Decay of correlations in one-dimensional dynamics

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Abstract

We consider (multimodal) interval maps f satisfying a summability condition on the derivatives D_n along the critical orbits which implies the existence of an absolutely continuous f-invariant probability measure μ. If f is non-renormalizable, μ is mixing and we show that the speed of mixing (decay of correlations) is strongly related to the rate of growth of the sequence $\{D_n\}$ as $n \to \infty$. We also give sufficient conditions for μ to satisfy the Central Limit Theorem.

1 Introduction

1.1 Statement of results

Let $f : I \to I$ be a C^2 interval or circle map with negative Schwarzian derivative (i.e. $1/\sqrt{f'}$ is locally convex) and a finite critical set $\mathcal{C}$. All critical points are assumed to have the same finite *critical order* $\ell \in (1, \infty)$. This means that for $c \in \mathcal{C}$, there exist a diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ fixing 0 such that for x close to c,

$$f(x) = \pm|\varphi(x-c)|^\ell + f(c),$$

where the $\pm$ may depend on $\mathrm{sgn}(x-c)$. For a critical point c, let

$$D_n(c) = |(f^n)'(f(c))|.$$

The aim of this paper is to prove the existence of an absolutely continuous invariant probability measure under sufficient growth conditions of $D_n(c)$ and to study its statistical properties (rate of mixing, Central Limit Theorem). Throughout the paper we need an additional assumption called *bounded backward contraction* (BBC):

> There exists $\kappa > 0$ such that for all $\delta_0 > 0$, there exists $\delta \in (0, \delta_0)$ such that for $\Delta = \cup_c (c-\delta, c+\delta)$ and every x
>
> $$|(f^n)'(x)| \geq \kappa \text{ for } n = \min\{i \geq 0; f^i(x) \in \Delta\}. \qquad \text{(BBC)}$$

For unimodal maps with negative Schwarzian derivative it is well known [3] that (BBC) holds (even with $\kappa = 1$ if f is symmetric). By a result of Kozlovski [7] also the assumption of negative Schwarzian derivative turns out to be redundant in the unimodal case. We say that f satisfies condition $(*)$ if the following holds:

> There exists a sequence $\{\gamma_n\}$, $0 < \gamma_n < \frac{1}{2}$, such that $\sum_n \gamma_n < \infty$ and
>
> $$\sum_n [\gamma_n^{\ell-1} D_n(c)]^{-1/\ell} < \infty \text{ for all } c \in \mathcal{C}. \qquad (*)$$

Theorem 1 (Existence of invariant probability measures). *If f satisfies conditions $(*)$ and (BBC), then there exists an f-invariant probability measure μ absolutely continuous with respect to Lebesgue measure.*

Moreover, as was shown in general in [5], supp(μ) is an interval or cycle of intervals.

By choosing (for example) $\gamma_n = n^{-(1+\epsilon)}$ for some sufficiently small ϵ, Theorem 1 clearly implies the existence of a measure μ as above whenever f satisfies (BBC) and there exists $C > 0$ such that for all $c \in \mathcal{C}$

$$D_n(c) \geq Cn^\tau \quad \text{for } \tau > 2\ell - 1 \text{ and all } n \geq 1. \qquad (**)$$

In the unimodal case, Nowicki & van Strien [15] proved the same result under the weaker assumption that $\sum_n D_n^{-1/\ell} < \infty$. In the holomorphic case, Przytycki proved that for every α-conformal measure μ on J (satisfying an additional assumption), assuming $D_n(c)$ grows exponentially, there exists an invariant probability measure which is absolutely continuous with respect to μ, see [16].

The measure μ need not be unique if f is multimodal and not Lebesgue ergodic. (Unimodal maps with negative Schwarzian derivative are Lebesgue ergodic, [2].) Let X be a closed f-invariant set of positive Lebesgue measure such that X contains no smaller set with these properties. Misiurewicz [13] proved that X contains a critical point. We do not know if X can be a Cantor set (however, cf. [17]), but if X supports the measure μ from Theorem 1, then X has a non-empty interior. By a result of Ledrappier [8], μ is mixing if and only if $f : X \to X$ is not *renormalizable*, i.e. X is not a cycle of intervals permuted by f. In this case it is natural to ask about the speed of mixing, quantified through the *correlation function*

$$\mathcal{C}_n = \mathcal{C}_n(\varphi, \psi) = \left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right|$$

where φ and ψ are respectively bounded and Hölder continuous functions on X.

Theorem 2 (Decay of correlations). *Let f satisfy (BBC) and $(*)$ and let μ be an absolutely continuous invariant probability measure with support supp(μ). If f is not renormalizable on supp(μ), then (supp(μ), μ, f) is mixing with the following rates: Write*

$$d_n(c) = \min_{i<n} [\gamma_i / D_i(c)]^{1/\ell} |f^i(c) - \mathcal{C}|.$$

Polynomial case: *If*

$$d_n(c) \leq Cn^{-\alpha}$$

for all $c \in \mathcal{C}$*, some* $\alpha > 1$ *and all* $n \geq 1$*, then for each*

$$\tilde{\alpha} < \alpha - 1.$$

there exist $\tilde{C} = \tilde{C}(\varphi, \psi) > 0$ *such that*

$$\mathcal{C}_n \leq \tilde{C} n^{-\tilde{\alpha}} \text{ for all } n \geq 1.$$

Stretched exponential case: *If*

$$d_n(c) \leq C e^{-\beta n^{\alpha}}$$

for all $c \in \mathcal{C}$*, some* $C, \beta > 0$*,* $\alpha \in (0,1)$ *and all* $n \geq 1$*, then for all* $\tilde{\alpha} \in (0, \alpha)$ *there exist* $\tilde{C} = \tilde{C}(\varphi, \psi), \tilde{\beta} > 0$ *such that*

$$\mathcal{C}_n \leq \tilde{C} e^{\tilde{\beta} n^{\tilde{\alpha}}} \text{ for all } n \geq 1.$$

Exponential case: *If*

$$d_n(c) \leq C e^{-\beta n}$$

for all $c \in \mathcal{C}$*, some* $C, \beta > 0$ *and* $n \geq 1$*, then there exist* $\tilde{C} = \tilde{C}(\varphi, \psi), \tilde{\beta} > 0$ *such that*

$$\mathcal{C}_n \leq \tilde{C} e^{-\tilde{\beta} n} \text{ for all } n \geq 1.$$

As an illustration of what Theorem 2 means, let us consider, as in $(**)$, the polynomial case: $D_n(c) \geq C n^{\tau}$, $\tau > 2\ell - 1$ for all $c \in \mathcal{C}$ and $n \geq 1$. The theorem then tells us that $\mathcal{C}_n \leq \tilde{C} n^{-\tilde{\tau}}$ for any $\tilde{\tau} < \frac{\tau - 1}{\ell - 1} - 1$.

If μ is an f-invariant probability measure, we say that the Central Limit Theorem holds if for any Hölder continuous function φ which is not a coboundary ($\varphi \neq \psi \circ f - \psi$ for any ψ) there exists $\sigma > 0$ such that for every interval $J \subset \mathbb{R}$,

$$\mu \left\{ x \in X : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\varphi(f^j(x)) - \int \varphi d\mu \right) \in J \right\} \to \frac{1}{\sigma \sqrt{2\pi}} \int_J e^{-t^2/2\sigma^2} dt.$$

This property is indicative of a certain regularity in the way Birkhoff averages of Hölder observable approach their expected asymptotic values.

Theorem 3 (Central Limit Theorem). *Let f satisfy conditions (BBC) and $(*)$. If f is not renormalizable and $d_n(c) \le Cn^{-\alpha}$, $\alpha > 2$ for all $c \in \mathcal{C}$ and $n \ge 1$, then the measure μ of Theorem 1 satisfies the Central Limit Theorem.*

The statements about decay of correlations and Central Limit Theorem in the unimodal exponential case were proved in [6, 18]. As far as we know the results in all other cases are new. Most known examples of systems with strictly subexponential rates of decay of correlations consist of maps which are uniformly expanding except for the presence of some neutral fixed point, see for example [9, 20]. The situation here is more subtle as the cause for the loss of exponential estimates is not so localized.

In fact in the unimodal case, $D_n \ge Ce^{\beta n}$ if and only if (X, μ, f) has exponential decay of correlations [14]. A remarkable consequence of this fact and Theorem 2 is

Corollary 1.1. *Let f be unimodal. Then exponential decay of*

$$d_n(c) = \min_{i<n} \frac{\gamma_i^{1/\ell} |f^i(c) - \mathcal{C}|}{D_i(c)^{1/\ell}}$$

plus condition $(*)$ is equivalent to the exponential growth of D_n, i.e. the Collet-Eckmann condition.

It is interesting to apply the results to the *Fibonacci maps*, i.e. the conjugacy class of unimodal maps characterized by the property that the sequence of closest return times is exactly the Fibonacci sequence. Lyubich & Milnor [10] proved that in the quadratic case, the Fibonacci map satisfies Nowicki's & van Strien's summability condition. Here we show

Corollary 1.2. *Let f be a Fibonacci map with quadratic critical point. Then one has (faster than) polynomial decay of correlations and the central limit theorem holds.*

Proof. In fact, the estimates in [10, Section 5] show that condition $(*)$ holds for $\gamma_i = 0.1\sqrt{1/D_i}$. So we can apply Theorem 1. Theorem 2 and 3 also hold. Indeed, if $S_r \approx \gamma^{-r}$ is the r-th Fibonacci number (with $\gamma = (\sqrt{5}-1)/2$), then $|f^{S_r}(c) - c| \approx e^{-\beta' r^2} \approx S_r^{-\beta \log S_r}$ for some $\beta', \beta > 0$.

Let $S_{r-1} < k \le S_r$, be arbitrary. Then $d_k \le S_{r-1}^{-\beta \log S_{r-1}} \le (\gamma k)^{-\gamma\beta \log(\gamma k)}$, which decreases faster than any polynomial, but more slowly than what we call stretched exponentially. In particular, the Central Limit Theorem holds. □

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1.2 Strategy of the proof

Our strategy is to define a Markov return map $\hat{f} = f^R : \Omega_0 \to \Omega_0$ on a suitable neighbourhood of one of the critical points. We shall obtain estimates on the tail $|\{x \in \Omega_0 : R > n\}|$ of the return times and apply the general framework of L.-S. Young [19] linking these estimates with bounds for the decay of correlation. The general philosophy is that intervals outside a neighbourhood Δ of the critical set grow exponentially fast (by a classical result of Misiurewicz [13], see [11] for a C^2 version) and therefore for these intervals during these times, the tail of the return times decays exponentially fast. Many intervals however fall into Δ before a good return to Ω_0 occurs and are strongly contracted in the next iterate. We estimate the time it takes for them to recover their original size in terms of the derivatives along the critical orbits, which in turn provides bounds for the decay of the tail of the return time function.

In Section 2 we consider intervals in Δ and use a binding argument to obtain estimates for their growth in terms of the derivative along the appropriate critical orbit. Similar arguments have been applied before, notably by Jakobson [4] and Benedicks & Carleson [1], under stronger conditions on D_n and on the recurrence of the critical orbit. Here we have generalized the argument to deal with slow derivative growth rates along the critical orbits and essentially arbitrary recurrence patterns.

In Section 3 we consider an arbitrary interval $J \subset I$ and show that there exists a partition $\hat{\mathcal{P}}$ of J and a stopping time function $\hat{p}$ such that the images $f^{\hat{p}(\omega)}(\omega)$ are uniformly large for all $\omega \in \hat{\mathcal{P}}$, i.e. almost every point of J belongs to an interval which achieves *large scale*. We describe a combinatorial structure of $f^{\hat{p}}$ on J which keeps track of the pattern of returns to Δ of each ω. By combining this information with some analytic estimates on the size of elements with given combinatorics, we obtain key estimates on the size of the tail $\{x \in J : \hat{p} > n\}$ of the stopping time function $\hat{p}$. A variety of arguments is used here to deal with the various possible rates (polynomial, stretched exponential or exponential).

In Section 4 we show that once an interval has achieved large scale there is a fixed proportion of it which has a full return to the original interval Ω_0 within a fixed number of iterates. It follows that the transition from large scale to full return occurs exponentially fast and does not significantly affect the tail estimates. We also state precisely the results of Young which we apply to our return map to obtain the conclusions of our theorems.

2 Inducing to small scales

We define a partition $\mathcal{P}$ of a critical neighbourhood Δ and a stopping time function p such that the induced map $F = f^p$ on Δ is expanding. The images of partition elements are not uniformly large, i.e. $\inf\{|f^{p(\omega)}(\omega)| : \omega \in \mathcal{P}\} = 0$, and therefore we call this *inducing to small scales.*

2.1 Definitions and notation

We use the symbol $\approx$ to indicate that two terms are equal up to a factor depending only on f. Because $\ell < \infty$,

$$|f'(x)| \approx |x - c|^{\ell-1}.$$

for all $x \in X$ close to c. Also there exists τ such that

$$\frac{|f'(x) - f'(y)|}{|f'(x)|} \leq \tau \frac{|x - y|}{|x - \mathcal{C}|}, \tag{1}$$

for all x, y such that $|x - y| \leq \frac{1}{2}\max\{|x - \mathcal{C}|, |y - \mathcal{C}|\}$. Here $|x - \mathcal{C}| = \min\{|x - c|; c \in \mathcal{C}\}$. Let $\Gamma = \exp(\tau \sum_{j=1}^{\infty} \frac{\gamma_j}{1-\gamma_j})$.

For $x \in X$, let $c = c(x) \in \mathcal{C}$ be the critical point closest to x. This is well defined for x sufficiently close to $\mathcal{C}$. Let

$$p(x) := \max\{p : |f^k(x) - f^k(c)| \leq \gamma_k |f^k(c) - \mathcal{C}| \quad \forall\ k \leq p - 1\}.$$

Clearly $p \to \infty$ monotonically as $x \to c(x)$. In order to choose the size of our critical neighbourhood Δ we need the following lemma.

Lemma 2.1. *Suppose that $G_p \geq 0$ and $\sum_p G_p < \infty$. Then for any $\zeta > 0$ there exists p_0 such that*

$$P = \sum_{s\geq 1} \sum_{\substack{(p_1,\dots,p_s)\\ p_i \geq p_0}} \prod_{p_i} \zeta G_{p_i} \leq 1.$$

Proof. Let $S_0 = \sum_{p\geq p_0} \zeta G_p$. Then both S_0 and $S := \sum_{s\geq 1} S_0^s$ tend to 0 as $p_0 \to \infty$. Developing term by term we see that $P \leq S$. This proves the lemma. □

Taking advantage of this lemma and condition $(*)$ we fix for the rest of the paper a critical neighbourhood $\Delta = \Delta_\delta = \cup_{c\in\mathcal{C}}(c-\delta, c+\delta)$ where $\delta > 0$ is such that (BBC) holds for δ and so small that

$$\sum_{s\leq n} \sum_{\substack{(p_1,\dots,p_s)\\ \sum_i p_i \leq n\\ p_i \geq p_\delta}} \prod_{p_i} \zeta(\gamma_i^{\ell-1} D_i(c))^{1/\ell} \leq 1 \tag{2}$$

for all $c \in \mathcal{C}$, $p_\delta := p(c \pm \delta)$, $\zeta = 64K_0/\kappa C_0$, C_0 the constant introduced in Lemma 2.3, and K_0 a fixed Koebe distortion constant, which turns out to be ≤ 16.

For $p \geq 0$ we let $I_p = \{x : p(x) = p\}$ denote the level sets of the function p. Let $\mathcal{P}$ denote the corresponding partition of Δ, which we extend to X by adding the components of $I_0 = X \setminus \Delta$ to $\mathcal{P}$, and putting $p = 0$ on them. Notice that I_p can be empty for some values of p, and that it has at most $2\#\mathcal{C}$ components. Define $F : X \to X$ by letting $F(x) = f^{p(x)}(x)$ for $x \in \Delta$ and $F(x) = f(x)$ for $x \in X \setminus \Delta$.

2.2 Expansion estimates

We have two main expansion estimates.

Lemma 2.2 (Derivative growth for pieces of orbit outside Δ). *There exist constants $C_\delta > 0$ and $\lambda_\delta > 0$ such that for every piece of orbit $\{f^i(x)\}_{i=0}^{k-1}$ lying completely outside Δ we have*

$$|(f^k)'(x)| \geq C_\delta e^{\lambda_\delta k}.$$

If moreover $f^k(x) \in \Delta$, then

$$|(f^k)'(x)| \geq \max\{\kappa, C_\delta e^{\lambda_\lambda k}\}.$$

Notice that the first estimate clearly implies the second if k is large. The second however is extremely useful when considering small values of k.

Proof. The first estimate is well known for maps with negative Schwarzian derivative (as in our case), see Chapter II in [12]. The second statement follows from assumption (BBC). We repeat that in the unimodal case (BBC) holds automatically (in absence of periodic attractors). $\square$

The following expansion bound will of importance. Let

$$F_p'(c) := \min\{|(f^p)'(x)|; x \in I_p \cap (c-\delta, c+\delta)\}.$$

Lemma 2.3 (Derivative growth for pieces of orbit starting in Δ). *There exists $C_0 > 0$ such that for every $c \in \mathcal{C}$ and $p \geq p_\delta$ with $I_p \neq \emptyset$ we have*

$$F_p'(c) \geq C_0[\gamma_p^{\ell-1} D_p(c)]^{1/\ell}. \tag{3}$$

In the sequel, we will write F_p' instead of $F_p'(c)$ when no confusion can arise. We shall need an intermediate result for the proof.

Lemma 2.4. *For $x \in \Delta$ we have*

$$\frac{|(f^i)'(y)|}{|(f^i)'(z)|} \leq \Gamma \textit{ for all } \ y, z \in [f(x), f(c)] \textit{ and all } \ i \leq p(x) - 1.$$

Proof. Letting $y_j = f^j(y)$ and $z_j = f^j(z)$ for $j \geq 0$ we have by the chain rule

$$\left|\frac{(f^i)'(y)}{(f^i)'(z)}\right| = \prod_{j=0}^{i-1} \left|\frac{f'(y_j)}{f'(z_j)}\right| = \prod_{j=0}^{i-1} \left(1 + \frac{|f'(z_j) - f'(y_j)|}{|f'(y_j)|}\right).$$

By (1), $|f'(z_j) - f'(y_j)|/|f'(y_j)| \leq \tau|z_j - y_j|/|y_j - \mathcal{C}|$ and so, using the elementary fact that $\log(1+x) \leq x$ for all $x > 0$ we get

$$\log\left|\frac{(f^i)'(y)}{(f^i)'(z)}\right| \leq \sum_{j=0}^{i-1} \log\left(1 + \tau\frac{|z_j - y_j|}{|y_j - \mathcal{C}|}\right) \leq \tau \sum_{j=0}^{i-1} \left(\frac{|z_j - y_j|}{|y_j - \mathcal{C}|}\right).$$

By definition of p we have $|z_j - y_j| \leq |f^{j+1}(x) - f^{j+1}(c)| \leq \gamma_{j+1}|f^{j+1}(c) - \mathcal{C}|$ and $|y_j - \mathcal{C}| \geq (1-\gamma_{j+1})|f^{j+1}(c) - \mathcal{C}|$. Here c is again the critical point closest to x. Substituting these inequalities into the last formula yields the desired statement. $\square$

Proof of Lemma 2.3. Let c be the critical point closest to x. By Lemma 2.4 and the fact that $|f'(x)| \approx |x-c|^{\ell-1} \approx |f(x)-f(c)|^{(\ell-1)/\ell}$ we have

$$|(f^p)'(x)| \geq \frac{|f'(x)|D_{p-1}(c)}{\Gamma} \approx \frac{|f(x)-f(c)|^{(\ell-1)/\ell}D_{p-1}(c)}{\Gamma}. \tag{4}$$

By the Mean Value Theorem, the definition of p and the distortion estimate in Lemma 2.4 we have

$$\Gamma D_{p-1}|f(x)-f(c)| \geq |f^p(x)-f^p(c)| \geq \gamma_p |f^p(c)-\mathcal{C}|$$

and therefore

$$|f(x)-f(c)| \geq \frac{\gamma_p|f^p(c)-\mathcal{C}|}{\Gamma D_{p-1}(c)}. \tag{5}$$

Substituting (5) into (4) gives

$$|(f^p)'(x)| \geq \mathcal{O}(\Gamma^{-2+1/\ell})\gamma_p^{(\ell-1)/\ell}D_{p-1}(c)^{1/\ell}|f^p(c)-\mathcal{C}|^{(\ell-1)/\ell}.$$

Because $|f'(f^p(c))| \approx |f^p(c)-\mathcal{C}|^{\ell-1}$, the Chain Rule gives

$$|(f^p)'(x)| \geq \mathcal{O}(\Gamma^{-2+1/\ell})\gamma_p^{(\ell-1)/\ell}D_p(c)^{1/\ell}.$$

This proves the lemma. □

3 Inducing to large scales

The main result of this section is the following

Proposition 3.1. *Suppose that f satisfies conditions (BBC) and $(*)$. Then there exist constants $\delta', \delta'' > 0$ with the following properties. For an arbitrary interval $J \subset X$ with $|J| \geq \delta''$ there exists a partition $\hat{\mathcal{P}}$ of J (mod 0) and a stopping time function $\hat{p}: \hat{\mathcal{P}} \to \mathbb{N}$ such that for all $\omega \in \hat{\mathcal{P}}$, $\hat{F}|\omega := f^{\hat{p}(\omega)}|\omega$ is a diffeomorphism with uniformly bounded distortion and $|\hat{F}(\omega)| = |f^{\hat{p}(\omega)}(\omega)| \geq \delta'$. Moreover the following estimates hold:*

Summable case:

$$\sum_n |\{\hat{p} > n|J\}| < \infty.$$

Polynomial case: *If* $d_n(c) \leq Cn^{-\alpha}$ *for all* $c \in \mathcal{C}$ *and* $n \geq 1$*, then there exists* $\hat{C} > 0$ *such that*

$$|\{\hat{p} > n | J\}| \leq \hat{C} n^{-\alpha}.$$

Stretched exponential case: *If* $d_n(c) \leq Ce^{-\beta n^\alpha}$, $\alpha \in (0,1)$, $\beta > 0$ *for all* $c \in \mathcal{C}$ *and* $n \geq 1$*, then for each* $\hat{\alpha} \in (0, \alpha)$ *there exist* $\hat{\beta}, \hat{C} > 0$ *such that*

$$|\{\hat{p} > n | J\}| \leq \hat{C} e^{-\hat{\beta} n^{\hat{\alpha}}}.$$

Exponential case: *If* $d_n(c) \leq Ce^{-\beta n}$, $\beta > 0$ *for all* $c \in \mathcal{C}$ *and* $n \geq 1$*, then there exist* $\hat{\beta}, \hat{C} > 0$ *such that*

$$|\{\hat{p} > n | J\}| \leq \hat{C} e^{-\hat{\beta} n}.$$

The expression $|\{\hat{p} > n|J\}|$ denotes the conditional probability: $\{\hat{p} > n|J\}| = |\{\hat{p} > n\}|/|J|$. In Section 3.1 we define and describe the combinatorics of the partition $\hat{\mathcal{P}}$ of J and the stopping time $\hat{p}$. In Section 3.2 we prove some key estimates on the size of an interval $\omega_{p_1,\dots,p_s} \in \hat{\mathcal{P}}$ with a given combinatorics. In Section 3.3 we combine these with some counting arguments to obtain estimates on $\{\hat{p} > n|J\}$.

In the previous section we have fixed δ. By the Contraction Principle (see e.g. [12, Section IV.5], there exists δ' such that for each component W of $\Delta \setminus \mathcal{C}$ and each $n \geq 0$, $|f^n(W)| \geq \delta'$. The constant δ'' will be defined below as $\min\{\delta'/3, |\Omega_0|\}$, where Ω_0 is an interval specified in Section 4.

3.1 Combinatorial structure of $\hat{F}$

We start with any interval $\omega \in \mathcal{P}|J$, i.e. $\omega = I_p \cap J$ for some $p \geq 0$, and let $\nu_1 = \min\{n \geq 0; f^n(\omega) \cap \Delta \neq \emptyset\}$ be the first visit to Δ. Write $\tilde{\omega} = f^{\nu_1}(\omega)$. There are two (mutually exclusive) cases:

- $|\tilde{\omega}| < \delta'$. We partition $\tilde{\omega}$ by intersecting $\tilde{\omega}$ with the elements $\{I_p\}$. Each interval $I_p \cap \tilde{\omega}$ for $p > 0$ is labeled as *Deep Return*. The interval $I_0 \cap \tilde{\omega}$ is taken together with the interval $I_p \cap \tilde{\omega}$ adjacent to it. Because $\tilde{\omega}$ is not too large compared to this particular component of I_p, the estimates of the bound period of I_p go through.

- $|\tilde{\omega}| \geq \delta'$. We cut off the outmost intervals of length ϵ from $\tilde{\omega}$, and stop with the remaining middle part; it has reached *large scale*. The subinterval $\omega_0 \subset \omega$ such that $f^{\nu_1}(\omega_0)$ equals this middle part of $\tilde{\omega}$ is added to the partition $\hat{\mathcal{P}}$. Here $\epsilon \ll \delta'$ is a constant to be fixed in the proof of Lemma 3.6. For the moment it suffices to know that ϵ is smaller than each component of Δ and each component of $X \setminus \Delta$.

  The outmost intervals $\tilde{\omega}_\pm$ of length ϵ are partitioned by intersecting them with the elements $\{I_p\}$. Each interval $I_p \cap \tilde{\omega}_\pm$ for $p > 0$ is labeled as *Deep Return*. The interval $I_0 \cap \tilde{\omega}_\pm$ (if it exists) is labeled as *Shallow Return*.

Now let ω' be an interval which results from this partitioning of $\tilde{\omega}$, which has not reached large scale. We first apply the binding period, i.e. we take $f^p(\omega')$ for the stopping time $p = p(\omega')$ (which is possibly 0, namely if $\omega' \cap \Delta = \emptyset$), and then take the second return $\nu_2 = \min\{n \geq p(\omega'); f^n(\omega') \cap \Delta \neq \emptyset\}$. Subdivide $f^{\nu_2}(\omega')$ according to the above rules, distinguishing between large and deep returns.

Notice that the procedure depends on the choice made for J (if J_1 and J_2 are two intersecting intervals, then one could get two different partitions created at a point $x \in J_1 \cap J_2$). Let $\hat{\mathcal{P}}_n$ be the partition which we obtain by only considering at most n iterates of f and $\hat{\mathcal{P}}$ the partition of J by considering all iterates of f. We should emphasize that this partition depends on J and also on ϵ. At points $x \in \omega$ where the procedure eventually stops, i.e., for which there exists $n > 0$ so that the n-th iterate of the interval in $\hat{\mathcal{P}}_n$ containing x has reached large scale, we define $\hat{p}_J(x) = n$. At other points $x \in J$ we define $\hat{p}_J(x) = \infty$. Note that if $x \in \omega$, then $f^n(x) \in \Delta$ only if n is in a binding period of x, x has a deep return or if $n = \hat{p}(x)$. At shallow return times n, $f^n(x) \notin \Delta$. Finally to define $\hat{F}_J$ we let $\hat{J} = \{x \in J : \hat{p}_J(x) < \infty\}$ and define $\hat{F}_J : \hat{J} \to I$ by $\hat{F}_J(x) = f^{\hat{p}_J(x)}(x)$. We shall prove that $\hat{J} = J$ up to sets of zero Lebesgue measure.

Take $n < \infty$. To each $\omega_n \in \hat{\mathcal{P}}_n$ we assign a formal *itinerary*

$$(\nu_1, p_1), \ldots, (\nu_s, p_s),$$

consisting of the stopping times and lengths of the corresponding binding periods; s is maximal for $\nu_s \leq n$. Depending on the depth of the return at time ν_s, $\nu_s + p_s$ can be arbitrarily large. If the return at time ν_j is shallow, then $p_j = 0$. If ω is an interval on which $(\nu_1, p_1), \ldots, (\nu_{j-1}, p_{j-1})$

is constant and for which $\nu_j = \nu_j(\omega)$ is the next return to Δ, then the set $\{x \in \omega; p_j(x) = p\}$ has at most 4 components. This maximum is attained when $|f^{\nu_j(\omega)}| \geq \delta$ and the outmost intervals of size ϵ both cover a critical point. It can happen that $f^{\nu_j}(\omega)$ covers many more critical points, but since ω has reached large scale, p_j is only define on the outmost intervals. We will take care of this multiplicity in the estimates in Subsection 3.3. But apart from this multiplicity, a sequence $p_1, \ldots, p_s$ uniquely determines a partition element $\omega_{p_1,\ldots,p_s} \in \hat{\mathcal{P}}_n$ (or perhaps the empty set). Indeed, p_i determines the position of the i-th return of $\omega_{p_1,\ldots,p_s}$, and from the previous $p_1, \ldots, p_{i-1}$ and the starting interval J one can compute the next return time. Hence the information $\nu_1, \ldots, \nu_s$ is strictly speaking superfluous. Observe however that there are many itineraries that do not correspond to partition elements. Note that f^n is a diffeomorphism on each interval from the partition $\hat{\mathcal{P}}_n$.

For a given sequence $(p_1, \ldots, p_s)$, let

$$S_d = \{i \leq s; \nu_i \text{ is a deep return } \} = \{i \leq s; p_i > 0\}$$

and

$$S_s = \{i \leq s; \nu_i \text{ is a shallow return } \} = \{i \leq s; p_i = 0\} = S \setminus S_d.$$

Moreover, let

$$S_{s,s} = \{i < s; p_i = 0 \text{ and } p_{i+1} = 0\}.$$

Notice that because each index in $S_s \setminus S_{s,s}$ either equals s or is followed by an index in S_d, we get

$$\#S_s \leq \#S_{s,s} + \#S_d + 1. \tag{6}$$

3.2 Metric and combinatorial estimates

Lemma 3.2. *Let $C = C_\delta$ and $\lambda = \lambda_\delta$ be as in Lemma 2.2. There exists $K_0 > 0$ independent of ϵ and $\rho \in (0,1)$ ($\rho \to 0$ as $\epsilon \to 0$), with the following properties. For a given sequence $(\nu_1, p_1), \ldots (\nu_s, p_s)$ with $\nu_s \leq n$ we have*

$$\frac{|\omega_{p_1,\ldots,p_s}|}{|f^m(\omega_{p_1,\ldots,p_s})|} \leq \min\{C^{-\#S_d} e^{-\lambda(m - \sum_{i=0}^{s} p_i)}, \left(\frac{K_0}{\kappa}\right)^{\#S_d} \rho^{\#S_{s,s}}\} \prod_{i \in S_d} (F'_{p_i})^{-1}$$

for $m = \max\{n, \nu_s + p_s\}$. Moreover there exists $T > 0$ which can be chosen arbitrarily large if ϵ is small, such that $\nu_{i+1} - \nu_i \geq T$ whenever $p_i = p_{i+1} = 0$.

Proof. By construction, $f^m|\omega_{p_1,\dots,p_s}$ is a diffeomorphism onto its image. Take $x \in \omega_{p_1,\dots,p_s}$ and divide its orbit into pieces separated by returns (both deep and shallow):

$$[\nu_1, \nu_2 - 1],\ [\nu_2, \nu_3 - 1],\ \dots,\ [\nu_s, m].$$

Let $\nu_i < \nu_{i'}$ be two consecutive deep returns. That is, assume that $p_i > 0$, $p_{i+1} = \dots = p_{i'-1} = 0$ and $p_{i'} > 0$ (with possibly $i' = i+1$). Because each such interval lasts at least the corresponding binding period, and in the remaining time the point x does not visit Δ, Lemma 2.2 and the definition of F'_p give

$$|(f^{\nu_{i'}-\nu_i})'(f^{\nu_i}(x))| \geq C e^{\lambda(\nu_{i'}-(\nu_i+p_i))} F'_{p_i}.$$

Hence the chain rule and the Mean Value Theorem show that $|\omega_{p_1,\dots,p_s}| \leq C^{-\#S_d} e^{-\lambda(m-\sum p_i)} \prod_i 1/F'_{p_i}$.

To prove the other inequality, let ν_i and $\nu_{i'}$ be subsequent deep return times. First let us tackle the step from $\nu_{i'-1} + p_{i'-1}$ to $\nu_{i'}$. Because $f^{p(y)}(y) \notin \Delta$ for any binding period, we have $\nu_{i'-1} + p_{i'-1} < \nu_{i'}$. Moreover, $f^{\nu_{i'-1}+p_{i'-1}}(\omega_{p_1,\dots,p_s}) \cap \Delta = \emptyset$, while there is at least one point $y \in \omega_{p_1,\dots,p_s}$ such that $f^{\nu_{i'}}(y) \in \Delta$. Therefore

$$|(f^{\nu_{i'}-(\nu_{i'-1}+p_{i'-1})})'(f^{\nu_{i'-1}+p_{i'-1}}(y))| \geq \kappa.$$

Because $\nu_{i'} < n$, $|f^{\nu_{i'}}(\omega_{p_1,\dots,p_s})| < \delta'$. Take $H \supset f^{\nu_{i'-1}+p_{i'-1}}(\omega_{p_1,\dots,p_s})$ the largest interval on which $f^{\nu_{i'}-(\nu_{i'-1}+p_{i'-1})}$ is monotone. Then by the choice of δ', $f^{\nu_{i'}-(\nu_{i'-1}+p_{i'-1})}(H)$ contains a $\frac{1}{3}$-scaled neighbourhood of $f^{\nu_{i'}}(\omega_{p_1,\dots,p_s})$. Therefore the derivative of $f^{\nu_{i'}-(\nu_{i'-1}+p_{i'-1})}$ has distortion bounded by some $K_0 = K_0(\frac{1}{3}) \leq 16$, cf. [12, Chapter IV]. This shows that

$$|(f^{\nu_{i'}-(\nu_{i'-1}+p_{i'-1})})'(f^{\nu_{i'-1}+p_{i'-1}}(x))| \geq \kappa/K_0.$$

If $i' \leq i+2$, the same argument gives $|(f^{\nu_{i'}-(\nu_i+p_i)})'(f^{\nu_{i'-1}+p_{i'-1}}(x))| \geq \kappa/K_0$, and indeed in this case there are no entries of $S_{s,s}$ between i and i'.

If $i' > i+2$, then the differences $\nu_{i+2} - (\nu_i - p_i)$, $\nu_{i+3} - \nu_{i+2}$, ..., $\nu_{i'-1} - \nu_{i'-2}$ are all large if ϵ is small. Indeed, in these times an interval of size ϵ must have expanded to an interval of size $\delta' \gg \epsilon$. Because x does not visit Δ during these iterates, the first part of Lemma 2.2 gives

$$|(f^{\nu_{i'-1}-(\nu_i+p_i)})'(f^{\nu_i+p_i}(x))| \geq \left(\frac{1}{\rho}\right)^{i'+i-2},$$

where $\rho \to 0$ as $\epsilon \to 0$. (In Lemma 3.6 we will fix ρ at $1/8$.) Because $\nu_{i-1} > \nu_{i'-1} + p_{i'-1}$, this proves the lemma. $\square$

Lemma 3.3. *There exists $K > 0$ depending only on ϵ such that for all starting intervals J and $\omega \in \hat{\mathcal{P}}$, the distortion of $\hat{F}_J|\omega$ is bounded by K.*

Proof. If $\hat{p}(\omega) = n$, then by construction, there is an interval $T \supset \omega$ such that f^n maps T monotonically onto an ϵ-scaled neighbourhood of $f^n(\omega)$, i.e. both components of $f^n(T) \setminus f^n(\omega)$ have size $\geq \epsilon |f^n(\omega)|$. The Koebe Principle (see e.g. [12]) gives the result. $\square$

The following lemma contains combinatorial estimates needed in the next section.

Lemma 3.4. *Let $N_{k,s}$ be the number of integer sequences $(p_1, \dots, , p_s)$ such that $p_1 + \dots + p_s = k$ and $p_i \geq 0$ for all i. Let $N^+_{k,s}$ be the same number for sequences with $p_i > 0$ for all i. Then if $s \leq k/2$,*

$$N_{k,s} \leq 2^s N^+_{k,s} < 2^{k+s-1}. \tag{7}$$

Given $\eta > 0$ small and $\alpha \in (0, 1]$, there exists $\hat{\eta} = \hat{\eta}(\eta, \alpha)$ with $\hat{\eta} \to 0$ as $\eta \to 0$ such that

$$N^+_{k,s} \leq \begin{cases} e^{\hat{\eta} k^\alpha \log k} & \text{if } s \leq \eta k^\alpha, \\ e^{\hat{\eta} k} & \text{if } s \leq \eta k. \end{cases} \tag{8}$$

Proof. Say $j \geq 1$ terms in the sum $p_1 + \dots + p_s$ are nonzero. There are $\binom{s}{j}$ ways to distribute them. This gives

$$N_{k,s} = \sum_{j=1}^{s} \binom{s}{j} N^+_{k,j} \leq \max_j N^+_{k,j} \sum_{j=1}^{s} \binom{s}{j} = (2^s - 1) \max_j N^+_{k,j}.$$

Next, supposing that $p_i > 0$ for all i, there are $s - 1$ partial sums $\sum_{i=1}^{j} p_i$ different from k. Therefore $N^+_{k,s} = \binom{k-1}{s-1}$. In particular, $\max_j N^+_{k,j} = N^+_{k,s} \leq 2^{k-1}$ if $s \leq k/2$. This proves (7). Let us estimate $N^+_{k,s}$ more precisely if $s \leq \eta k^\alpha \leq k/2$. By Stirling's formula

$$\binom{k}{s} \leq \frac{k^k}{(s^s)(k-s)^{k-s}} \leq \left(\frac{k}{k-s}\right)^k \cdot \left(\frac{k-s}{s}\right)^s \leq (1 + \frac{2s}{k})^k \cdot \left(\frac{k}{s}\right)^s.$$

Because $(1 + \frac{2s}{k})^k \leq \exp(k \log(1 + \frac{2s}{k})) \leq \exp(2s)$, it follows that

$$N^+_{k,s} < e^{2s} \left(\frac{k^{1-\alpha}}{\eta}\right)^{\eta k^\alpha} \leq e^{\eta(2 - \log \eta + (1-\alpha)\log k) k^\alpha}$$

if $s \leq \eta k^\alpha$. If $s \leq \eta k$, this simplifies to $N^+_{k,s} < e^{\eta(2 - \log \eta)k}$, proving (8). $\square$

3.3 Stopping time estimates

The aim of this section is to estimate the tail behaviour of the return time function $\hat{p}$, i.e. to obtain an upper bound for the Lebesgue measure of the set $\{x : \hat{p}(x) > n\}$ which we shall henceforth denote by $\{\hat{p} > n\}$. We shall always assume the notation of Proposition 3.1, particularly when referring to the polynomial, stretched exponential and exponential cases.

We fix n for the rest of this section. For each $\omega \in \hat{\mathcal{P}}$ we consider the sequence $p_1, \ldots, p_s$ as defined above, with some terms possibly equal to 0. Recall that s is given by the number of returns occurring before time n. Let $\eta > 0$ be a small constant to be determined below. The set of partition elements $\omega \in \hat{\mathcal{P}}_n$ with $\hat{p}|\omega > n$ can be divided into

$$\hat{\mathcal{P}}'_n = \left\{ \omega \in \hat{\mathcal{P}}_n;\ \hat{p}|\omega > n,\ \sum_{i=1}^{s} p_i \leq \eta n \right\}$$

and

$$\hat{\mathcal{P}}''_n = \left\{ \omega \in \hat{\mathcal{P}}_n;\ \hat{p}|\omega > n,\ \sum_{i=1}^{s} p_i > \eta n \right\}.$$

Clearly we have

$$|\{\hat{p} > n\}| = \sum_{\omega \in \hat{\mathcal{P}}'_n} |\omega| + \sum_{\omega \in \hat{\mathcal{P}}''_n} |\omega|.$$

To treat the exponential and stretched exponential case we shall need to subdivide $\hat{\mathcal{P}}''_n$ further into

$$\hat{\mathcal{P}}''_{n-} = \{\omega \in \hat{\mathcal{P}}''_n; s \leq \rho n^{\hat{\alpha}}\} \quad \text{and} \quad \hat{\mathcal{P}}''_{n+} = \{\omega \in \hat{\mathcal{P}}''_n; s > \rho n^{\hat{\alpha}}\},$$

where $\hat{\alpha} \in (0, 1]$ and $\rho > 0$ will be chosen below.

Intuitively elements in $\hat{\mathcal{P}}'_n$ spend most of their time in the "uniformly expanding" region $X \setminus \Delta$. Thus intervals are growing in size at a uniform exponential rate and achieve large scale exponentially fast. Elements of $\hat{\mathcal{P}}''_n$ on the other hand spend much time in binding periods. In this case the upper bound will more closely reflect the expanding properties of the critical orbit. We shall apply various combinations of the estimates obtained in Section 3.2 to obtain bounds on the total measure of the elements of the subpartitions defined above under the required assumptions on the growth of D_n.

Lemma 3.5. *For any $\theta > 0$ there exists $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$ and for all n sufficiently large,*

$$\sum_{\omega \in \hat{\mathcal{P}}'_n} |\omega| \leq e^{-(\lambda-\theta)n}.$$

Proof. As in Lemma 3.4, let $N_{k,s}$ denote the number of possible sequences $(p_1, \ldots, p_s)$ with $p_i \geq 0$ and $p_1 + \cdots + p_s = k$. Then by the definition of $\hat{\mathcal{P}}'$ and the first statement of Lemma 3.2 we have

$$\sum_{\omega \in \hat{\mathcal{P}}'} |\omega| \leq \sum_{s=1}^{n} \sum_{k=0}^{\eta n} \sum_{\substack{(p_1,\ldots,p_s) \\ \sum p_i = k}} |\omega_{p_1,\ldots,p_s}| \leq \sum_{s=1}^{n} \sum_{k=0}^{\eta n} 4^s N_{k,s} C^{-s} e^{-\lambda(1-\eta)n}. \tag{9}$$

Here the factor 4^s expresses the maximal number of components of I_p for each return. We use the bound $N_{k,s} < 2^{k+s}$ from Lemma 3.4. Recall that $k \leq \eta n$. Since $\nu_{i+1} - \nu_i \geq T$ when $p_i = p_{i+1} = 0$ (see the previous lemma) formula (6) gives $s = \#S_d + \#S_s \leq 2\#S_d + \#S_{s,s} + 1 \leq 2\eta n + n/T + 1$. Writing $\eta' = (3\eta + 1/T + 1/n)(3 \log 2 + \log C^{-1})$, we get $4^s N_{k,s} C^{-s} \leq e^{\eta' n}$. Taking $\theta = 2(\eta + \eta'/\lambda)$ and substituting in (9) gives

$$\sum_{\omega \in \hat{\mathcal{P}}'_n} |\omega| \leq n \sum_{k=1}^{\eta n} e^{\eta' n} e^{-\lambda(1-\eta)n} \leq \eta n^2 e^{-(\lambda - \frac{\theta}{2})n} \leq e^{-(\lambda-\theta)n}$$

provided η and η' are sufficiently small and n sufficiently large. $\square$

Lemma 3.6. *Recall that $d_n(c) = \min_{i<n}(\gamma_i/D_i(c))^{1/\ell}|f^i(c) - \mathcal{C}|$. Let*

$$\hat{d}_{n,s}(c) = \max_{i \geq \eta n/2s^2} d_i(c).$$

For any $\eta > 0$ there exists $C_1 > 0$ such that

$$\sum_{\omega \in \hat{\mathcal{P}}''_n} |\omega| \leq C_1 \max_{c \in \mathcal{C}} \sum_{s=1}^{n} 2^{-s} \hat{d}_{n,s}(c).$$

Proof. Given a sequence $(p_1, \ldots, p_s)$, let p_j be the first term such that $p_j \geq \eta n/(2j^2)$. Because $p_1 + \cdots + p_s \geq \eta n$, such j exists. Let $\tilde{\omega}_{p_1,\ldots,p_j}$ be the union of adjacent intervals $\omega_{p_1,\ldots,p_j}$ with common return times $\nu_1, \ldots, \nu_j$ and $p_j \geq p$. Then f^{ν_j} maps $\tilde{\omega}_{p_1,\ldots,p_j}$ diffeomorphically into an interval (x, y) such that $p(x), p(y) \geq p_j$. Assume without loss of generality that $|x-c| \geq |y-c|$. Therefore, for each $i < p_j$,

$$\begin{aligned}
\gamma_i |f^i(c) - \mathcal{C}| &\geq |f^i(x) - f^i(c)| \\
&\geq \Gamma^{-1} D_{i-1}(c) |f(x) - f(c)| \\
&\geq \mathcal{O}(1/\Gamma) D_{i-1}(c) |x-c|^\ell \\
&\geq \mathcal{O}(1/\Gamma) \frac{D_i(c)|x-c|^\ell}{|f^i(c) - \mathcal{C}|^{\ell-1}}.
\end{aligned}$$

This gives

$$\begin{aligned}
|x-y| \leq 2|x-c| &\leq \mathcal{O}(2\Gamma^{1/\ell})\, d_p(c) \\
&\leq \mathcal{O}(2\Gamma^{1/\ell}) \max_{p \geq \eta n/2j^2} d_p(c) = \mathcal{O}(2\Gamma^{1/\ell})\, \hat{d}_{n,j}(c).
\end{aligned}$$

Let S_s and S_d be the indices $\leq j$ corresponding to large respectively deep returns. Also let $S'_d = S_d \setminus \{j\}$ and let $S_{s,s}$ be the indices $\leq j$ of shallow returns that are followed by another shallow return. Now Lemma 3.2 applied to $\tilde{\omega}_{p_1,\dots,p_j}$ and the iterate ν_j gives

$$\begin{aligned}
\sum_{\omega \in \hat{\mathcal{P}}''_n} |\omega| &\leq \sum_{j=1}^{n} \sum_{(p_1,\dots,p_j)} |\tilde{\omega}_{p_1,\dots,p_j}| \\
&\leq \sum_{j=1}^{n} \mathcal{O}(2\Gamma^{1/\ell}) \max_{c \in \mathcal{C}} \hat{d}_{n,j}(c) \sum_{(p_1,\dots,p_{j-1})} 4^j \left(\frac{K_0}{\kappa}\right)^{\#S_d} \rho^{\#S_{s,s}} \prod_{i \in S'_d} \frac{1}{F'_{p_i}}.
\end{aligned}$$

The factor 4^j expresses the different components of the level sets I_p that intersect forward iterates of ω, and the factor $(K_0/\kappa)^{\#S_d} \rho^{\#S_{s,s}}$ comes from Lemma 3.2. Using (6) and the fact that $\#S'_d = \#S_d - 1$ we can write $4^s = 2^{-s} 8^s$ and

$$8^s = 8^{\#S_s + \#S_d} \leq 8^{\#S_{s,s} + 2\#S_d + 1} = 8^{\#S_{s,s}} 64^{\#S_d} 8 = 512\ 8^{\#S_{s,s}} 64^{\#S'_d}.$$

Then

$$\sum_{(p_1,\dots,p_{j-1})} 4^j \left(\frac{K_0}{\kappa}\right)^{\#S_d} \rho^{\#S_{s,s}} \prod_{i \in S'_d} \frac{1}{F'_{p_i}} \leq 2^{-j} \frac{512 K_0}{\kappa} \sum_{(p_1,\dots,p_{j-1})} (8\rho)^{\#S_{s,s}} \prod_{i \in S'_d} \frac{64 K_0}{\kappa F'_{p_i}}.$$

Take ϵ in Lemma 3.2 so small that $\rho \leq \frac{1}{8}$ and recall that $p_i \geq p_\delta$ for all $i \in S_d$. Therefore Lemma 2.3 and formula (2) (with $\zeta = 64K_0/\kappa$) give that

$$\sum_{(p_1,\ldots,p_{j-1})} (8\rho)^{\# S_{s,s}} \prod_{i \in S_d'} \frac{64K_0}{\kappa F'_{p_i}} \leq 1.$$

By (3), the lemma follows with $C_1 = \mathcal{O}(2\Gamma^{1/\ell})512K_0/\kappa$. ☐

The previous lemma is not so useful in the exponential and stretched exponential cases for relatively small values of s. Indeed, consider for example the situation that $d_p = e^{-\beta p}$. Then the term in the sum in Lemma 3.6 corresponding to $s = \log n$ gives

$$C_1 2^{-\log n} \cdot e^{-\eta\beta n/2\log^2 n}.$$

Clearly this decreases merely subexponentially in n. Let us improve on this.

Lemma 3.7. *Assume that there exists $C, \beta > 0$ and $\alpha \in (0,1]$ such that $D_n^{-1/\ell} \leq Ce^{-\beta n^\alpha}$ for all n. Then for each $\hat{\alpha} \in (0,\alpha)$ (or $\hat{\alpha} = 1$ if $\alpha = 1$) there exists $\rho, C', \beta' > 0$ such that*

$$\sum_{\omega \in \hat{\mathcal{P}}''_{n-}} |\omega| \leq C' e^{-\beta' n^\alpha}$$

for all n. Note that the set $\mathcal{P}''_{n-}$ depends on ρ and $\hat{\alpha}$.

Proof. First notice that since $\alpha \in (0,1]$ one has $p_1^\alpha + p_2^\alpha \geq (p_1+p_2)^\alpha$. Using Lemma 3.2 this gives that there exists $\beta'' > 0$ and C such that

$$|\omega_{p_1,\ldots,p_s}| \ \leq \ C^{-s} e^{-\beta''(\sum p_i)^\alpha}.$$

Reasoning as in the proof of Lemma 3.5, we write $k = p_1 + \cdots + p_s$ and we obtain

$$\sum_{\omega \in \hat{\mathcal{P}}''_{n-}} |\omega| \leq \sum_{s=1}^{\rho n^\alpha} \sum_{k=\eta n}^{\infty} 4^s N_{k,s} C^{-s} e^{-\beta'' k^\alpha}. \tag{10}$$

Lemma 3.4 assures that for some $\hat{\rho} = \hat{\rho}(\rho,\alpha)$ with $\hat{\rho} \to 0$ as $\rho \to 0$ holds

$$N_{k,s} \leq \begin{cases} 2^s e^{\hat{\rho} k^{\hat{\alpha}} \log k} & \text{if } s \leq \rho k^{\hat{\alpha}}, \\ 2^s e^{\hat{\rho} k} & \text{if } s \leq \rho k. \end{cases}$$

Because $\hat{\alpha} < \alpha$ and taking ρ and therefore $\hat{\rho}$ sufficiently small, we get in either case

$$\sum_{k=\eta n}^{\infty} 4^s N_{k,s} C^{-s} e^{-\beta'' k^\alpha} \le \mathcal{O}(\eta n) 8^s C^{-s} e^{-\beta''(\eta n)^\alpha/2}.$$

Using again that $\hat{\alpha} < \alpha$, inequality (10) gives

$$\begin{aligned}\sum_{\omega \in \hat{\mathcal{P}}''_{n-}} |\omega| &\le \sum_{s=1}^{\rho n^{\hat{\alpha}}} \sum_{k=\eta n}^{\infty} 4^s N_{k,s} C^{-s} e^{-\beta'' k^\alpha} \\ &\le \mathcal{O}(\eta n)(8e^2 C^{-1})^{\rho n^{\hat{\alpha}}} e^{-\beta''(\eta n)^\alpha/2} \le e^{-\beta''(\eta n)^\alpha/4}.\end{aligned}$$

This proves the lemma with $\beta' = \eta^\alpha \beta''/4$. $\square$

Proof of Proposition 3.1. We show first of all that $\hat{J}$ has full measure in J, i.e. $|\{\hat{p} > n\}| \to 0$ as $n \to \infty$. By (for example) Lemma 2.2, it follows that almost all $x \in J$, $f^n(x)$ accumulates to c. Hence x has infinitely many deep return times, and it is contained in sets of the form $\omega_{p_1,\dots,p_s}$ for itineraries of arbitrary length s. Because $\hat{p}(\omega_{p_1,\dots,p_s}) \ge s \to \infty$ as $s \to \infty$, the proofs of Lemmas 3.5 and 3.6 show that $\sum_{(p_1,\dots,p_s)} |\omega_{p_1,\dots,p_s}| \to 0$ as $s \to \infty$. Therefore $|J \setminus \hat{J}| = 0$.

To prove the remaining estimates in the four cases mentioned in the proposition, notice that we have exponential bounds for $\hat{\mathcal{P}}'_n$ and therefore we only need to concentrate here on $\hat{\mathcal{P}}''_n$. The sequence $\{\hat{d}_{n,s}(c)\}$ is decreasing in n, so for each k there are at most $\#\{n; k-1 \le \eta n/(2s^2) \le k\} \le 2s^2/\eta$ numbers n such that $k = [\eta n/(2s^2)]$. Therefore

$$\begin{aligned}\sum_{n\ge1}\sum_{s=1}^{n} 2^{-s}\hat{d}_{n,s}(c) &\le \sum_{s\ge1} \frac{2s^2}{\eta} 2^{-s} \sum_{n\ge1} [\gamma_n/D_n(c)]^{1/\ell}\ |f^n(c) - \mathcal{C}| \\ &\le \frac{12}{\eta} \sum_{n\ge1} [\gamma_n/D_n(c)]^{1/\ell} \\ &\le \frac{12}{\eta} \sum_{n\ge1} [\gamma_n^{\ell-1} D_n(c)]^{-1/\ell}.\end{aligned}$$

Hence the summable case follows from $(*)$. Lemma 3.6 gives for the polynomial case

$$\sum_{\omega \in \mathcal{P}''_n} |\omega| \le C_1 \max_{c\in\mathcal{C}} \sum_{s=1}^{n} 2^{-s} \hat{d}_{n,s}(c) \le C_1 \sum_{s=1}^{n} 2^{-s} \left(\frac{2s^2}{\eta n}\right)^\alpha \le 12 C_1 \eta^{-\alpha} n^{-\alpha}$$

as required. In the exponential and stretched exponential cases we use Lemma 3.6 applied to $\hat{\mathcal{P}}''_{n+}$ (so $s > \rho n^{\alpha}$) to get

$$\sum_{\omega \in \mathcal{P}''_{n+}} |\omega| \leq C_1 \max_{c \in \mathcal{C}} \sum_{s \geq \eta n^{\hat{\alpha}}} 2^{-s} \hat{d}_{n,s}(c) \leq C_2 e^{-(\log 2) n^{\hat{\alpha}}}$$

for some $C_2 > 0$. Lemma 3.7 takes care of the remaining collection $\mathcal{P}''_{n-}$. $\square$

4 The full return map

Proposition 4.1. *Suppose that f satisfies condition (BBC) and $(*)$. Then for any $c \in \mathcal{C} \cap X$ there exists a neighbourhood Ω_0 of c, a countable partition $\mathcal{Q}$ of Ω_0 (mod 0) and a return time function $R : \mathcal{Q} \to \mathbb{N}$ with the following properties. For each $\omega \in \mathcal{Q}$, f^R maps ω to Ω_0 diffeomorphically with bounded distortion: letting*

$$s(x,y) = \min\{n; \hat{f}^n(x), \hat{f}^n(y) \textit{ belong to different elements of } \mathcal{Q}\}, \tag{11}$$

there exists $\beta \in (0,1)$ and $C > 0$ such that for all $\omega \in \mathcal{Q}$ and all $x, y \in \omega$,

$$\left| \frac{\hat{f}'(x)}{\hat{f}'(y)} - 1 \right| \leq C \beta^{s(x,y)}. \tag{12}$$

Moreover the tail $|\{R > n\}|$ of the return times satisfy the following estimates:

Summable case:

$$\sum_n |\{R > n\}| < \infty.$$

Polynomial case: *If $d_n(c) \leq C n^{-\alpha}$ for all $c \in \mathcal{C}$ and $n \geq 1$, then there exists $\tilde{C} > 0$ such that*

$$|\{R > n\}| \leq \tilde{C} n^{-\alpha}.$$

Stretched exponential case: *If $d_n(c) \leq C e^{-\beta n^{\alpha}}$, $\alpha \in (0,1)$, $\beta > 0$ for all $c \in \mathcal{C}$ and $n \geq 1$, then for each $\tilde{\alpha} \in (0, \alpha)$ there exist $\tilde{\beta}, \tilde{C} > 0$ such that*

$$|\{R > n\}| \leq \tilde{C} e^{-\tilde{\beta} n^{\tilde{\alpha}}}.$$

Exponential case: *If* $d_n(c) \le Ce^{-\beta n}$, $\beta > 0$ *for all* $c \in \mathcal{C}$ *and* $n \ge 1$, *then there exist* $\tilde{\beta}, \tilde{C} > 0$ *such that*

$$|\{R > n\}| \le \tilde{C} e^{-\tilde{\beta} n}.$$

In Section 4.1 we explain how to choose Ω_0 and how to define the partition $\mathcal{Q}$ and the return time function R. Notice that R is not a first return time. In Section 4.2 we prove the distortion bound and in Section 4.3 we prove the estimates on the return times.

4.1 Large scales and full returns

Let $\Omega_0 \subset \Delta$ be a small neighbourhood of a point $c \in \mathcal{C}$ (the precise requirements on its size will be given in the proof of Lemma 4.2 below). Let $J \subset X$ be an arbitrary interval. Consider the map $F = f^{\hat{p}} : J \to X$ and the associated partition $\hat{\mathcal{P}}$ on J with the stopping time function $\hat{p}$ as defined in Section 3.

Lemma 4.2. *There exist* $t_0 \in \mathbb{N}$ *and* $\xi > 0$ *independent of* J *such that for every* $\omega \in \hat{\mathcal{P}}$ *there exists* $\tilde{\omega} \subset \omega$ *satisfying the following properties:*

- $f^{\hat{p}(\omega)+t}$ *maps* $\tilde{\omega}$ *diffeomorphically onto* Ω_0 *for some* $t \le t_0$*;*
- $|\tilde{\omega}| \ge \xi |\omega|$.
- *both components of* $f^{\hat{p}(x)}(\omega \setminus \tilde{\omega})$ *have length* $\ge \delta'/3$.

Proof. By definition of X, the preimages of c are dense in X. Therefore there exists $t_0 \ge 1$ such that every interval of length $\ge \delta'$ contains a point $x \in \cup_{t \le t_0} f^{-t}(c)$ in its middle fifth. Say $f^t(x) = c$. Now choose sufficiently small neighbourhoods ω_x of each such x not containing any points of $f^{-j}(\mathcal{C})$ for any $j < t$. Clearly f^t maps ω_x diffeomorphically to some critical neighbourhood. By adjusting the size of ω_x we can make sure that they all (i.e. for all points x) map onto exactly the same critical neighbourhood Ω_0 and that $|\omega_x| \le \delta'/15$. This proves the first and third statement.

Now apply the Koebe principle twice, first to obtain a distortion bound for $f^{\hat{p}(\omega)}|\omega$, and second for the remaining t iterates. Indeed, in choosing ω_x we can assume that there are larger intervals $H_x \supset \omega_x$ such that f^t maps H_x diffeomorphically onto a $\frac{1}{3}$-scaled neighbourhood of Ω_0. The second statement follows immediately. $\square$

We now define $\hat{f} : \Omega_0 \to \Omega_0$, the associated partition $\mathcal{Q}$ and the stopping time function R constant on elements of $\mathcal{Q}$ such that $\hat{f} = f^{R(\omega)}$ on $\omega \in \mathcal{Q}$. For each ω in the partition $\hat{\mathcal{P}}$ of Ω_0, let $\tilde{\omega}$ denote the subinterval given in Lemma 4.2, so $|f^{\hat{p}(\omega)}(\tilde{\omega})| \geq \delta'$. We put $\tilde{\omega} \in \mathcal{Q}$ by definition and $R(\tilde{\omega}) = \hat{p}(\omega)+t$. Both components of $f^{\hat{p}(\omega)}(\omega)\setminus\omega_x$ have size at least $\delta'/3$. Considering them as new starting intervals we carry out the construction of Section 3 and repeat the procedure described above. This determines all the necessary objects. In this way each $\omega \in \mathcal{Q}$ also has an associated sequence of *large scale times* before a *full return*. We write $\hat{p}_1 = \hat{p}(x)$ and $\hat{p}_{i+1}(x) = \hat{p}_i(x) + \hat{p}(f^{\hat{p}_i(x)}(x))$ so that $\hat{p}_{i+1}(x)$ denotes the total number of iterates making up the first $i+1$ large scale stopping times associated to the point x. We have $R(\omega) = \hat{p}_s(\omega) + t$ for some $s \geq 1, t \leq t_0$.

We prove two easy but important consequences of the construction.

Lemma 4.3. *For each $n \geq 0$ and each interval ω on which $\hat{f}^n$ is continuous, the distortion of $\hat{f}^n|\omega$ is uniformly bounded (independently of ω).*

Proof. The statement follows directly from the construction and Lemma 3.3. The additional t iterates do not significantly affect the distortion, see also the remarks in the proof of Lemma 4.2. $\square$

Lemma 4.4. *For every i,*

$$|\{x; \hat{p}_{i+1}(x) \text{ exists and } \hat{p}_{i+1} > \hat{p}_i + k|\hat{p}_i\}| \leq \frac{3K}{\delta'}|\{\hat{p} > k\}|.$$

Here the expression on the left denotes the conditional probability of $\hat{p}_{i+1} > \hat{p}_i + k$ on the set of intervals on which $\hat{p}_i$ is defined.

Proof. The statement follows immediately from Lemmas 4.2 and 4.3. Indeed, let ω be a maximal interval on which $\hat{p}_i$ is defined and constant, say $f^{\hat{p}_i(\omega)}(\omega) = J \supset \omega_x$, where ω_x is as in Lemma 4.2. Let $\omega' \subset \omega$ be such that $J' = f^{\hat{p}_i(\omega)}(\omega')$ is a component of $J \setminus \omega_x$. By construction $|J'| \geq \delta'/3$. As the transformation $f^{\hat{p}_i(\omega)}|\omega$ has distortion bounded by $K = K(\epsilon)$, we get

$$|\{x \in \omega'; \hat{p}_{i+1}(x) > \hat{p}_i(x) + k\}| \leq K\frac{|\omega'|}{|J'|}|\{y \in J'; \hat{p}_{J'}(y) > k\}|$$

Because $|\{\hat{p}_{J'} > k\}| \leq |\{\hat{p} > k\}|$ the result follows by summing over all the intervals ω'. $\square$

4.2 Bounded distortion

The function s from (11) is called the *separation time function*. Notice that $s(x,y)$ is finite for all $x \neq y$, because otherwise $f^n|(x,y)$ would be homeomorphic for all n. The assumptions on D_n imply that $|(f^n)'(x)|$ does not converge to 0 for any $x \in X \setminus \cup_n f^{-n}(\mathcal{C})$, so this cannot happen. By the same token one can show that some iterate of $\hat{f}$ is uniformly expanding, i.e. there exists N such that $|(\hat{f}^N)'(x)| \geq 2$ wherever it is defined.

Lemma 4.5. *There exists $\beta \in (0,1)$ and $C > 0$ such that for all $\omega \in \mathcal{Q}$ and all $x, y \in \omega$,*

$$\left| \frac{\hat{f}'(x)}{\hat{f}'(y)} - 1 \right| \leq C\beta^{s(x,y)}. \tag{13}$$

Proof. For small values of $s(x,y)$, (13) follows immediately from Lemma 4.3 Otherwise, uniform expansion of $\hat{f}^N$ and Lemma 4.3 imply that $|\hat{f}(x) - \hat{f}(y)| \leq |\Omega_0| K 2^{-s(x,y)/N}$. Because the Koebe space of $\hat{f}|\omega$ is at least ϵ, we get

$$\begin{aligned} \left| \frac{\hat{f}'(x)}{\hat{f}'(y)} - 1 \right| &\leq \left| \left(\frac{\epsilon/|\hat{f}(x) - \hat{f}(y)| + 1}{\epsilon/|\hat{f}(x) - \hat{f}(y)|} \right)^2 - 1 \right| \\ &\leq \left| \left(\frac{\epsilon + |\Omega_0| K 2^{-s(x,y)/N}}{\epsilon} \right)^2 - 1 \right| \leq C 2^{-s(x,y)/N}, \end{aligned}$$

where $C = \frac{2K|\Omega_0|}{\epsilon} + \frac{|\Omega_0|^2 K^2}{\epsilon^2}$, see [12, Chapter IV] for the precise Koebe distortion constant that we used here. $\square$

4.3 Return time estimates

We fix $n \geq 1$ and consider the tail $\{R > n\}$ of the return times for $\hat{f}$ on Ω_0. Let us agree to use the notation $|\{\hat{p} > n\}|$ to indicate the *estimates* obtained in Section 3 for an arbitrary interval J rather than the actual value. As mentioned above such estimates are independent of the specific starting interval, although clearly the actual values do depend on the starting interval.

Before starting the proof we introduce some notation. Recall that by construction each $\omega \in \mathcal{Q}$ has an associated sequence

$$0 = \hat{p}_0 < \hat{p}_1 < \hat{p}_2 < \cdots < \hat{p}_{s(\omega)} < R(\omega)$$

with $R(\omega) = \hat{p}_{s(\omega)} + t$ and clearly $s \le R$. Write $\mathcal{Q}^{(n)} = \{\omega \in \mathcal{Q}; R(\omega) > n\}$ and let

$$\mathcal{Q}_i^{(n)} = \{\omega \in \mathcal{Q}^{(n)}; \hat{p}_{i-1} < n \le \hat{p}_i\}$$

denote the set of elements of $\mathcal{Q}$ with $R(\omega) > n$ and having at least $i-1$ and at most i large scale times before time n. Moreover, for each i and every sequence $(k_1, \dots, k_i)$ of positive integers with $\sum k_j = n$ we write

$$\mathcal{Q}_i^{(n)}(k_1, \dots, k_i) = \{\omega \in \mathcal{Q}_i^{(n)}; k_j = \hat{p}_j - \hat{p}_{j-1} \text{ for } \ j \le i-1, \ k_i = n - \hat{p}_{i-1}\}.$$

Finally we let

$$|\mathcal{Q}_i^{(n)}| = \sum_{\omega \in \mathcal{Q}_i^{(n)}} |\omega| \quad \text{and} \quad |\mathcal{Q}^{(n)}| = \sum_{i \le n} |\mathcal{Q}_i^{(n)}|.$$

Obviously $|\{R > n\}| = |\mathcal{Q}^{(n)}|$. We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. In the stretched exponential case, take $\tilde{\alpha} < \hat{\alpha} < \alpha$, where $\hat{\alpha}$ is as in Proposition 3.1. Both $\tilde{\alpha}$ and $\hat{\alpha}$ can be arbitrarily close to α. In the exponential case take $\tilde{\alpha} = \alpha = 1$. Let $\eta \in (0,1)$ be a small number to be determined below, depending on α and β but not on n. We write

$$|\{R > n\}| = \sum_{i \le n} |\mathcal{Q}_i| = \sum_{i < \eta n^{\tilde{\alpha}}} |\mathcal{Q}_i| + \sum_{\eta n^{\tilde{\alpha}} \le i \le n} |\mathcal{Q}_i|. \tag{14}$$

Lemma 4.2 says that a fixed proportion ξ of every element in $\mathcal{Q}_{i-1}^{(n)}$ has a full return to Ω_0 before its next large scale time. Therefore

$$|\mathcal{Q}_i^{(n)}| / |\mathcal{Q}_{i-1}^{(n)}| \le 1 - \xi.$$

This implies $|\mathcal{Q}_i^{(n)}| \le (1-\xi)^i$ and therefore the second term in (14) satisfies

$$\sum_{\eta n^{\tilde{\alpha}} \le i \le n} |\mathcal{Q}_i^{(n)}| \le \sum_{\eta n^{\tilde{\alpha}} \le i \le n} (1-\xi)^i \le \frac{1}{\xi} (1-\xi)^{\eta n^{\tilde{\alpha}}}. \tag{15}$$

For the first term write

$$\sum_{i < \eta n^{\tilde{\alpha}}} |\mathcal{Q}_i^{(n)}| = \sum_{i < \eta n^{\tilde{\alpha}}} \sum_{\substack{(k_1, \dots, k_i) \\ \sum k_j = n}} |\mathcal{Q}_i^{(n)}(k_1, \dots, k_i)|.$$

For a given sequence $(k_1, \ldots, k_i)$, Lemma 4.4 and Proposition 3.1 imply

$$\begin{aligned}
|\mathcal{Q}_i^{(n)}(k_1, \ldots, k_i)| &\le |\{\hat{p}_i > \hat{p}_{i-1} + k_{i-1} - 1 | \hat{p}_{i-1}\}| \cdots |\{\hat{p}_1 > k_{i-1} - 1\}| \\
&\le \tilde{K}^i \prod_{j=1}^{i} |\{\hat{p} > k_j - 1\}| \\
&\le \tilde{K}^i \prod_{j=1}^{i} e^{-\hat{\beta}(k_j - 1)^{\hat{\alpha}}} \le (\tilde{K} e^{\hat{\beta}})^i e^{-\hat{\beta} n^{\hat{\alpha}}}
\end{aligned}$$

Here $\tilde{K} = 3K/\delta'$ is the constant in the statement of Lemma 4.4. From Lemma 3.4 we have that the number sequences $(k_1, \ldots, k_i)$ as above equals $N^+_{n,i}$ and satisfies

$$N^+_{n,i} \le \begin{cases} e^{\hat{\eta} n^{\tilde{\alpha}} \log n} & \text{if } i \le \eta n^{\tilde{\alpha}},\ \tilde{\alpha} < 1, \\ e^{\hat{\eta} n} & \text{if } i \le \eta n, \end{cases}$$

for some $\hat{\eta} = \hat{\eta}(\eta, \tilde{\alpha})$ tending to 0 as $\eta \to 0$. In the stretched exponential case

$$\begin{aligned}
\sum_{i < \eta n^{\tilde{\alpha}}} |\mathcal{Q}_i| &= \sum_{i < \eta n^{\tilde{\alpha}}} \sum_{\substack{(k_1, \ldots, k_i) \\ \sum k_j = n}} |\mathcal{Q}_i^{(n)}(k_1, \ldots, k_i)| \\
&\le \sum_{i < \eta n^{\tilde{\alpha}}} e^{\hat{\eta} n^{\tilde{\alpha}} \log n} (K e^{\hat{\beta}})^i e^{-\hat{\beta} n^{\hat{\alpha}}} \le \hat{C} e^{-\beta' n^{\hat{\alpha}}}
\end{aligned}$$

for some $\hat{C}, \beta' > 0$ as long as $\hat{\eta}$ is sufficiently small. In precisely the same way we get $\sum_{i < \eta n} |\mathcal{Q}_i| \le \hat{C} e^{-\beta' n}$ in the exponential case.

To treat the summable and polynomial case we write $|\{R > n\}|$ as in (14), with $\hat{\alpha} = 1$ and $\eta = 1/2$. The same argument gives an exponential estimate as in (15) for the second term. To estimate the first term, notice that for each i and each sequence $k_1, \ldots, k_i$ with $\sum k_j = n$, the largest k_j satisfies $k_j \ge n/i$. Thus letting

$$\mathcal{Q}_{i,j}^{(n)} = \{\omega \in \mathcal{Q}_i^{(n)}; k_{j'} < n/i \text{ for } j' < j \text{ and } k_j \ge n/i\}$$

we have

$$\sum_{i < n/2} |\mathcal{Q}_i^{(n)}| = \sum_{i < n/2} \sum_{j=1}^{i} |\mathcal{Q}_{i,j}^{(n)}| \le \sum_{i < n/2} i(1 - \xi)^{i-1} |\{\hat{p} > n/i\}|$$

Because $|\{\hat{p} > k\}|$ is decreasing in k, the above is bounded by $\sum_{i<n/2} i^2(1-\xi)^i|\{\hat{p} > n\}| = \mathcal{O}(|\{\hat{p} > n\}|)$ and therefore (use Proposition 3.1) summable over n.

In the polynomial case we get

$$\sum_{i<n/2} |\mathcal{Q}_i^{(n)}| \leq \mathcal{O}(n^{-\alpha}) \sum_{i<n/2} i^{1+\alpha}(1-\xi)^i = \mathcal{O}(n^{-\alpha}).$$

Together with the exponential estimate for the term $\sum_{i\geq n/2} |\mathcal{Q}_i^{(n)}|$, this yields the proposition. □

4.4 Proof of Theorems 1, 2 and 3

We now state the assumptions and results of Young which we want to apply. Together with the estimates obtained in Proposition 4.1, they easily imply Theorems 1, 2 and 3. Let m denote Lebesgue measure on X. L.-S. Young applies the following tower construction for her results. Given a countably piecewise monotone and onto map $\hat{f} : \cup_{\omega\in\mathcal{Q}}\omega \to \Omega_0$, $\hat{f}|\omega = f^{R(\omega)}$, define a tower

$$\Omega = \bigsqcup_{\substack{\omega\in\mathcal{Q} \\ 0\leq i<R(\omega)}} (\omega, i),$$

with an action

$$g(x,i) = \begin{cases} (x, i+1) & \text{if } x \in \omega,\ i+1 < R(\omega), \\ (\hat{f}(x), 0) & \text{if } x \in \omega,\ i+1 = R(\omega). \end{cases}$$

The connection with the original map f is established by means of the projection $\pi(x,i) = f^i(x)$. Because f^i is smooth and has bounded distortion on each $\omega \in \mathcal{Q}$, $i < R(\omega)$, this projection has bounded distortion. Also $\pi \circ g = f \circ \pi$. Therefore, if ν is a g-invariant absolutely continuous probability measure on Ω, $\mu := \nu \circ \pi^{-1}$ is an invariant absolutely continuous probability measure on the interval.

We summarize Young's results from [19] as far as we need them. For a fixed $\beta \in (0,1)$ as in Lemma 4.5, let

$$\mathcal{C}_\beta = \{\varphi : \Omega \to \mathbb{R};\ \exists\, C > 0\, \forall\, x, y\ |\varphi(x) - \varphi(y)| \leq C\beta^{s(x,y)}\}$$

and

$$\mathcal{C}_\beta^+ = \{\varphi \in \mathcal{C}_\beta; \varphi \geq 0\}.$$

Here we have extended the separation time s to Ω in the obvious way. Also let m_Ω be Lebesgue measure on Ω. (A priori, m_Ω can be infinite.)

Theorem (Young [19]). *Suppose that $\hat{f} : \Omega_0 \to \Omega_0$ is as above, i.e. $m(\Omega_0 \setminus \cup_{\omega \in \mathcal{Q}} \omega) = 0$ and (13) holds. Let $\{\rho_n\}$ be a sequence of positive reals related to the tail behaviour of R as follows. If $m(\{R > n\}) \leq n^{-\alpha}$, then $\rho_n = n^{1-\alpha}$, if $m(\{R > n\}) \leq e^{-\beta n}$, then $\rho_n = e^{-\beta' n}$ for some (any) $\beta' < \beta$ and if $m(\{R > n\}) \leq e^{-n^\alpha}$ for some $\alpha \in (0,1)$, then $\rho_n = e^{-n^{\alpha'}}$ for some (any) $\alpha' < \alpha$. Then*

1. *If $\sum_n m(\{R > n\}) < \infty$, then Ω carries an g-invariant absolutely continuous probability measure ν (Kac's Theorem) and $\frac{d\nu}{dm_\Omega} \in \mathcal{C}^+_\beta$.*

2. *For any measure $\tilde{\nu}$ with $\frac{d\tilde{\nu}}{dm_\Omega} \in \mathcal{C}^+_\beta$, $g^n_* \tilde{\nu} \to \nu$ and there exists $C_{\tilde{\nu}} > 0$ such that $|g^n_* \tilde{\nu} - \nu| \leq C_{\tilde{\nu}} \rho_n$.*

3. *For any pair of functions $\varphi \in L^\infty(\Omega, m_\Omega)$ and $\psi \in \mathcal{C}_\beta$, there exists $C_{\varphi,\psi} > 0$ such that*
$$| \int (\varphi \circ g^n) \psi d\nu - \int \varphi d\nu \int \psi d\nu | \leq C_{\varphi,\psi} \rho_n.$$

4. *If $m(\{R > n\}) \leq \mathcal{O}(n^{-\alpha})$ for some $\alpha > 2$, then for any $\varphi \in \mathcal{C}_\beta$ which is not a coboundary ($\varphi \neq \psi \circ g - \psi$ for any ψ), the Central Limit Theorem holds, i.e. there exists $\sigma > 0$ such that $\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ g^i$ converges to the normal distribution $\mathcal{N}(\int \varphi d\nu, \sigma)$.*

Remark: Young states this theorem in terms of a stopping time $\hat{R}$ which is the extension of R to the entire tower Ω. As it happens $m_\Omega(\{\hat{R} > n\}) = \sum_{k \geq n} m(\{R > k\})$, so that $m_\Omega(\{\hat{R} > n\}) \leq \mathcal{O}(n^{-\alpha})$ if $m(\{R > k\}) \leq \mathcal{O}(n^{-\alpha-1})$. This explains why the exponent in the polynomial case at first glance looks different from the ones in Young's version.

Using the projection π, these results immediately carry over to the original map f with measure $\mu = \nu \circ \pi^{-1}$. Using the projection π, immediately carry over to the original map f with measure $\mu = \nu \circ \pi^{-1}$. With respect to the support of the measure, note that $\hat{f} : \Omega_0 \to \Omega_0$ is a mixing map, and its invariant measure $\frac{1}{\nu(\Omega_0)} \nu | \Omega_0$ has the whole interval Ω_0 as support. The formula $\mu = \nu \circ \pi^{-1}$ shows that $\Omega_0 \subset \text{supp}(\mu)$.

Finally, recall from Proposition 4.1 how the tail $m(\{R > n\})$ is related to $d_n(c)$. Therefore Theorem 4.4 immediately gives Theorems 1, 2 and 3.

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