**Ergodicity of the 2D Navier-Stokes Equations with Random Forcing**[1]

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*Dedicated to Joel L. Lebowitz for his $70^{th}$ birthday.*

**Abstract**

We consider the Navier-Stokes equation on a two dimensional torus with a random force, acting at discrete times and analytic in space, for arbitrarily small viscosity coefficient. We prove the existence and uniqueness of the invariant measure for this system as well as exponential mixing in time.

# 1 Introduction

A convenient mathematical model for the study of homogenous isotropic turbulence is to consider the Navier-Stokes equation subject to a random stationary (in space and time) forcing. The turbulent situation is modelled by a smooth force, i.e. one whose Fourier transform decays fast for large wave numbers. One is then interested in various properties of the correlation functions of the velocity field in a stationary state of the ensuing stochastic process. An obvious first question concerns the large time convergence to such a stationary state starting from an arbitrary initial condition of the velocity field, i.e. the uniqueness of the stationary state. In this paper we prove the existence, uniqueness and exponential mixing of the stationary state in the case of two dimensional turbulence.

We consider the Navier-Stokes equation for an incompressible velocity field $\mathbf{u}(t, \mathbf{x})$ defined on the torus $\mathbf{T} = (\mathbf{R}/2\pi\mathbf{Z})^2$:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} = \mathbf{f} - \nabla p \tag{1}$$

supplemented with the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

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The external force $\mathbf{f}(t,\mathbf{x})$ consists of random kicks at discrete times

$$\mathbf{f}(t,\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} e^{-2\pi i\mathbf{x}\cdot\mathbf{k}}\mathbf{f}_{\mathbf{k}}(t) \tag{3}$$

with

$$\mathbf{f}_{\mathbf{k}}(t) = \sum_{n\in\mathbf{Z}} \delta(t-n)\mathbf{f}_{\mathbf{k},n}. \tag{4}$$

The random variables $\mathbf{f}_{\mathbf{k},n}$ will be taken Gaussian, with mean zero, $\bar{\mathbf{f}_{\mathbf{k}}} = \mathbf{f}_{-\mathbf{k}}$ and covariance

$$Ef^{\alpha}_{\mathbf{k}}(m)f^{\beta}_{\mathbf{l}}(n) = \delta_{\mathbf{k},-\mathbf{l}}\delta_{m,n}\delta^{\alpha\beta}\,\phi_{\mathbf{k}}.$$

Furthermore, we will assume $\phi_0 = 0$, which implies the vanishing of the average force over the torus: $\int_{\mathbf{T}}\mathbf{f}(t,\mathbf{x}) = 0$. Assuming also zero average initial velocity $\int_{\mathbf{T}}\mathbf{u}(0,\mathbf{x}) = 0$ we conclude that $\int_{\mathbf{T}}\mathbf{u}(t,\mathbf{x}) = 0$ for all times $t$.

It is convenient to solve the incompressibility condition (2) by expressing the Navier-Stokes equation (1) in terms of the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ which satisfies the transport equation

$$\partial_t\omega + (\mathbf{u}\cdot\nabla)\omega - \nu\Delta\omega = g, \tag{5}$$

where $g = \partial_1 f_2 - \partial_2 f_1$.

Going to the Fourier transform $\omega_{\mathbf{k}}(t) = (2\pi)^{-2}\int_{\mathbf{T}} e^{i\mathbf{k}\cdot\mathbf{x}}\omega(t,\mathbf{x})d\mathbf{x}$ with $\mathbf{k}\in\mathbf{Z}^2$, we may solve the velocity in terms of the vorticity as

$$\mathbf{u}_{\mathbf{k}} = i\tfrac{(-k_2,k_1)}{\mathbf{k}^2}\omega_{\mathbf{k}}$$

and write the vorticity equation as

$$\partial_t\omega_{\mathbf{k}} = -\nu\mathbf{k}^2\omega_{\mathbf{k}} + \sum_{\mathbf{l}\in\mathbf{Z}^2\setminus\{0,\mathbf{k}\}} (\mathbf{k}\times\mathbf{l})|\mathbf{l}|^{-2}\omega_{\mathbf{k}-\mathbf{l}}\omega_{\mathbf{l}} + \sum_{n\in\mathbf{Z}}\delta(t-n)g_{\mathbf{k}}(n) \tag{6}$$

where $\mathbf{k}\times\mathbf{l} = k_1l_2 - l_1k_2$ and $g_{\mathbf{k}}(n)$ are Gaussian with mean zero, $\bar{g_{\mathbf{k}}} = g_{-\mathbf{k}}$ and covariance

$$Eg_{\mathbf{k}}(m)g_{\mathbf{l}}(n) = \delta_{\mathbf{k},-\mathbf{l}}\delta_{m,n}\,\gamma_{\mathbf{k}}.$$

with

$$\gamma_{\mathbf{k}} = \mathbf{k}^2\phi_{\mathbf{k}}.$$

We assume

$$b^{-1}e^{-\kappa_\gamma^{-1}|\mathbf{k}|} \le \gamma_{\mathbf{k}} \le be^{-\kappa_\gamma^{-1}|\mathbf{k}|} \tag{7}$$

where $\kappa_\gamma > 0$, and we think of $b$ as being large. We will be interested in the turbulent region $\nu\to 0$; therefore, when it is convenient, *we will always assume below that $\nu$ is small enough*, although our results hold for all $\nu$.

Before stating our result, we need some definitions. First, we define the enstrophy as (a multiple of) the square of the $L^2$ norm

$$\Phi = \tfrac{1}{2}\sum_{\mathbf{k}}|\omega_{\mathbf{k}}|^2 = \tfrac{1}{2}||\omega||^2_{L^2}. \tag{8}$$

Next, we fix a number $r > 1$ and consider the Banach space

$$\Omega = \{\omega \mid \|\omega\| \equiv \sup_{\mathbf{k}}|\omega_{\mathbf{k}}||\mathbf{k}|^r < \infty\}$$

as our probability space, with $\mathcal{B}$ the product $\sigma$-algebra. Note that $\Omega$ is a subspace of $L^2$.

Finally, due to the analyticity (with probability one) of the noise, $\omega(t)$ also will turn out to be analytic with probability one and it will be useful to introduce norms capturing this property. For any positive number $\kappa$, we define a norm (that we shall call the $\kappa$-norm),

$$||\omega||_\kappa = \sup_{\mathbf{k}} |\omega_{\mathbf{k}}||\mathbf{k}|^r e^{\kappa^{-1}|\mathbf{k}|}. \tag{9}$$

Functions with $||\omega||_\kappa < \infty$ are analytic in a $\kappa^{-1}$ neighbourhood of the torus. The factor $|\mathbf{k}|^r$ is useful technically (and was already used in [5]).

The stochastic equation (6) gives rise to a Markov chain $\omega(n)$, $n \in \mathbf{N}$ defined by

$$\omega(n+1) = F(\omega(n)) + g(n+1) \tag{10}$$

where $F$ is the map at time 1 of the Navier-Stokes flow (6) without the forcing. We denote by $P(\omega, E)$ the transition probability of this chain.

Our main result is the

**Theorem.** *The Markov chain* (10) *is defined on* $(\Omega, \mathcal{B})$ *and has a unique invariant measure* $\mu$ *there. It satisfies*

$$\int 1_{(||\omega||_\kappa \geq \nu\kappa)} \mu(d\omega) \leq C \exp(-c\nu^4 \kappa^{\frac{2}{\alpha}}) \tag{11}$$

*for any* $\alpha > 1 + r$, *and* $C, c < \infty$, *depending on* $\alpha$. *Moreover,* $\forall \omega \in \Omega$ *and* $\forall E \in \mathcal{B}$, *we have,*

$$|P^t(\omega, E) - \mu(E)| \leq C(\omega) e^{-mt} \tag{12}$$

*where* $m = m(\nu) > 0$ *for all* $\nu$, *and* $C(\omega) \leq C(\frac{||\omega||+1}{\nu})^C$.

**Remark 1.** Since $||\cdot||_{\kappa'} < ||\cdot||_\kappa$ for $\kappa' > \kappa$, (11) holds for all $\kappa'$-norms with $\kappa' > \kappa$ too, including the norm $||\cdot||$ defining $\Omega$, which corresponds to $\kappa' = \infty$. Estimate (11) means that with high probability $\omega$ is analytic in a $\nu^{2\alpha}$-neighbourhood of the torus and bounded there by $\nu^{1-2\alpha}$. By taking $r$ close to 1, $\alpha$ can be taken close to 2.

**Remark 2.** Here and below, we denote by $C$ or $c$ a "generic" constant that can vary from place to place, even in the same equation.

**Remark 3.** We obtain a lower bound on $m$ in (12) of the form $m \geq \exp(-C\nu^{-3}(\log \nu^{-1})^c)$ (see Proposition 2 and Lemma 4 below), which means, however, that our estimate on the rate of convergence is unphysically small for $\nu$ small.

Let us finish this section by a brief comment on previous work on the uniqueness question. There is a long history of proofs in cases that do not correspond to the turbulence problem. Either the forcing is taken to decay very slowly for large $|\mathbf{k}|$, i.e. with a lower bound of the form $|\mathbf{k}|^{-p}$ (see [3] and references therein), or the viscosity is taken large [6]. The only proof of uniqueness we know of in the turbulent situation is the recent one [4] where one considers a model like ours but with bounded noise (each $g_{\mathbf{k}}$ has compact support).

The proof of the Theorem, given in Section 4, will be based on probabilistic estimates (Section 3) and on properties of the deterministic Navier-Stokes equation, which we discuss now.

## 2 The deterministic Navier-Stokes flow.

In this section we derive some properties of the flow of the deterministic Navier-Stokes equation, i.e. (6) without the forcing term $g_{\mathbf{k}}(t)$. Let us define a family of subsets of $\Omega$ that impose constraints on the size of the $L^2$-norm and of the $\kappa$-norm:

$$U(\kappa, \phi, A) = \{\omega | \Phi \leq \phi, \ ||\omega||_\kappa \leq A\}. \tag{13}$$

Then, we introduce a one-parameter subfamily of $U(\kappa,\phi,A)$:

$$U_\kappa \equiv U(\kappa,\phi(\kappa),A(\kappa)) \tag{14}$$

where, $\phi(\kappa)=\nu^2\varphi\kappa^{\frac{2}{\alpha}}$ and $A(\kappa)=\nu a\kappa$. This family is useful because, as we shall see, the flow maps one $U_\kappa$ in that family into another one with a smaller $\kappa$. The parameter $\alpha$ will be taken to satisfy $\alpha>1+r$ and $\varphi$ and $a$ will be chosen small depending on some "geometric" constants that will appear in the course of the proof. Thus, if $\omega\in U_\kappa$, then for all $\mathbf{k}$ we have

$$|\omega_{\mathbf{k}}| \le \nu a\kappa|\mathbf{k}|^{-r}e^{-\kappa^{-1}|\mathbf{k}|} \tag{15}$$

and

$$\Phi \le \nu^2\varphi\kappa^{\frac{2}{\alpha}} \tag{16}$$

Let now

$$\kappa(t)=\frac{\kappa}{1+\eta\nu t\min(1,\kappa)} \tag{17}$$

where $\eta$ will be chosen suitably small below, and denote also by $\omega(t)$ the solution of (6) without the forcing term $g_{\mathbf{k}}(t)$.

**Proposition 1.** (a) *Let $\omega(0)\in U_\kappa$, then for all $0\le t\le 1$, $\omega(t)\in U_{\kappa(t)}$.*
(b) *Suppose $\omega(0)\in\Omega$ with $\|\omega(0)\|\le D\nu$. Then $\omega(1)\in U_\kappa$ for $\kappa=C(D^\alpha+\frac{1}{\nu})$.*

The point of part (a) of this Proposition is that the domain of analyticity of the solution of the unforced Navier-Stokes equation increases with time and its $L^2$ and $\kappa$-norms decrease with time. Part (b) says that, even if $\omega(0)$ is not analytic, but belongs to $\Omega$, the solution after time 1 is analytic and its $L^2$ and $\kappa$-norms are bounded in terms of the norm of the initial data in $\Omega$. Our proof of Proposition 1 is inspired by [5] (see also [1]).

For the proof we rewrite (6) (without forcing) in integral form

$$\omega_{\mathbf{k}}(t)=e^{-t\nu\mathbf{k}^2}\omega_{\mathbf{k}}(0)+\int_0^t ds\, e^{(s-t)\nu\mathbf{k}^2}\sum_{\mathbf{l}\in\mathbf{Z}^2\setminus\{\mathbf{0},\mathbf{k}\}}(\mathbf{k}\times\mathbf{l})|\mathbf{l}|^{-2}\omega_{\mathbf{k}-\mathbf{l}}(s)\omega_{\mathbf{l}}(s) \tag{18}$$

and solve this in a suitable Banach space.

Let $Y_\kappa$ be the Banach space equiped with the norm $||\cdot||_\kappa$ and

$$X_{\kappa,\tau}=\{\omega\in C^0([0,\tau],Y_\kappa)\mid\ |||\omega|||\equiv\sup_{t\in[0,\tau]}|||\omega(t)||_{\kappa(t)}<\infty\} \tag{19}$$

We have the following existence lemma.

**Lemma 1.** *Let $\omega(0)\in U_\kappa$ then the solution $\omega$ of equation (18) exists in the set $X_{\kappa,\tau}$, for $\tau\le(C\nu^{\frac{1}{2}}\kappa)^{-2}$. Moreover, $|||\omega|||\le 2\nu a\kappa$.*

**Proof.** Let

$$\omega^0_{\mathbf{k}}(t)=e^{-t\nu\mathbf{k}^2}\omega_{\mathbf{k}}(0), \tag{20}$$

and write (18) as a fixed point equation

$$\omega(t)=\omega^0(t)+N(\omega)(t)\equiv\mathcal{F}(\omega)(t) \tag{21}$$

with

$$N_{\mathbf{k}}(\omega)(t)\equiv\int_0^t ds\, e^{(s-t)\nu\mathbf{k}^2}\sum_{\mathbf{l}\in\mathbf{Z}^2\setminus\{\mathbf{0},\mathbf{k}\}}(\mathbf{k}\times\mathbf{l})|\mathbf{l}|^{-2}\omega_{\mathbf{k}-\mathbf{l}}(s)\omega_{\mathbf{l}}(s). \tag{22}$$

We show that the map $\mathcal{F}$ is a contraction in the ball

$$B = \{\omega \in X_{\kappa,\tau} | \ |||\omega - \omega^0||| \leq \nu a \kappa\} \tag{23}$$

Let us first show that $\mathcal{F}$ maps $B$ into itself. Obviously, $|||\omega^0||| \leq \nu a\kappa$ and if $\omega \in B$, then $|||\omega||| \leq 2\nu a\kappa$, which means,

$$|\omega_{\mathbf{k}}(t)| \leq 2\nu a\kappa |\mathbf{k}|^{-r} e^{-\kappa(t)^{-1}|\mathbf{k}|}. \tag{24}$$

We must prove that $|||N(\omega)||| \leq \nu a\kappa$, i.e.

$$|N_{\mathbf{k}}(\omega)(t)| \leq \nu a\kappa |\mathbf{k}|^{-r} e^{-\kappa(t)^{-1}|\mathbf{k}|}, \tag{25}$$

for all $\mathbf{k} \in \mathbf{Z}^2 \backslash \mathbf{0}$ (recall that $N_{\mathbf{k}} = 0$ for $\mathbf{k} = \mathbf{0}$) and for all $t \in [0, \tau]$.

Inserting (24) and $|\mathbf{k} \times \mathbf{l}|\,|\mathbf{l}|^{-2} \leq |\mathbf{k}||\mathbf{l}|^{-1}$ in (22), we get:

$$|N_{\mathbf{k}}(\omega)(t)| \leq (2\nu a\kappa)^2 |\mathbf{k}| \int_0^t ds\, e^{(s-t)\nu\mathbf{k}^2} \sum_{\mathbf{l}\in\mathbf{Z}^2\backslash\{0,\mathbf{k}\}} e^{-\kappa(s)^{-1}|\mathbf{k}-\mathbf{l}|} e^{-\kappa(s)^{-1}|\mathbf{l}|} |\mathbf{k}-\mathbf{l}|^{-r} |\mathbf{l}|^{-r-1} \tag{26}$$

Writing (17) as $\kappa(t)^{-1} = \kappa^{-1} + \eta\nu t \min(1,\kappa)\kappa^{-1}$, we obtain, since $\mathbf{k} \neq \mathbf{0}$ means $|\mathbf{k}| \geq 1$, that

$$\tfrac{1}{2}(s-t)\nu\mathbf{k}^2 \leq \eta(s-t)\nu|\mathbf{k}| \leq (\kappa(s)^{-1} - \kappa(t)^{-1})|\mathbf{k}| \tag{27}$$

holds for $0 \leq s \leq t \leq 1$ and $\eta \leq \frac{1}{2}$. Since $-|\mathbf{k}-\mathbf{l}| - |\mathbf{l}| \leq -|\mathbf{k}|$ and

$$\sum_{\mathbf{l}\in\mathbf{Z}^2\backslash\{0,\mathbf{k}\}} |\mathbf{k}-\mathbf{l}|^{-r} |\mathbf{l}|^{-r-1} \leq C|\mathbf{k}|^{-r}, \tag{28}$$

(since $r > 1$), we get

$$\begin{aligned} |N_{\mathbf{k}}(\omega)(t)| &\leq C(2\nu a\kappa)^2 |\mathbf{k}| \Big(\int_0^t ds\, e^{\frac{1}{2}(s-t)\nu\mathbf{k}^2}\Big) |\mathbf{k}|^{-r} e^{-\kappa(t)^{-1}|\mathbf{k}|} \\ &= C\nu(2a\kappa)^2 (2|\mathbf{k}|^{-1}(1 - e^{-\frac{1}{2}t\nu\mathbf{k}^2})) |\mathbf{k}|^{-r} e^{-\kappa(t)^{-1}|\mathbf{k}|}. \end{aligned} \tag{29}$$

Since $|\mathbf{k}|^{-1}(1-e^{-\frac{1}{2}t\nu\mathbf{k}^2}) \leq (\nu t)^{\frac{1}{2}}$, (25) follows for $\tau \leq (C\nu^{\frac{1}{2}}\kappa)^{-2}$. The contractive property is proven similarily. Thus we obtain a unique solution of (21) in $B$, which satisfies (24), hence $\|\omega\|_{\kappa(t)} \leq 2\nu a\kappa$, $\forall t \leq (C\nu^{\frac{1}{2}}\kappa)^{-2}$. □

**Proof of Proposition 1.** (a) It suffices to show that the solution constructed in Lemma 1 on the interval $[0,\tau]$ satisfies the two bounds of the Proposition there: the one on the $\kappa(t)$ norm of the solution $\omega$ and the one on its enstrophy. This implies trivially that the solution can be extended to the whole interval $[0,1]$ and satisfies also there the bounds of the Proposition.

The bound on the enstrophy is easy to prove; as is well known, the enstrophy satisfies

$$\tfrac{d}{dt}\Phi(t) = -\sum_{\mathbf{k}\neq 0} \nu\mathbf{k}^2 |\omega_{\mathbf{k}}|^2 \leq -\nu\Phi(t)$$

leading to $\Phi(t) \leq \Phi(0)e^{-\nu t}$. Since $e^{-\nu t} \leq (1 + \eta\nu t\min(1,\kappa))^{-\frac{2}{\alpha}}$ for $\eta$ small, we get

$$\Phi(t) \leq \nu^2 \varphi \kappa(t)^{\frac{2}{\alpha}} \tag{30}$$

i.e. the claim of the Proposition concerning $\Phi(t)$.

To prove the bound $\|\omega\|_{\kappa(t)} \leq \nu a\kappa(t)$, we consider separately the cases $\kappa < 1$ and $\kappa \geq 1$.

If $\kappa < 1$, it is enough to use the bound (29) which, since $|\mathbf{k}| \geq 1$, gives

$$|N_{\mathbf{k}}(\omega)(t)| \leq \nu a\kappa\lambda(1 - e^{-\frac{1}{2}t\nu\mathbf{k}^2}) |\mathbf{k}|^{-r} e^{-\kappa(t)^{-1}|\mathbf{k}|}, \tag{31}$$

where $\lambda$ can be chosen arbitrarily small by decreasing $a$. Now inserting this bound and

$$e^{-\frac{1}{2}t\nu\mathbf{k}^2}|\mathbf{k}|^{-r}e^{-(\kappa)^{-1}|\mathbf{k}|} \leq |\mathbf{k}|^{-r}e^{-\kappa(t)^{-1}|\mathbf{k}|}, \tag{32}$$

which follows from (27) with $s = 0$, in (21), we conclude that

$$|\omega_{\mathbf{k}}(t)| \leq \nu a\kappa(e^{-\frac{1}{2}t\nu\mathbf{k}^2} + \lambda(1 - e^{-\frac{1}{2}t\nu\mathbf{k}^2}))|\mathbf{k}|^{-r}e^{-\kappa(t)^{-1}|\mathbf{k}|} \leq \nu a\kappa(t)|\mathbf{k}|^{-r}e^{-\kappa(t)^{-1}|\mathbf{k}|}, \tag{33}$$

since

$$e^{-\frac{1}{2}t\nu\mathbf{k}^2} + \lambda(1 - e^{-\frac{1}{2}t\nu\mathbf{k}^2}) \leq (1 + \eta\nu t\min(1,\kappa))^{-1}, \tag{34}$$

which holds, since $|\mathbf{k}| \geq 1$, for $\lambda$ and $\eta$ small enough and $0 \leq t \leq 1$. Inequality (33) is the claim of part (a) of the Proposition concerning the $\kappa(t)$ norm of the solution, namely $\|\omega(t)\|_{\kappa(t)} \leq \nu a\kappa(t)$.

Turning to $\kappa \geq 1$, fix a number $1 < \beta < \frac{\alpha-1}{r}$ (recall that $\alpha > 1 + r$) and consider first $1 \leq |\mathbf{k}| \leq \kappa^{\frac{\beta}{\alpha}}$. Using

$$|\omega_{\mathbf{k}}(t)| \leq (2\Phi(t))^{\frac{1}{2}}\ ,\ \ \Phi(t) \leq \nu^2\varphi\kappa(t)^{\frac{2}{\alpha}},$$

we get immediately,

$$|\omega_{\mathbf{k}}(t)| \leq (2\nu^2\varphi)^{\frac{1}{2}}\kappa(t)^{\frac{1}{\alpha}} \leq \nu a\kappa(t)|\mathbf{k}|^{-r}e^{-\kappa(t)^{-1}|\mathbf{k}|} \tag{35}$$

for $\varphi$ small enough and $t \leq 1$ since $|\mathbf{k}|^r \leq \kappa^{\frac{r\beta}{\alpha}} \leq \kappa^{1-\frac{1}{\alpha}}$ and $\kappa(t)^{-1}|\mathbf{k}| \leq 1$.

To conclude the proof, it suffices to show

$$|N_{\mathbf{k}}(\omega)(t)| \leq \nu a\kappa\lambda(1 - e^{-\frac{1}{2}t\nu\mathbf{k}^2})|\mathbf{k}|^{-r}e^{-\kappa(t)^{-1}|\mathbf{k}|}, \tag{36}$$

for $|\mathbf{k}| \geq \kappa^{\frac{\beta}{\alpha}}$ since we may then proceed as in (32–34).

Consider first the case $\kappa^{\frac{\beta}{\alpha}} < |\mathbf{k}| \leq \kappa$. We bound $|\mathbf{k}\times\mathbf{l}||\mathbf{l}|^{-2} \leq |\mathbf{k}||\mathbf{l}|^{-1}$ and split the sum in (22) into

$$\Big(\sum_{0\neq|\mathbf{l}|\leq\frac{|\mathbf{k}|}{2}} + \sum_{\mathbf{l}\neq\mathbf{k},|\mathbf{l}|>\frac{|\mathbf{k}|}{2}}\Big)|\omega_{\mathbf{k}-\mathbf{l}}(s)||\omega_{\mathbf{l}}(s)||\mathbf{k}||\mathbf{l}|^{-1} \equiv \Sigma_1 + \Sigma_2. \tag{37}$$

In the first sum, we bound, using Lemma 1,

$$|\omega_{\mathbf{k}-\mathbf{l}}(s)| \leq 2\nu a\kappa|\mathbf{k}-\mathbf{l}|^{-r} \leq C\nu a\kappa|\mathbf{k}|^{-r}$$

since $|\mathbf{k}-\mathbf{l}| \geq \frac{1}{2}|\mathbf{k}|$. Then Schwartz' inequality and (30) yield

$$\sum_{0\neq|\mathbf{l}|\leq\frac{|\mathbf{k}|}{2}}|\omega_{\mathbf{l}}(s)||\mathbf{l}|^{-1} \leq (2\nu^2\varphi)^{\frac{1}{2}}\kappa^{\frac{1}{\alpha}}\Big(\sum_{0\neq|\mathbf{l}|\leq\frac{|\mathbf{k}|}{2}}|\mathbf{l}|^{-2}\Big)^{\frac{1}{2}} \leq C\nu\varphi^{\frac{1}{2}}\kappa^{\frac{1}{\alpha}}(\log|\mathbf{k}|)^{\frac{1}{2}} \tag{38}$$

Combining these two bounds, we get

$$\Sigma_1 \leq C\nu^2|\mathbf{k}|\varphi^{\frac{1}{2}}\kappa^{\frac{1}{\alpha}}(\log|\mathbf{k}|)^{\frac{1}{2}}a\kappa|\mathbf{k}|^{-r}. \tag{39}$$

For the second sum, we use $|\omega_{\mathbf{l}}(s)| \leq 2\nu a\kappa|\mathbf{l}|^{-r}$ (coming from Lemma 1 again), together with (30) and Schwartz' inequality to bound it by

$$\Sigma_2 \leq C\nu^2|\mathbf{k}|\varphi^{\frac{1}{2}}\kappa^{\frac{1}{\alpha}}a\kappa\Big(\sum_{\mathbf{l}\neq\mathbf{k},|\mathbf{l}|>\frac{|\mathbf{k}|}{2}}|\mathbf{l}|^{-2(r+1)}\Big)^{\frac{1}{2}} \leq C\nu^2|\mathbf{k}|\varphi^{\frac{1}{2}}\kappa^{\frac{1}{\alpha}}a\kappa|\mathbf{k}|^{-r}. \tag{40}$$

Inserting (37), (39) and (40) into $N_{\mathbf{k}}(\omega)(t)$ and performing the integral over time, we get the bound

$$\begin{aligned}|N_{\mathbf{k}}(\omega)(t)| &\leq C\nu\varphi^{\frac{1}{2}}\kappa^{\frac{1}{\alpha}}(\log|\mathbf{k}|)^{\frac{1}{2}}|\mathbf{k}|^{-1}(1-e^{-t\nu\mathbf{k}^2})a\kappa|\mathbf{k}|^{-r} \\ &\leq C\nu e^{1+\eta\nu}\varphi^{\frac{1}{2}}\kappa^{\frac{1}{\alpha}}\kappa^{-\frac{\beta}{\alpha}}(\log\kappa)^{\frac{1}{2}}(1-e^{-\frac{1}{2}t\nu\mathbf{k}^2})a\kappa|\mathbf{k}|^{-r}e^{-\kappa(t)^{-1}|\mathbf{k}|} \qquad (41)\end{aligned}$$

where we used $\kappa^{\frac{\beta}{\alpha}} < |\mathbf{k}| \leq \kappa$ and

$$1 \leq e^{-\kappa(t)^{-1}|\mathbf{k}|}e^{1+\eta\nu}$$

which holds since $|\mathbf{k}| \leq \kappa$ and, see (17), $\kappa(t)^{-1} \leq \kappa^{-1}(1+\eta\nu)$ if $0 \leq t \leq 1$. Thus we obtain (36) for $\varphi$ small enough, because $\beta > 1$, $\log x \leq \frac{1}{\epsilon}x^{\epsilon}$, for $x > 1$ and $\epsilon > 0$.

Finally, in the case $|\mathbf{k}| > \kappa$ the bound (29) yields immediately (36) for $a$ small. This finishes the proof of part (a) of Proposition 1.

For part (b), we can proceed as in Lemma 1, but replace $a\kappa$ by $D$ and in the definition (19) $\kappa(t)$ by $\frac{2}{\nu t}$. The inequality (27) is then replaced by $\frac{1}{2}(s-t)\nu\mathbf{k}^2 \leq \frac{1}{2}(s-t)\nu|\mathbf{k}|$ and the proof goes as before to the conclusion

$$\|\omega(t)\|_{\frac{2}{\nu t}} \leq 2\nu D \qquad (42)$$

for $t \leq (C\nu^{\frac{1}{2}}D)^{-2} \equiv \tau$. We want to rewrite this bound in the form $\|\omega(\tau)\|_{\kappa} \leq \nu a\kappa$, for a suitable $\kappa$. If $\tau \leq 1$, i.e. $D \geq C^{-1}\nu^{-\frac{1}{2}}$ we have $\frac{2}{\nu\tau} = CD^2$ and we can write (42) as $\|\omega(\tau)\|_{\rho} \leq 2\nu D$ where $\rho = CD^2$ (remember that $C$ is allowed to vary). Choosing now $\kappa = C\rho = C'D^2$, we obtain (since $D$ is bounded away from zero, hence $D \leq CD^2$) that $\|\omega(\tau)\|_{\kappa} \leq \nu a\kappa$. Applying now part (a) yields the same claim for $\omega(1)$. For $\tau > 1$ i.e. $D < C^{-1}\nu^{-\frac{1}{2}}$ we get $\|\omega(1)\|_{\frac{2}{\nu}} \leq 2\nu D \leq C$, given the bound on $D$; so, $\|\omega(1)\|_{\kappa} \leq \nu a\kappa$, with $\kappa = \frac{C}{\nu}$.

Finally, for the enstrophy, we have, by (30),

$$\Phi(t) \leq \Phi(0) \leq \nu^2 CD^2 \leq \nu^2\varphi\kappa^{\frac{2}{\alpha}}$$

if we take $\kappa > CD^{\alpha}$. Since $\alpha > 2$, taking $\kappa = C(D^{\alpha}+\frac{1}{\nu})$ gives an upper bound covering all cases, i.e. $\omega(1) \in U_{\kappa}$. □

## 3 Probabilistic estimates.

We define a region $U \equiv U_{\nu^{-p}}$, where $p > \frac{7}{2}\alpha$, and in which the solution of (6) is confined with high probability. Let us divide the transition probability into a likely and unlikely part:

$$P(\omega, E) = Q(\omega, E) + R(\omega, E) \qquad (43)$$

where

$$Q(\omega, E) = \chi_U(\omega)P(\omega, U \cap E). \qquad (44)$$

The following Proposition about the dynamics in $U$ and the unprobability of excursions outside $U$ will play a central role in the proof of our uniqueness result[2].

**Proposition 2.** (a) *There exist constants $c, C < \infty$, $c' > 0$, such that for all $\omega \in U$, $E \in \mathcal{B}$,*

$$|Q^t(\omega, E) - Q^t(0, E)| \leq 4e^{-mt} \qquad (45)$$

*where $m \geq \exp(-C\nu^{-3}(\log\nu^{-1})^c)$ and $t \leq c'm^{-1}\nu^{-q}$, with $q \equiv \frac{2p}{\alpha} - 4 > 3$.*
(b) *There exists $\zeta < 1$, $c > 0$, $C < \infty$, such that $\forall\kappa \geq 0$, for all $\omega \in U_{\kappa}$ and for $\kappa' \geq \zeta\kappa$,*

[2] Here and below, the kernel $AB(\omega, E)$ is defined in the obvious way by $\int A(\omega, d\omega')B(\omega', E)$.

$$P(\omega, U_{\kappa'}^c) \leq C \exp(-c\nu^4 \kappa'^{\frac{2}{\alpha}}) \tag{46}$$

The proof of (45) is based on a standard argument for exponential convergence of Markov chains (given in Doob [2]), and the idea is fairly simple. If $Q$ was a genuine transition probability, it would be enough, in order to prove the Proposition, to show that $Q$ has good mixing properties. The precise properties are stated in the Lemmas below.

First, Lemma 2 says that, for any point in $U$ there is a nonzero probability to go in a finite time to a smaller region $\bar{U} \subset U$ determined by the covariance of the noise and thus by $\kappa_\gamma$:

$$\bar{U} \equiv U_{2\kappa_\gamma + \rho\nu}. \tag{47}$$

where $\rho > 0$ will be chosen below (sufficiently small)[3]. This is an easy consequence of Proposition 1. On each time interval, the solution increases its domain of analyticity (which is determined by $\kappa$, i.e. $\kappa$ decreases); then, if the "kicks" of the noise are sufficiently small (but not too small, so that this event is not too unprobable), the solution reaches $\bar{U}$ in a finite time (of order $\nu^{-1}\log\nu^{-1}$).

Secondly, we show in Lemma 3 that, in the region $\bar{U}$, the stochastic dynamics is sufficiently mixing; this is again due to the fact that the deterministic Navier-Stokes evolution increases the domain of analyticity of the solution.

Third, the fact that $Q$ is not a bona fide transition probability is what limits the Proposition to finite times. For longer times, we will need to have some estimate on the probability of escaping the region $U$, which follows from part (b) of the Proposition. Indeed, the latter implies, using (43, 44) and taking $\kappa = \kappa' = \nu^{-p}$ that, for all $\omega \in U$,

$$P(\omega, U^c) = R(\omega, \Omega) \leq e^{-c\nu^{-q}}, \tag{48}$$

with $q = \frac{2p}{\alpha} - 4 > 3$ (remember that $p > \frac{7}{2}\alpha$ and that $\nu$ is small).

**Lemma 2.** *There exist constants $c, C < \infty$, such that $\forall \omega \in U$,*

$$P^{T_1}(\omega, \bar{U}) \geq \exp(-C\nu^{-3}(\log\nu^{-1})^c) \tag{49}$$

with $T_1 = C\nu^{-1}\log\nu^{-1}$.

**Lemma 3.** *There exist constants $c, C < \infty$, such that, $\forall \omega, \omega' \in \bar{U}$, $\forall B \subset \bar{U}$,*

$$P(\omega, B) + P(\omega', \bar{U}\backslash B) \geq \exp(-C\nu^{-2}(\log\nu^{-1})^c) \tag{50}$$

Lemmas 2, 3 imply that there exist

$$\delta(\nu) \equiv \exp(-C\nu^{-3}(\log\nu^{-1})^c)$$

and

$$T \equiv T(\nu) = C\nu^{-1}\log\nu^{-1}$$

with $C, c < \infty$, such that $\forall \omega, \omega' \in U$ and $\forall B \subset \bar{U}$,

$$P^T(\omega, B) + P^T(\omega', \bar{U}\backslash B) \geq \delta(\nu) \tag{51}$$

which implies in turn, since $\bar{U} \subset U$, that $\forall \omega, \omega' \in U$ and $\forall B \subset U$,

$$P^T(\omega, B) + P^T(\omega', U\backslash B) \geq \delta(\nu) \tag{52}$$

This the main inequality that we shall use now.

[3] Similar ideas were used by Kuksin and Shirikyan in [4].

### 3.1 Proof of Proposition 2.

We start with the proof of part (a), where we shall use (48), which is a consequence of part (b), to be proven independently below.

To get (45) we follow, with slight modifications, an argument in [2], p. 197–198. Let

$$\underline{Q}(t,E) = \inf_{\omega\in U} Q^t(\omega,E), \quad \overline{Q}(t,E) = \sup_{\omega\in U} Q^t(\omega,E).$$

Fix $\omega,\omega'\in U$ and consider the function defined on subsets $E\subset\Omega$:

$$\psi_{\omega,\omega'}(E) = Q^T(\omega,E) - Q^T(\omega',E).$$

Let $S^+$ be the set such that $\psi_{\omega,\omega'}(E)\ge 0$ for $E\subset S^+$ and $\psi_{\omega,\omega'}(E)\le 0$ for $E\subset U\backslash S^+\equiv S^-$ ($S^\pm$ depend on $\omega,\omega'$, but we suppress this dependence). Observe that writing, see (43), $P=Q+R$, and using (48), we have, for any $\omega\in U$, $E\subset\Omega$, that

$$|P^T(\omega,E)-Q^T(\omega,E)| = |\sum_{t=0}^{T-1} Q^t R P^{T-t-1}(\omega,E)| \le T e^{-c\nu^{-q}} \equiv \tfrac{1}{2}\epsilon(\nu). \tag{53}$$

Then,

$$|\psi_{\omega,\omega'}(S^+)+\psi_{\omega,\omega'}(S^-)| = |Q^T(\omega,\Omega)-Q^T(\omega',\Omega)| \le \epsilon(\nu). \tag{54}$$

since $P^T(\omega,\Omega)=P^T(\omega',\Omega)=1$. Moreover, using (53, 52),

$$\begin{aligned}&\psi_{\omega,\omega'}(S^+) = Q^T(\omega,S^+) - Q^T(\omega',S^+)\\ &\le 1-(P^T(\omega,S^-)+P^T(\omega',S^+))+\epsilon(\nu) \le 1-\delta(\nu)+\epsilon(\nu).\end{aligned} \tag{55}$$

Thus,

$$\begin{aligned}\overline{Q}(t+T,E)-\underline{Q}(t+T,E) &= \sup_{\omega,\omega'}\int (Q^T(\omega,d\omega'')-Q^T(\omega',d\omega''))Q^t(\omega'',E)\\ &= \sup_{\omega,\omega'}\int \psi_{\omega,\omega'}(d\omega'')Q^t(\omega'',E)\\ &\le \sup_{\omega,\omega'}(\psi_{\omega,\omega'}(S^+)\overline{Q}(t,E)+\psi_{\omega,\omega'}(S^-)\underline{Q}(t,E))\\ &\le (1-\delta(\nu)+\epsilon(\nu))(\overline{Q}(t,E)-\underline{Q}(t,E))+\epsilon(\nu)\end{aligned}$$

where, to get the last inequality, we write $\psi_{\omega,\omega'}(S^-) = -\psi_{\omega,\omega'}(S^+)+\psi_{\omega,\omega'}(S^+)+\psi_{\omega,\omega'}(S^-)$, bound $\psi_{\omega,\omega'}(S^+)$ by (55), $|\psi_{\omega,\omega'}(S^+)+\psi_{\omega,\omega'}(S^-)|$ by (54) and use $\underline{Q}(t,E)\le 1$. We conclude that, for $\epsilon(\nu)<\delta(\nu)$,

$$|Q^{nT}(\omega,E)-Q^{nT}(0,E)| \le \overline{Q}(nT,E)-\underline{Q}(nT,E) \le 2(1-\delta(\nu)+\epsilon(\nu))^{n-1}+\frac{\epsilon(\nu)}{\delta(\nu)-\epsilon(\nu)}.$$

Recall that $\delta(\nu)=\exp(-C\nu^{-3}(\log\nu^{-1})^c)$ and that $T=T(\nu)=C\nu^{-1}\log\nu^{-1}$, hence, see (53), $\epsilon(\nu)\le e^{-c'\nu^{-q}}$; so, since we assume $\nu$ to be small and $q>3$, part (a) of the Proposition follows.

Let us now prove part (b). It suffices to assume $\kappa\ge C\nu^{-2\alpha}$ since the LHS of (46) is bounded by one. Using (10) for $n=0$, we have

$$\|\omega(1)\|_{\kappa'} \le \|F(\omega(0))\|_{\kappa'}+\|g(1)\|_{\kappa'} \tag{56}$$

and

$$\Phi(1)=\tfrac{1}{2}\|\omega(1)\|^2_{L^2} \le \|F(\omega(0))\|^2_{L^2}+\|g(1)\|^2_{L^2} \tag{57}$$

and, by Proposition 1, we know that, if $\omega(0) \in U_\kappa$, $F(\omega(0)) \in U_{\kappa(1)}$, with $\kappa(1) = \frac{\kappa}{1+\eta\nu}$ (recall that $\kappa \geq C\nu^{-2\alpha} \geq 1$ here). Then, letting $\zeta = (1+\eta\nu)^{-\frac{1}{2}} < 1$, we get, for $\kappa' \geq \zeta\kappa$, that

$$\|F(\omega(0))\|_{\kappa'} \leq (1+\eta\nu)^{-\frac{1}{2}} \nu a \kappa'$$

and

$$\|F(\omega(0))\|_{L^2}^2 \leq (1+\eta\nu)^{-\frac{2}{\alpha}} \nu^2 \varphi \kappa'^{\frac{2}{\alpha}}.$$

Now, assume that $g(1)$ satisfies, $\forall \mathbf{k}$,

$$|g_{\mathbf{k}}(1)| \leq \epsilon_1 \nu^2 \kappa'^{\frac{1}{\alpha}} b^{-\frac{1}{2}} |\mathbf{k}| (\gamma_{\mathbf{k}})^{\frac{1}{2}}, \tag{58}$$

with $\epsilon_1$ small (depending on $\eta$ but independent of $\nu$ and $\kappa$). Then, we get that $\omega(1) \in U_{\kappa'}$, using the upper bound in (7) and the fact that, for $\nu$ small, $\kappa' \geq \zeta\kappa \geq C\zeta\nu^{-2\alpha}$ is much larger than $2\kappa_\gamma$.

Hence, the probability in (46) is bounded by the probability that at least one of the inequalities in (58) is violated. Since the $g_{\mathbf{k}}$'s are Gaussian random variables with covariance $\gamma_{\mathbf{k}}$, this event has a probability less than:

$$1 - \prod_{\mathbf{k}} (1 - C\exp(-C\epsilon_1^2 \nu^4 (\kappa')^{\frac{2}{\alpha}} b^{-1} |\mathbf{k}|^2)), \tag{59}$$

For $\nu^4(\kappa')^{\frac{2}{\alpha}}$ large, each exponential in the product is small. The factor $|\mathbf{k}|^2$ controls the sum over $\mathbf{k}$ of $\exp(-C\epsilon_1^2\nu^4(\kappa')^{\frac{2}{\alpha}} b^{-1}|\mathbf{k}|^2)$, sum which is small for $\nu^4(\kappa')^{\frac{2}{\alpha}}$ large enough, and therefore (59) is bounded from above by $1 - (1 - \exp(-c\nu^4\kappa'^{\frac{2}{\alpha}}))$, i.e. by the RHS of (46). □

### 3.2 Proofs of Lemmas 2 and 3.

**Proof of Lemma 2.** Let $\omega(0) \in U_\kappa \subseteq U$ and consider (10) for $n = 0$. Choose $g(1)$ such that, $\forall \mathbf{k}$,

$$|g_{\mathbf{k}}(1)| \leq \epsilon_1 \nu^2 e^{\epsilon_2 \nu |\mathbf{k}|} b^{-\frac{1}{2}} (\gamma_{\mathbf{k}})^{\frac{1}{2}}. \tag{60}$$

¿From Proposition 1, (60) and (7), one obtains,

$$|\omega_{\mathbf{k}}(1)| \leq \nu a \kappa(1) |\mathbf{k}|^{-r} e^{-\kappa(1)^{-1}|\mathbf{k}|} + \epsilon_1 \nu^2 e^{\epsilon_2 \nu |\mathbf{k}|} e^{-\kappa_\gamma^{-1} \frac{|\mathbf{k}|}{2}} \tag{61}$$

Then, from (17), one gets, for any $\rho > 0$, by choosing $\epsilon_1, \epsilon_2$ small enough, that $\exists \bar{\lambda} < e^{-c\nu} < 1$ such that, $\forall \mathbf{k}$,

$$|\omega_{\mathbf{k}}(1)| \leq \nu a \kappa' |\mathbf{k}|^{-r} e^{-(\kappa')^{-1}|\mathbf{k}|} \tag{62}$$

with $\kappa' = \max(\bar{\lambda}\kappa, 2\kappa_\gamma + \rho\nu)$.

¿From (30), (57), (58), one also easily obtains that

$$\Phi(1) \leq \nu^2 \varphi (\kappa')^{\frac{2}{\alpha}}$$

and thus that $\omega(1) \in U_{\kappa'}$. Thus,

$$P(\omega, U_{\kappa'}) \geq \prod_{\mathbf{k}} P(|g_{\mathbf{k}}(1)| \leq \epsilon_1 \nu^2 e^{\epsilon_2 \nu |\mathbf{k}|} b^{-\frac{1}{2}} (\gamma_{\mathbf{k}})^{\frac{1}{2}}).$$

Now, since the $g_{\mathbf{k}}$'s are Gaussian random variables with covariance $\gamma_{\mathbf{k}}$, we have that,

$$P(|g_{\mathbf{k}}(1)| \leq \epsilon_1 \nu^2 e^{\epsilon_2 \nu |\mathbf{k}|} b^{-\frac{1}{2}} (\gamma_{\mathbf{k}})^{\frac{1}{2}}) \geq 1 - \exp(-c\epsilon_1^2 \nu^4 e^{2\epsilon_2 \nu |\mathbf{k}|} b^{-1}) \tag{63}$$

for $|\mathbf{k}| \geq C\nu^{-1}\log\nu^{-1}$, if $C$ is chosen so that $b\nu^{-4} \leq e^{\epsilon_2\nu|\mathbf{k}|}$ (note that the product over such $\mathbf{k}$'s of the RHS of (63) is strictly positive uniformly in $\nu$), while for $|\mathbf{k}| \leq C\nu^{-1}\log\nu^{-1}$,

$$P(|g_{\mathbf{k}}(1)| \leq \epsilon_1\nu^2 e^{\epsilon_2\nu|\mathbf{k}|} b^{-\frac{1}{2}}(\gamma_{\mathbf{k}})^{\frac{1}{2}}) \geq P(|g_{\mathbf{k}}(1)| \leq \epsilon_1\nu^2 b^{-\frac{1}{2}}(\gamma_{\mathbf{k}})^{\frac{1}{2}}) \geq C\nu^4 \tag{64}$$

which follows from the fact that the $g_{\mathbf{k}}$'s are (complex) Gaussian random variables with covariance $\gamma_{\mathbf{k}}$ and therefore that

$$P(|g_{\mathbf{k}}(1)| \leq \epsilon_1\nu^2 b^{-\frac{1}{2}}(\gamma_{\mathbf{k}})^{\frac{1}{2}}) \geq \int_0^{\widetilde{\epsilon}_{\mathbf{k}}} \frac{2r\,dr}{\gamma_{\mathbf{k}}} \geq C\nu^4$$

with $\widetilde{\epsilon}_{\mathbf{k}} \equiv \epsilon_1\nu^2 b^{-\frac{1}{2}}(\gamma_{\mathbf{k}})^{\frac{1}{2}}$.

The bound (64) readily implies that there are constants $C, c_1 < \infty$ such that $\forall\omega \in U_\kappa$,

$$P(\omega, U_{\kappa'}) \geq \exp(-C\nu^{-2}(\log\nu^{-1})^{c_1}). \tag{65}$$

Since $U = U_{\nu^{-p}}$, and since $\kappa$ decreases by a factor $\bar{\lambda} < 1$ at each step, as long as $\bar{\lambda}\kappa \geq 2\kappa_\gamma + \rho\nu$, one may iterate the above argument and reach $\bar{U} = U_{2\kappa_\gamma+\rho\nu}$, see (47), in a time less than $T_1(\nu) = C\nu^{-1}\log\nu^{-1}$, $\forall\omega(0) \in U$. Therefore, the claim of the lemma follows (with a different $C$ than in (65), and with $c = c_1 + 1$). □

**Proof of Lemma 3.** Let $\omega_0 \in \bar{U}$ and $B \subset \bar{U}$. Since the $g_{\mathbf{k}}$'s are Gaussian random variables with covariance $\gamma_{\mathbf{k}}$, we have,

$$P(\omega_0, B) = \int_B \prod_{\mathbf{k}} \frac{d\bar{\omega}_{\mathbf{k}} \wedge d\omega_{\mathbf{k}}}{2\pi i\gamma_{\mathbf{k}}} \exp(-\frac{|\omega_{\mathbf{k}} - F_{\mathbf{k}}(\omega_0)|^2}{\gamma_{\mathbf{k}}}) \tag{66}$$

where we recall from (10) that $F(\omega_0)$ denotes the value at time 1 of the solution of (18) with initial condition $\omega_0$. In view of Proposition 1, and the definition of $\bar{U} = U_{2\kappa_\gamma+\rho\nu}$, we can bound, $\forall\omega_0 \in \bar{U}$,

$$|F_{\mathbf{k}}(\omega_0)| \leq C\nu a e^{-\frac{|\mathbf{k}|}{2\kappa_\gamma}} e^{-\rho\nu|\mathbf{k}|} \equiv \epsilon_{\mathbf{k}}, \tag{67}$$

provided we choose $\rho$ sufficiently small so that

$$\frac{1 + \eta\nu\min(1, 2\kappa_\gamma + \rho\nu)}{2\kappa_\gamma + \rho\nu} \geq \frac{1}{2\kappa_\gamma} + \rho\nu.$$

Thus, we can bound $|\omega_{\mathbf{k}} - F_{\mathbf{k}}(\omega_0)|^2 \leq (|\omega_{\mathbf{k}}| + \epsilon_{\mathbf{k}})^2$; this gives a lower bound on (66) independent of $\omega_0$ and we may use this bound on each term of the LHS of (50), with $\omega_0 = \omega, \omega'$. We get that the LHS of (50) is bounded from below by

$$\int_{\bar{U}} \prod_{\mathbf{k}} \frac{d\bar{\omega}_{\mathbf{k}} \wedge d\omega_{\mathbf{k}}}{2\pi i\gamma_{\mathbf{k}}} \exp(-\frac{(|\omega_{\mathbf{k}}| + \epsilon_{\mathbf{k}})^2}{\gamma_{\mathbf{k}}}) \tag{68}$$

In order to estimate that latter integral, observe that, by (7), $\omega \in \bar{U} = U_{2\kappa_\gamma+\rho\nu}$ provided that, $\forall\mathbf{k}$,

$$|\omega_{\mathbf{k}}| \leq \epsilon_1\nu e^{\epsilon_2\nu|\mathbf{k}|} b^{-\frac{1}{2}}(\gamma_{\mathbf{k}})^{\frac{1}{2}} \equiv \bar{\epsilon}_{\mathbf{k}} \tag{69}$$

if we take $\epsilon_1, \epsilon_2$ small enough. Thus, by restricting the domain of integration, we get a lower bound on (68):

$$\prod_{\mathbf{k}} [\int_0^{\bar{\epsilon}_{\mathbf{k}}} \frac{2r\,dr}{\gamma_{\mathbf{k}}} \exp(-\frac{(r + \epsilon_{\mathbf{k}})^2}{\gamma_{\mathbf{k}}})] \tag{70}$$

Each factor is bounded from below by

$$1 - C\frac{\epsilon_{\mathbf{k}}^2}{\gamma_{\mathbf{k}}} - \exp(-c\epsilon_1^2\nu^2 e^{2\epsilon_2|\mathbf{k}|}(b)^{-1}) \tag{71}$$

for $|\mathbf{k}| \geq C\nu^{-1}\log\nu^{-1}$. To bound the product over those $\mathbf{k}$'s of the factors given by (71) by a strictly positive constant, independent of $\nu$, observe first that the last term is summable over $\mathbf{k}$, for $|\mathbf{k}| \geq C\nu^{-1}\log\nu^{-1}$ and that the sum is small. Moreover, using (67) and the lower bound in (7), we get,

$$\frac{\epsilon_{\mathbf{k}}^2}{\gamma_{\mathbf{k}}} \leq Cab\nu^2 \exp(-2\rho\nu|\mathbf{k}|). \tag{72}$$

Then, (72) is also summable over $\mathbf{k}$, for $|\mathbf{k}| \geq C\nu^{-1}\log\nu^{-1}$ and the sum is also small. Finally, for $|\mathbf{k}| \leq C\nu^{-1}\log\nu^{-1}$, each factor in (70) is bounded from below, using (7, 69), by $C\int_0^{\bar{\epsilon}_{\mathbf{k}}} \frac{2rdr}{\gamma_{\mathbf{k}}} \geq C\nu^2$, which yields the claim of the Lemma. □

# 4 Proof of the Theorem.

We deduce the Theorem from Proposition 2.

Let us choose a number $\bar{\epsilon}$ small enough and a time $\tau$, large enough, i.e $\tau = cm^{-1}\nu^{-q}$ so that (45) is less than $\frac{\bar{\epsilon}}{2}$. Then, for $T$ an integer multiple of $\tau$, write

$$P^T(\omega, E) = (P^\tau)^{\frac{T}{\tau}}(\omega, E). \tag{73}$$

Next, let

$$\pi(\omega, E) \equiv \pi(E) = Q^\tau(0, E) \tag{74}$$

and

$$\begin{aligned} \bar{R}(\omega, E) &= P^\tau(\omega, E) - Q^\tau(\omega, E) \\ R'(\omega, E) &= Q^\tau(\omega, E) - Q^\tau(0, E) = Q^\tau(\omega, E) - \pi(E) && (75) \\ r(\omega, E) &= \bar{R}(\omega, E) + R'(\omega, E). && (76) \end{aligned}$$

One may then write

$$P^T(\omega, E) = (\pi + r)^{\frac{T}{\tau}}(\omega, E). \tag{77}$$

We can expand $(\pi + r)^{\frac{T}{\tau}}$ in powers of $r$:

$$(\pi + r)^{\frac{T}{\tau}} = \sum_{k_i} \pi^{k_1} r^{k_2} \ldots \pi^{k_l} + r^{\frac{T}{\tau}} \equiv \Sigma^1 + r^{\frac{T}{\tau}} \tag{78}$$

where the sum $\Sigma^1$ runs over $k_i \geq 0$, $\sum k_i = \frac{T}{\tau}$ and collects all the terms with at least one factor $\pi$. Now observe that, using (77) with $T = \tau$, we have that

$$r(\omega, \Omega) = P^\tau(\omega, \Omega) - \pi(\Omega) = 1 - \pi(\Omega) = 1 - Q^\tau(0, U)$$

is independent of $\omega$; hence, by (74),

$$(r\pi)(\omega, d\omega_2) = \int r(\omega, d\omega_1)\pi(\omega_1, d\omega_2) = r(\omega, \Omega)\pi(d\omega_2) = (1 - Q^\tau(0, U))\pi(d\omega_2) \tag{79}$$

is also independent of $\omega$. From this, we conclude that, since there is at least one factor $\pi$ in each term of $\Sigma^1$, $\Sigma^1(\omega, E) = \Sigma^1(\omega', E)$, $\forall \omega, \omega'$, $\forall E$, and, using (77), that

$$|P^T(\omega, E) - P^T(\omega', E)| \leq |r^{\frac{T}{\tau}}(\omega, E)| + |r^{\frac{T}{\tau}}(\omega', E)| \tag{80}$$

where the RHS is controlled by:

**Lemma 4.** *For $T$ an integer multiple of $\tau$,*

$$|r^{\frac{T}{\tau}}(\omega, E)| \leq C(\omega) e^{-\overline{m}T} \tag{81}$$

*where $\overline{m} = \overline{m}(\nu) \geq \exp(-C\nu^{-3}(\log \nu^{-1})^c)$ and where $C(\omega) \leq C(\frac{||\omega||+1}{\nu})^c$, for $C, c < \infty$.*

To conclude the proof of (12), it is enough to show that $\lim_{T\to\infty} P^T(0, E) = \mu(E)$ exists. And, to prove that, we write, for $T > T'$,

$$|P^T(0,E) - P^{T'}(0,E)| \leq \int P^{T-T'}(0, d\omega)|P^{T'}(\omega, E) - P^{T'}(0,E)|. \tag{82}$$

We may write the integral as an integral over $U$ plus a sum over $\kappa \in \mathbf{N}$, $\kappa \geq \nu^{-p}$, of integrals over $U_{\kappa+1} \backslash U_\kappa$ and, combining (80), Lemma 4, $C(\omega) \leq C(\frac{||\omega||+1}{\nu})^c \leq C(\frac{||\omega||_\kappa+1}{\nu})^c$, $\forall \kappa$, and (46) (which implies a similar bound for $P^{T-T'}(0, U^c_{\kappa'})$), we bound (82) by

$$Ce^{-\overline{m}T'}(\nu^{-(p+1)c} + \sum_{\kappa\in\mathbf{N},\kappa\geq\nu^{-p}} \kappa^c \exp(-c'\nu^4\kappa^{\frac{2}{\alpha}}) \leq C(\nu)e^{-\overline{m}T'} \tag{83}$$

which proves the existence of $\lim_{T\to\infty} P^T(0,E)$. Finally, the bound (11) follows from (46), for $\kappa$ large, and we bound the LHS of (11) by 1 for $\kappa$ small. □

**Proof of Lemma 4.** Define, for $n \geq 0$,

$$U(n) \equiv U_{\zeta^{-n}\nu^{-p}},$$

(so that $U = U(0)$) with $\zeta < 1$ as in Proposition 2, and define $V(n)$ by $V(n) = U(n) \backslash U(n-1)$, for $n \geq 1$, and $V(0) = U(0)$. Next, let

$$\rho_{mn} \equiv \sup_{\omega\in V(m)} |r(\omega, V(n))|,$$

where $r$ is defined in (76). Observe that we have the following bounds on $\rho_{mn}$:

$$\begin{aligned} \rho_{00} &\leq \bar{\epsilon} \\ \rho_{mn} &\leq \exp(-c\xi^n\nu^{-q}) \quad n \geq m \\ \rho_{mn} &\leq 4 \quad n < m \end{aligned} \tag{84}$$

where $\xi \equiv \zeta^{-\frac{2}{\alpha}} > 1$. To check this, use, for $m, n = 0$, (53) to bound $\bar{R}$ and (45) to bound $R'$. For the second inequality, $n \neq 0$, only $P$ contributes to $r$ and the bound follows immediately from (46), with $\kappa' = \zeta^{1-n}\nu^{-p}$ (remember that $q = \frac{2p}{\alpha} - 4$). Finally, for $n < m$, we use the fact that $r$ is the sum of four terms, each less than 1.

Write now

$$r^N(\omega, E) = \int \prod_{i=0}^{N-1} r(\omega_i, d\omega_{i+1}) \chi(\omega_N \in E) \tag{85}$$

with $\omega_0 = \omega$, and insert a decomposition of the identity for each $i = 1, \ldots, N$

$$1 = \sum_{n_i\geq 0} \chi(\omega_i \in V(n_i)).$$

This leads to

$$\sup_{\omega\in U(n_0)} \sup_E |r^N(\omega, E)| \leq \sum_{(n_i)_{i=1}^N, n_i\geq 0} \prod_{i=0}^{N-1} \rho_{n_i n_{i+1}} \equiv \sum_n \rho^N_{n_0 n}. \tag{86}$$

Note that the RHS describes "random walks" on nonnegative integers, where only steps strictly down ($n_{i+1} < n_i$) are not suppressed. To estimate it, write $\rho = d + u$ where $d$ is the "down" part of $\rho$, i.e. the matrix whose elements are given by $\rho_{mn}$ with $n < m$ and zero otherwise, and u is the rest ("up"). We shall first prove the simple estimates

$$\sum_n d^k_{mn} \leq C^m \quad k \leq m \tag{87}$$

and zero otherwise (where the restriction $k \leq m$ comes from the fact that the indices of $d_{mn}$ must be positive and, whenever $d_{mn} \neq 0$, must satisfy $n < m$), and

$$\sum_n (u^k d^l)_{mn} \leq (C\bar{\epsilon})^{k+l}. \tag{88}$$

Indeed, (87) is estimated by

$$\sum_{n_1 \ldots n_k} d_{mn_1} \ldots d_{n_{k-1}n_k} \leq \sum_{\sum_1^k p_i \leq m} 4^k = 4^k \sum_{l=k}^{m} \binom{l-1}{k-1} \leq 4^k 2^m$$

with $p_i = n_{i-1} - n_i \geq 1$, $n_0 = m$, yielding the claim since $k \leq m$. To prove (88), write

$$(u^k d^l)_{mn} = \sum_{l \leq m'} u^k_{mm'} d^l_{m'n}$$

where the constraint in the sum comes from the second inequality in (87), and note that $u^k_{mm'}$ is bounded by $(\bar{\epsilon})^k$ if $m = m' = 0$ and by

$$\exp(-c\xi^{m'}\nu^{-q})(C\bar{\epsilon})^{k-1}$$

otherwise (both bounds following from (84) and the fact that, by definition, $u_{mn} = 0$, unless $m \geq n$). The bound (88) follows by combining these with (87), since $l \leq m'$, we can therefore use the factor $\exp(-c\xi^{m'}\nu^{-q})$ and $\nu$ small to obtain the factor $\bar{\epsilon}^{l+1}$ in (88) ($\bar{\epsilon}$ is a fixed small number).

Inserting (87), (88) into

$$\rho^N = (u+d)^N = \sum d^{l_0} u^{k_1} d^{l_1} \ldots u^{k_s} d^{l_s}$$

where $l_i \geq 0$, $l_i > 0$ for $i \neq 0, s$, we obtain the bound

$$\sum_n \rho^N_{n_0 n} \leq C^N \bar{\epsilon}^{N-n_0}.$$

where $\bar{\epsilon}^{-n_0}$ comes from the fact that we have no $\bar{\epsilon}$ bound on $d^{l_0}$, but we can use $l_0 \leq n_0$.

This proves the Lemma, if we choose in (81)

$$\overline{m} = -\frac{1}{\tau}\log C\bar{\epsilon} = -cm^{-1}\nu^{-q}\log C\bar{\epsilon} \geq \exp(-C\nu^{-3}(\log\nu^{-1})^c)$$

(given our choice of $\tau$ at the beginning of this section, our bound on $m$ in Proposition 2, and changing the constants), and, for $\omega \in U(n_0)$, choose $C(\omega) = (C\bar{\epsilon})^{-n_0}$; indeed, let, for $\omega \in U_\kappa$, $n_0$ be the smallest integer such that $\kappa \leq \zeta^{-n_0}\nu^{-p}$; then, $n_0 \leq C\log\kappa$ and $C(\omega) \leq C\kappa^c$. Moreover, from part (b) of Proposition 1, we know that, $\forall\omega \in \Omega$, $F(\omega) \in U_\kappa$, with $\kappa \leq C((\frac{||\omega||}{\nu})^\alpha + \frac{1}{\nu})$; so, altogether, $C(\omega) \leq C(\frac{||\omega||+1}{\nu})^c$. □

## References


[1] J. Bricmont, A. Kupiainen, R. Lefevere: *Probabilistic estimates for the two dimensional stochastic Navier-Stokes equations.* Preprint, to appear in J. Stat. Phys. (http://mpej.unige.ch/mp-arc/e/00-113.).

[2] J.L. Doob: *Stochastic Processes*, John Wiley, New-York, 1953.

[3] F. Flandoli, B. Maslowski: *Ergodicity of the 2-D Navier-Stokes equation under random perturbations.* Commun. Math. Phys. **171** (1995), 119-141.

[4] S. Kuksin, A. Shirikyan: *Stochastic dissipative PDE's and Gibbs measures.* Preprint, to appear in Commun. Math. Phys.

[5] J. C. Mattingly, Y. Sinai: *An elementary proof of the existence and uniqueness theorem for the Navier-Stokes equations.* Preprint.

[6] J. C. Mattingly: *Ergodicity of 2D Navier-Stokes equations with random forcing and large viscosity.* Commun. Math. Phys. **206** (1999), 273-288.