \,\exists\, \sigma : I \to I \text{ s.th. } & m(A) > 1 - \kappa, \sigma \text{ is a diffeo-} \\ & \text{morphism, } T_{\epsilon|_A} = T \circ \sigma_{|_A}, \text{ and } \forall x \in I\text{: } |\sigma(x) - x| < \kappa, \\ & |1/\sigma'(x) - 1| < \kappa\} \end{aligned}$$

In this case $\tau_\epsilon = 12\, d(T, T_\epsilon)$, see [21].

b) P_ϵ is the transition operator of the stochastically perturbed map T, where ϵ is the "size" of the perturbation, see [21, 4, 2, 5]. Typically, $\tau_\epsilon = O(\epsilon)$.

c) P_ϵ is the transition operator for the Ulam-type discretization of T with grid size ϵ. Again $\tau_\epsilon = O(\epsilon)$, see [25, 21, 3, 9, 5] and for related work also [14, 26, 11, 17].

For all these families $(P_\epsilon)_{\epsilon\geq 0}$, inequality (3) is satisfied uniformly in ϵ provided there is an iterate T^k of T with $\inf|(T^k)'| > 2$ and such that there are no discontinuities or turning points c, c' of T with $T^j c = c'$ for some $0 < j \leq k$. But even if this condition is violated certain types of perturbations, notably Ulam discretizations, satisfy (3) uniformly in ϵ, see [5] for a detailed discussion.

Remark 2 Since in all these situations the P_ϵ can be interpreted as positive operators on L^1_m, their peripheral eigenvalues form a finite cyclic group, see [31, 30, 13].

Remark 3 For some applications it is interesting to note that the compactness assumption (7) can be replaced by the following weaker set of assumptions:

There is a sequence of linear operators $\pi_k : B \to B$ with $\sup_k \|\pi_k\| < \infty$ and such that

$$\sup\{|f - \pi_k f| : f \in B, \|f\| \leq 1\} \leq \text{const} \cdot \left(\frac{\alpha}{M}\right)^k$$

and

$$P_\epsilon^k \pi_k \text{ is a compact operator for all } k.$$

A simple calculation based only on these assumptions and on (3) shows that there is a constant $C > 0$ such that

$$\|P_\epsilon^k - P_\epsilon^k \pi_k\| \leq C\alpha^k \quad \forall k > 0\, \forall \epsilon \geq 0\ .$$

It follows from [10, Lemma VIII.8.2] that in this case all $P_\epsilon : B \to B$ are quasicompact with essential spectral radius $\leq \alpha$ (in particular (4) holds). This is in fact the previously mentioned approach of Rychlik [30]. Related questions for function spaces of higher smoothness are discussed in [1].

2 The results

For $\delta > 0$ and $r > \alpha$ let

$$V_{\delta,r} := \{z \in \mathbb{C} : |z| \leq r \text{ or } \operatorname{dist}(z, \sigma(P_0)) \leq \delta\}.$$

The main results of this paper are the following bounds on the resolvents $(z - P_\epsilon)^{-1}$.

Theorem 1 *Suppose that $(P_\epsilon)_{\epsilon\geq 0}$ is a family of linear operators on B satisfying (2) – (5). Fix $\delta > 0$ and $r \in (\alpha, M)$ and let $\eta := \frac{\log r/\alpha}{\log M/\alpha}$. Then $\eta > 0$ and there are constants $\epsilon_0 = \epsilon_0(\delta, r) > 0$, $a = a(r) > 0$, $b = b(\delta, r) > 0$, $c = c(\delta, r) > 0$ and $d = d(\delta, r) > 0$ such that for $0 \leq \epsilon \leq \epsilon_0$ and $z \in \mathbb{C} \setminus V_{\delta,r}$*

$$\|(z - P_\epsilon)^{-1} f\| \leq a\,\|f\| + b\,|f| \quad \textit{for all } f \in B \tag{8}$$

and

$$\left|\left|\left|(z - P_\epsilon)^{-1} - (z - P_0)^{-1}\right|\right|\right| \leq \tau_\epsilon^\eta \left(c\,\|(z - P_0)^{-1}\| + d\,\|(z - P_0)^{-1}\|^2\right). \tag{9}$$

Explicit bounds on the constants ϵ_0, a, b, c, d are given in the proof. They all depend on the operators P_ϵ via the constants $M, C_1, \ldots, C_4$; ϵ_0 and b depend also on the functions $\epsilon \mapsto \tau_\epsilon$ and $z \mapsto \|(z-P_0)^{-1}\|$.

Denote $\sigma_\alpha(P_\epsilon) := \{z \in \mathbb{C} : |z| \leq \alpha\} \cup \sigma(P_\epsilon)$. An immediate consequence of (8) is that

$$S_{\delta,r} := \sup\left\{\|(z-P_\epsilon)^{-1}\| : 0 \leq \epsilon \leq \epsilon_0(\delta,r), z \in \mathbb{C}\setminus V_{\delta,r}\right\} < \infty \tag{10}$$

for all $\delta > 0$ and $r \in (\alpha, M)$. Therefore, all accumulation points (as $\epsilon \to 0$) of spectral values in $\sigma_\alpha(P_\epsilon)$ are contained in $\sigma_\alpha(P_0)$. A more elementary proof of this fact was previously given in [6]. But much more can be deduced from (9). If λ is an isolated eigenvalue of P_0 with $|\lambda| > \alpha$, then $\delta > 0$ can be chosen so small that $B_\delta(\lambda) \cap \sigma_\alpha(P_0) = \{\lambda\}$ and we can define

$$\Pi_\epsilon^{(\lambda,\delta)} := \frac{1}{2\pi\imath}\int_{\partial B_\delta(\lambda)} (z-P_\epsilon)^{-1}\,dz\ .$$

$\Pi_0^{(\lambda,\delta)}$ does not depend on δ as long as $B_\delta(\lambda) \cap \sigma_\alpha(P_0) = \{\lambda\}$, and as we just saw, also the projections $\Pi_\epsilon^{(\lambda,\delta)}$ are well defined and independent of δ for sufficiently small ϵ.

Corollary 1 *In the situation of Theorem 1, if λ is an isolated eigenvalue of P_0 with $|\lambda| > r$ and if $\delta > 0$ is such that $B_\delta(\lambda) \cap \sigma_\alpha(P_0) = \{\lambda\}$, we have:*

1) There is a constant $K_1 = K_1(\delta, r) > 0$ such that $|||\Pi_\epsilon^{(\lambda,\delta)} - \Pi_0^{(\lambda,\delta)}||| \leq K_1 \cdot \tau_\epsilon^\eta$ for all $\epsilon \in [0, \epsilon_0]$.

2) There are constants $K_2 = K_2(\delta, r) > 0$ and $\delta_0 = \delta_0(r) > 0$ such that $\|\Pi_\epsilon^{(\lambda,\delta)} f\| \leq K_2 \cdot |\Pi_\epsilon^{(\lambda,\delta)} f|$ for all $f \in B$, $\delta \in (0, \delta_0]$ and $\epsilon \in [0, \epsilon_1]$.

3) If $\delta \in (0, \delta_0]$, then $\operatorname{rank}(\Pi_\epsilon^{(\lambda,\delta)}) = \operatorname{rank}(\Pi_0^{(\lambda,\delta)})$ for ϵ small enough.

Remark 4 If the isolated eigenvalue λ of P_0 has finite multiplicity, this means in particular that, for ϵ and δ small enough, $\sigma_\alpha(P_\epsilon) \cap B_\delta(\lambda)$ consists of eigenvalues $\lambda_j^{(\epsilon)}$, such that $\lim_{\epsilon\to 0} \lambda_j^{(\epsilon)} = \lambda$ for all j, and the total multiplicity of the $\lambda_j^{(\epsilon)}$ equals the multiplicity of λ.

For $r > \alpha$ denote by γ_r the circle of radius r around the origin and define

$$\Pi_\epsilon^{(r)} := \frac{1}{2\pi\imath}\int_{\gamma_r} (z-P_\epsilon)^{-1}\,dz\ .$$

Corollary 2 *If, in the situation of Theorem 1, $\sigma(P_0) \cap \{z \in \mathbb{C} : |z| = r\} = \emptyset$, then*

1) $\lim_{\epsilon\to 0} |||\Pi_\epsilon^{(r)} - \Pi_0^{(r)}||| = 0$

2) There is $K_3 = K_3(\delta, r) > 0$ such that for all $\epsilon \in [0, \epsilon_0]$

$$\|P_\epsilon^n \Pi_\epsilon^{(r)}\| \leq K_3 \cdot r^n \quad \forall n \in \mathbb{N}\ .$$

Remark 5 In many cases the P_ϵ are $|\cdot|$-contractions, i.e. $M = 1$. If, in such a situation, we considered $r = 1$, i.e. a case which is not covered by Theorem 1, then η as given there would evaluate to 1. A look at the proofs reveals, however, that in this case the order of convergence in (9) is only $\tau_\epsilon |\log \tau_\epsilon|$ instead of τ_ϵ. This is in agreement with results in [21, 17] on the stability of eigenfunctions belonging to eigenvalues of modulus 1.

3 Proofs

In the proofs we use the following abbreviating notation:

$$Q_\epsilon := (z - P_\epsilon)$$

where we suppress the dependence of Q_ϵ on z. Recall that since $z \notin V_{\delta,r}$ always $|z| \geq r > \alpha$ and $Q_0^{-1} = (z - P_0)^{-1}$ exists as a bounded linear operator on B.

Lemma 1 *For each $r \in (\alpha, M)$ holds*

$$\|f\| \leq C_5 \, \|Q_\epsilon f\| + C_6 \, |f| \quad \textit{for all } f \in B \ \textit{and} \ \epsilon \geq 0$$

where

$$C_5 := \frac{2C_4}{M-r}\left(\frac{M}{r}\right)^{n_1} \quad \textit{and} \quad C_6 := 2C_3\left(\frac{M}{r}\right)^{n_1} \quad \textit{with} \quad n_1 := \left\lceil \frac{\log 2C_2}{\log r/\alpha} \right\rceil$$

and $\lceil x \rceil$ denotes the smallest integer greater or equal to x.

Proof: For $f \in B$ holds

$$\begin{aligned} |z|^n \, \|f\| &= \|z^n f\| \leq \|(z^n - P_\epsilon^n)f\| + \|P_\epsilon^n f\| \\ &\leq \sum_{i=0}^{n-1} |z^i| \, \|P_\epsilon^{n-1-i}\| \, \|(z - P_\epsilon)f\| + C_2 \alpha^n \, \|f\| + C_3 M^n \, |f| \end{aligned}$$

in view of (3). Observing (6), this yields

$$\|f\| \leq \frac{C_4}{M-r}\left(\frac{M}{r}\right)^n \|Q_\epsilon f\| + C_2\left(\frac{\alpha}{r}\right)^n \|f\| + C_3\left(\frac{M}{r}\right)^n |f|.$$

Let $n = n_1$. Then $C_2\left(\frac{\alpha}{r}\right)^{n_1} \leq \frac{1}{2}$ so that

$$\|f\| \leq \frac{2C_4}{M-r}\left(\frac{M}{r}\right)^{n_1} \|Q_\epsilon f\| + 2C_3\left(\frac{M}{r}\right)^{n_1} |f|.$$

□

Proof of Theorem 1: Consider any $h \in B$ and let $g := Q_0 h$. As

$$Q_0^{-1} = (z - P_0)^{-1} = z^{-n} Q_0^{-1} P_0^n + \sum_{j=0}^{n-1} z^{-j-1} P_0^j$$

we have

$$\begin{aligned} |h| = |Q_0^{-1} g| &\leq |z|^{-n} \|Q_0^{-1} P_0^n g\| + \sum_{j=0}^{n-1} |z|^{-j-1} |P_0^j g| \\ &\leq \|Q_0^{-1}\| \, C_2\left(\frac{\alpha}{r}\right)^n \|g\| + \|Q_0^{-1}\| \, C_3\left(\frac{M}{r}\right)^n |g| \\ &+ \frac{C_1}{M-r}\left(\frac{M}{r}\right)^n |g| \; . \end{aligned} \tag{11}$$

As $g = (Q_0 - Q_\epsilon)Q_0^{-1} g + Q_\epsilon Q_0^{-1} g = (P_\epsilon - P_0)h + Q_\epsilon h$, we have

$$|g| \leq |(P_\epsilon - P_0)h| + |Q_\epsilon h| \leq \tau_\epsilon \|h\| + |Q_\epsilon h| \; .$$

Combining these two estimates yields

$$|h| \;\leq\; \|Q_0^{-1}\|\, C_2 \left(\frac{\alpha}{r}\right)^n \|g\| + \left(\|Q_0^{-1}\|\, C_3 + \frac{C_1}{M-r}\right) \left(\frac{M}{r}\right)^n (\tau_\epsilon \|h\| + |Q_\epsilon h|)\,.$$

Recall that $g = Q_0 h = (z - P_0)h$ so that $\|g\| \leq (C_4 M + |z|)\|h\|$. Therefore,

$$\begin{aligned} |h| \;&\leq\; \left(\|Q_0^{-1}\| C_2 (C_4 M + |z|) \left(\frac{\alpha}{r}\right)^n + \left(\|Q_0^{-1}\| C_3 + \frac{C_1}{M-r}\right) \tau_\epsilon \left(\frac{M}{r}\right)^n\right) \|h\| \\ &\quad + \left(\|Q_0^{-1}\| C_3 + \frac{C_1}{M-r}\right) \left(\frac{M}{r}\right)^n |Q_\epsilon h|\,. \end{aligned} \tag{12}$$

For the proof of assertion (8) we may assume that $|z| \leq 2M$, because for $|z| > 2M$ a von Neumann series representation gives immediately that $\|(z - P_\epsilon)^{-1} f\| \leq C_4 M^{-1} \|f\|$. Let

$$\begin{aligned} H = H(\delta, r) &:= \sup\{\|(z - P_0)^{-1}\| : z \in \mathbb{C} \setminus V_{\delta,r}\}\;, \\ n = n_2(\delta, r) &:= \left\lceil \frac{\log 4 C_6 H C_2 (C_4 + 2) M}{\log r/\alpha} \right\rceil \end{aligned} \tag{13}$$

and

$$\epsilon_1 = \epsilon_1(\delta, r) := \sup\left\{\epsilon > 0 : \tau_\epsilon \left(H C_3 + \frac{C_1}{M-r}\right) \left(\frac{M}{r}\right)^{n_2} \leq \frac{1}{4C_6}\right\}\;. \tag{14}$$

Since $\|h\| \leq C_5\, \|Q_\epsilon h\| + C_6\, |h|$ by Lemma 1, this and (12) yields for $0 \leq \epsilon \leq \epsilon_1$

$$|h| \leq \frac{1}{2}|h| + \frac{C_5}{2C_6}\|Q_\epsilon h\| + \frac{1}{4\tau_{\epsilon_1} C_6}\, |Q_\epsilon h|$$

so that

$$|h| \leq \frac{C_5}{C_6}\|Q_\epsilon h\| + \frac{1}{2\tau_{\epsilon_1} C_6}\, |Q_\epsilon h|\;.$$

Applying Lemma 1 once more, we arrive at

$$\|h\| \leq 2C_5\, \|Q_\epsilon h\| + \frac{1}{2\tau_{\epsilon_1}}\, |Q_\epsilon h| \leq \left(2C_5 + \frac{1}{2\tau_{\epsilon_1}}\right) \|Q_\epsilon h\| \tag{15}$$

for $0 \leq \epsilon \leq \epsilon_1$. By (15) it follows that Q_ϵ is invertible, its range is closed, and the inverse is bounded, hence Q_ϵ^{-1} exists as a bounded operator on B since z cannot belong to the residual spectrum by assumption (4). Applying this inequality to $h := Q_\epsilon^{-1} f$, this is (8) with $a := \max\{2C_5, C_4 M^{-1}\}$ and $b := \frac{1}{2\tau_{\epsilon_1}}$.

We turn to the proof of (9) and keep the abbreviation $h = Q_\epsilon^{-1} f$. Consider estimate (12) for $n = \lceil \frac{\log \tau_\epsilon}{\log \alpha/M} \rceil = \lceil \eta\, \frac{\log \tau_\epsilon}{\log \alpha/r} \rceil$. Then

$$\begin{aligned} |h| \;&\leq\; \tau_\epsilon^\eta \left(\|Q_0^{-1}\|\, (C_2 (C_4 M + |z|) + C_3) + \frac{C_1}{M-r}\right) \|h\| \\ &\quad + \tau_\epsilon^{\eta-1} \left(\|Q_0^{-1}\|\, C_3 + \frac{C_1}{M-r}\right) |f|\;. \end{aligned}$$

As $|z|\, \|Q_0^{-1}\| \leq 2C_4$ if $|z| \geq 2M$, this yields

$$\begin{aligned} |h| \;&\leq\; \tau_\epsilon^\eta \left(\|Q_0^{-1}\|\, \underbrace{(C_2 (C_4 + 2) M + C_3)}_{=:A} + \underbrace{\left(2C_2 C_4 + \frac{C_1}{M-r}\right)}_{=:B}\right) \|h\| \\ &\quad + \tau_\epsilon^{\eta-1} \left(\|Q_0^{-1}\|\, C_3 + \frac{C_1}{M-r}\right) |f|\;. \end{aligned}$$

Since $\|h\| \le C_5 \|f\| + C_6 |h|$ by Lemma 1 again, this yields

$$\begin{aligned} |h| &\le \tau_\epsilon^\eta \left(\|Q_0^{-1}\| A + B\right) C_5 \|f\| + \tau_\epsilon^\eta \left(\|Q_0^{-1}\| A + B\right) C_6 |h| \\ &\quad + \tau_\epsilon^{\eta-1} \left(\|Q_0^{-1}\| C_3 + \frac{C_1}{M-r}\right) |f| \,. \end{aligned}$$

Recall $H = H(\delta, r)$ from (13) and let

$$\epsilon_0 = \epsilon_0(\delta, r) := \sup \left\{ \epsilon \in (0, \epsilon_1] : \tau_\epsilon^\eta (HA + B) C_6 \le \frac{1}{2} \right\} \,.$$

Then

$$|Q_\epsilon^{-1} f| = |h| \le 2\, \tau_\epsilon^\eta \left(\left(\|Q_0^{-1}\| A + B\right) C_5 \|f\| + \tau_\epsilon^{-1} \left(\|Q_0^{-1}\| C_3 + \frac{C_1}{M-r}\right) |f| \right)$$

for $0 \le \epsilon \le \epsilon_0$.

We apply this estimate to $(P_\epsilon - P_0) Q_0^{-1} f$ instead of f:

$$\begin{aligned} |(Q_\epsilon^{-1} - Q_0^{-1}) f| &= |Q_\epsilon^{-1} (P_\epsilon - P_0) Q_0^{-1} f| \\ &\le 2\tau_\epsilon^\eta \left(\left(\|Q_0^{-1}\| A + B\right) C_5 \|(P_\epsilon - P_0) Q_0^{-1} f\| \right. \\ &\quad \left. + \tau_\epsilon^{-1} \left(\|Q_0^{-1}\| C_3 + \frac{C_1}{M-r}\right) |(P_\epsilon - P_0) Q_0^{-1} f| \right) \\ &\le 2 \|Q_0^{-1}\|^2 \, \tau_\epsilon^\eta \, (2AC_4C_5M + C_3) \|f\| \\ &\quad + 2\|Q_0^{-1}\| \, \tau_\epsilon^\eta \left(2BC_4C_5M + \frac{C_1}{M-r} \right) \|f\| \end{aligned}$$

which proves (9) with $c := 2(2BC_4C_5M + \frac{C_1}{M-r})$ and $d := 2(2AC_4C_5M + C_3)$ for $0 \le \epsilon \le \epsilon_0$. $\square$

Remark 6 In the preceding proof assumption (4) was used only to conclude that if $z \in \mathbb{C} \setminus V_{\delta,r}$ then the range of the operator $Q_\epsilon = z - P_\epsilon$ must be all B. In some situations the following argument might be used instead of assumption (4): Let A denote a connected component of $\mathbb{C} \setminus V_{\delta,r}$, so A is open. We claim that

$$\text{either } A \subseteq \sigma(P_\epsilon) \text{ or } A \cap \sigma(P_\epsilon) = \emptyset. \tag{16}$$

Indeed, estimate (15) guarantees that $\mathrm{dist}(z, \sigma(P_\epsilon)) \ge \frac{1}{C}$ for $z \in \mathbb{C} \setminus (\sigma(P_\epsilon) \cup V_{\delta,r})$ where $C = 2C_5 + (2\tau_{\epsilon_1})^{-1}$, and it is an easy exercise to derive (16) from this. In view of the alternative (16) the unbounded component of $\mathbb{C} \setminus V_{\delta,r}$ is certainly disjoint from $\sigma(P_\epsilon)$.

If $\mathbb{C} \setminus \sigma_\alpha(P_0)$ is connected and if $z \in \mathbb{C} \setminus \sigma_\alpha(P_0)$, then z belongs to the unbounded connected component of $\mathbb{C} \setminus V_{\delta,r}$ provided $r - \alpha$ and δ are sufficiently small, and it follows that $z \in \mathbb{C} \setminus \sigma(P_\epsilon)$ if $\epsilon < \epsilon_0(\delta, r)$. In other words: Given $z \in \mathbb{C} \setminus \sigma_\alpha(P_0)$, there are $\delta, r > 0$ (possibly depending on z) such that $(z - P_\epsilon)^{-1}$ exists as a bounded linear operator for $\epsilon \in [0, \epsilon_0(\delta, r)]$. In particular, (4) can be replaced by " $\mathbb{C} \setminus V_{\delta,r}$ is connected for each $r - \alpha$ and δ sufficiently small."

Proof of Corollary 1:
1)

$$\begin{aligned} |||\Pi_\epsilon^{(\lambda,\delta)} - \Pi_0^{(\lambda,\delta)}||| &\le \frac{1}{2\pi} \int_{\partial B_\delta(\lambda)} |||(z - P_\epsilon)^{-1} - (z - P_0)^{-1}||| \, dz \\ &\le \delta\, \tau_\epsilon^\eta \left(c\, H(\delta, r) + d\, H(\delta, r)^2\right) \quad \text{with } H(\delta, r) \text{ from (13).} \end{aligned}$$

Note that this implies $|||\Pi_\epsilon^{(\lambda,\delta)} - \Pi_0^{(\lambda,\delta)}||| \to 0$ as $\epsilon \to 0$.

2) Using (8) follows

$$\begin{aligned}
\|\Pi_\epsilon^{(\lambda,\delta)} f\| &\leq \frac{1}{2\pi}\int_{\partial B_\delta(\lambda)} \|(z-P_\epsilon)^{-1}f\|\,dz \\
&\leq \frac{1}{2\pi}\int_{\partial B_\delta(\lambda)} (a(r)\,\|f\| + b(\delta,r)\,|f|)\;dz \\
&\leq \delta\, a(r)\,\|f\| + \delta\, b(\delta,r)\,|f|\ .
\end{aligned}$$

Fixing r and choosing $\delta > 0$ such that $\delta\, a(r) \leq \frac{1}{2}$ this yields for sufficiently small ϵ

$$\|\Pi_\epsilon^{(\lambda,\delta)} f\| \leq \frac{1}{2}\|f\| + \text{const}\cdot|f|\ .$$

Applied to $\Pi_\epsilon^{(\lambda,\delta)} f$ instead of f we can conclude that $\|\Pi_\epsilon^{(\lambda,\delta)} f\| \leq K_2 \cdot |\Pi_\epsilon^{(\lambda,\delta)} f|$ for a suitable constant $K_2 > 0$.

3) Let us consider a n-dimensional subspace $\mathbb{V}_n$ of $\Pi_\epsilon^{(\lambda,\delta)}(B)$. In view of part (1) we can choose ϵ and δ small enough such that $|||\Pi_\epsilon^{(\lambda,\delta)} - \Pi_0^{(\lambda,\delta)}||| < \frac{1}{2K_2}$. Then, for $f \in \mathbb{V}_n$,

$$|f - \Pi_0^{(\lambda,\,\delta)} f| = |\Pi_\varepsilon^{(\lambda,\,\delta)} f - \Pi_0^{(\lambda,\,\delta)} f| \leq \frac{1}{2K_2}\|f\| \leq \frac{1}{2}|f|,$$

where we have used the result of part (2) above.

This means that the unit ball of the subspace $\mathbb{V}_n$ is contained in a $\frac{1}{2}$-neighborhood of the subspace $\Pi_0^{(\lambda,\,\delta)}(B)$. In this situation $n \leq \text{rank}(\Pi_0^{(\lambda,\delta)})$ by Tichomirov's theorem (e.g. [29, Theorem 1.5]), and, by the arbitrariness of n, $\text{rank}(\Pi_\epsilon^{(\lambda,\delta)}) \leq \text{rank}(\Pi_0^{(\lambda,\delta)})$. The reverse inequality follows by interchanging the roles of $\Pi_\epsilon^{(\lambda,\delta)}$ and $\Pi_0^{(\lambda,\delta)}$. □

Proof of Corollary 2:

1) This is proved just like the first assertion of Corollary 1.

2) As observed in (10), $\|(z-P_\epsilon)^{-1}\| \leq S_{\delta,r}$ for $z \in \gamma_r$ and $\epsilon \in [0,\epsilon_0]$. Therefore

$$\|P_\epsilon^n \Pi_\epsilon^{(r)}\| \leq \frac{1}{2\pi}\int_{\gamma_r} |z|^n\,\|(z-P_\epsilon)^{-1}f\|\,dz \leq S_{\delta,r}\,r^{n+1}\ .$$

□

Acknowledgments

We like to thank Viviane Baladi for pointing out a mistake in an earlier version of the proof of Corollary 1.

References

[1] V. Baladi, M. Holschneider, *Approximation of nonessential spectrum of transfer operators,* Preprint (1998).

[2] V. Baladi, L.S. Young, *On the spectra of randomly perturbed expanding maps,* Comm. Math. Phys. **156** (1993), 355-385; **166** (1994), 219-220.

[3] M.L. Blank, *Stochastic properties of deterministic dynamical systems,* Sov. Sci. Rev. C. Math./Phys. **6** (1987), 243-271.

[4] M.L. Blank, *Small perturbations of chaotic dynamical systems,* Usp. Mat. Nauk **44** (1989), 3-28.

[5] M.L. Blank, G. Keller, *Stochastic stability versus localization in chaotic dynamical systems,* Nonlinearity **10** (1997), 81-107.

[6] M.L. Blank, G. Keller, *Random perturbations of chaotic dynamical systems: Stability of the spectrum,* Preprint (1998).

[7] A. Boyarsky, P. Góra, *Absolutely continuous invariant measures for piecewise expanding C^2 transformations in $\mathbb{R}^N$,* Israel J. Math. **67** (1989), 272-286.

[8] A. Boyarsky, P. Góra, *Laws of Chaos. Invariant Measures and Dynamical Systems in One Dimension,* Birkhäuser (1997).

[9] C. Chiu, Q. Du, T.Y. Li, *Error estimates of the Markov finite approximation to the Frobenius-Perron operator,* Nonlinear Analysis, Theory, Methods & Appl. **19** (1992), 291-308.

[10] N. Dunford, J.T. Schwartz, *Linear Operators, Part I: General Theory,* Wiley (1957).

[11] G. Froyland, *Computer-assisted bounds for the rate of decay of correlations,* Commun. Math. Phys. **189** (1997), 237-257.

[12] H. Hennion, *Sur un théorème spectral et son application aux noyaux Lipchitziens,* Proc. Amer. Math. Soc. **118** (1993), 627-634.

[13] F. Hofbauer, G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations,* Math. Zeitschrift **180** (1982), 119-140.

[14] F.Y. Hunt, W. Miller, *On the approximation of invariant measures,* J. Stat. Phys. **66** (1992), 535-548.

[15] C.T. Ionescu Tulcea, G. Marinescu, *Thórie ergodique pour des classes d'opérations non complètement continues,* Ann. Math. **52** (1950), 140-147.

[16] M. Iosifescu, R. Theodorescu, *Random Processes and Learning,* Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Vol. 150, Springer (1969).

[17] M. Keane, R. Murray, L.S. Young, *Computing invariant measures for expanding circle maps,* Nonlinearity **11** (1998), 27-46.

[18] G. Keller, *Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan,* C. R. Acad. Sci. Paris, Série A, **289** (1979), 625-627 (Kurzfassung der Dissertation).

[19] G. Keller, *Un théorème de la limite centrale pour une classe de transformations monotones par morceaux,* C. R. Acad. Sci. Paris, Série A, **291** (1980), 155-158.

[20] G. Keller, *On the rate of convergence to equilibrium in one-dimensional systems,* Commun. Math. Phys. **96** (1984), 181-193.

[21] G. Keller, *Stochastic stability in some chaotic dynamical systems,* Monatshefte Math. **94** (1982), 313-333.

[22] G. Keller, M. Künzle, *Transfer operators for coupled map lattices,* Ergod. Th. & Dynam. Sys. **12** (1992), 297-318.

[23] G. Keller, T. Nowicki, *Spectral theory, zeta functions and the distribution of periodic orbits for Collet-Eckmann maps,* Commun. Math. Phys. **149** (1992), 31-69.

[24] A. Lasota, J.A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations,* Transact. Amer. Math. Soc. **186** (1973), 481-488.

[25] T.Y. Li, *Finite approximations for the Frobenius-Perron operator: A solution to Ulam's conjecture,* J. Approx. Th. **17** (1976), 177-186.

[26] W. Miller, *Stability and approximation of invariant measures for a class of nonexpanding transformations,* Nonlinear Analysis, Theory, Methods & Appl. **23** (1994), 1013-1025.

[27] M.F. Norman, *Markov Processes and Learning Models,* Mathematics in Science and Engineering, Vol. 84, Academic Press (1972).

[28] W. Parry, M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics,* Astérisque, Vol. 187-188, (1990).

[29] A. Pinkus, *n-Widths in Approximation Theory,* Springer (1985).

[30] M. Rychlik, *Bounded variation and invariant measures,* Studia Math. **76** (1983), 69-80.

[31] H.H. Schaefer, *Banach Lattices and Positive Operators,* Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Vol. 215, Springer (1974).