An Example of a Nontrivial Bubble Tree in the Harmonic Map Heat Flow *

Peter Topping

Abstract

We present the first example of the formation of a nontrivial bubble tree in the harmonic map heat flow. In other words, we give a flow in which more than one bubble develops at the same point. The bubbles occur at infinite time and develop at different scales.

Let $(\mathcal{M}, \gamma)$ be a compact Riemannian surface, and $(\mathcal{N}, g)$ a compact Riemannian manifold without boundary. The harmonic map heat flow is L^2-gradient descent for the harmonic map energy

$$E(v) = \int_{\mathcal{M}} \frac{1}{2}|dv|^2,$$

and was introduced in 1964 by Eells and Sampson [4]. Explicitly, the flow is a solution $u : \mathcal{M} \times [0, \infty) \to \mathcal{N}$ to the parabolic equation

$$\begin{cases} \dfrac{\partial u^l}{\partial t} = \Delta u^l + \gamma^{\alpha\beta}\Gamma^l_{ij}(u)\dfrac{\partial u^i}{\partial x^\alpha}\dfrac{\partial u^j}{\partial x^\beta}; \\ u(\cdot, 0) = u_0; \\ u(\cdot, t)|_{\partial\mathcal{M}} = u_0|_{\partial\mathcal{M}}, \end{cases} \tag{1}$$

where Δ is the Laplace-Beltrami operator on $\mathcal{M}$ and Γ^l_{ij} denote the Christoffel symbols of the target $\mathcal{N}$. We refer to this equation as the 'heat equation,' to the map u_0 as the 'initial map,' and to the map $u_0|_{\partial\mathcal{M}}$ as the boundary values. Maps u_0 whose flows u do not vary in time are known as harmonic maps.

For an introduction to the harmonic map flow, the reader should refer to [10, Chapter 1]. However, we briefly survey the main results required for this work.

The basic existence result we will use is due to Struwe [8].

Theorem 1 *Given a regular initial map and boundary values, there exists a solution* $u \in W^{1,2}_{loc}(\mathcal{M} \times [0, \infty), \mathcal{N})$ *of the heat equation* (1) *which is smooth in* $\mathcal{M} \times (0, \infty)$ *away from at most a finite number of singular points.*

The work of Struwe also gave us the first information on the asymptotics of the flow at infinite time.

Theorem 2 *Let* u *be the solution of the heat equation* (1) *introduced in Theorem 1. Then there exist a sequence of times* $t_i \to \infty$*, a harmonic map* $u_\infty : \mathcal{M} \to \mathcal{N}$ *and a finite set of points* $\{x^1, \ldots x^m\} \subset \mathcal{M}$ *such that*

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(i) $u(\cdot,t_i) \rightharpoonup u_\infty$, *weakly in* $W^{1,2}(\mathcal{M})$ *(and hence strongly in* L^p *for any* $1 \leq p < \infty$*) as* $i \to \infty$,

(ii) $u(\cdot,t_i) \to u_\infty$, *strongly in* $W^{2,2}_{loc}(\mathcal{M}\backslash\{x^1,\ldots x^m\})$ *as* $i \to \infty$.

The asymptotic behaviour of the flow at the points $\{x^i\}$ and the behaviour near the finite number of singularities mentioned in Theorem 1 is similar. In both cases the flow 'blows up', and 'bubbling' occurs; for this reason we also refer to the singular points as 'blow-up' points. To describe the bubbling, let us assume that $\mathcal{M}$ is the standard 2-disc D.

Theorem 3 *Let* u *be a solution of* (1) *from Theorem 1. Let* $(x_0,T) \in \mathcal{M} \times (0,\infty]$ *be a singular point of the flow - in other words either one of the singular points with* $T < \infty$ *mentioned in Theorem 1 or a point* (x_0,∞) *with* $x_0 \in \{x^1,\ldots x^m\}$*. Then there exist sequences* $a_i \to x_0$, $t_i \uparrow T$, $R_i \downarrow 0$ *and a nonconstant harmonic map* $\overline{u_0} : \mathbb{R}^2 \to \mathcal{N}$ *such that as* $i \to \infty$,

$$u(a_i + R_i x, t_i) \to \overline{u_0} \text{ in } W^{2,2}_{loc}(\mathbb{R}^2,\mathcal{N}).$$

Moreover, $\overline{u_0}$ *extends via stereographic projection to a smooth harmonic map* $S^2 \to \mathcal{N}$ *which we refer to as a 'bubble'.*

An immediate question that this result of Struwe poses is whether the bubble extracted accounts entirely for the singularity. In other words, writing $E_U(v)$ for the energy of v over a small neighbourhood U of x_0, do we have equality in the inequality

$$\lim_{i\to\infty} E_U(u(\cdot,t_i)) \geq E_U(u(\cdot,T)) + E(\overline{u_0}), \tag{2}$$

and does the bubble account for the change in homotopy class of the flow? If this were not the case, then more than one bubble would have to develop at the same point - possibly at different scales - and the sum of the energies of the bubbles would account for the total loss in energy. This is expressed in the following result of Ding-Tian [3], Qing [7] and Wang [11].

Theorem 4 *Let* (x_0,T) *be a singular point of the flow (with* $T = \infty$ *permitted) and suppose that* U *is a neighbourhood of* x_0 *small enough so that* $U \times \{T\}$ *contains no other singular points. Then there exist finitely many nonconstant harmonic maps* $\{\omega_k\}_{k=1}^m$ *from* S^2 *to* $\mathcal{N}$ *which we see as maps from* $\mathbb{R}^2$ *by stereographic projection, together with sequences*

(i) $\{t_i\}$ *with* $t_i \uparrow T$,

(ii) $\{\{a_i^k\}\}_{k=1}^m$ *in* $\mathbb{R}^2$ *with* $\lim_{i\to\infty} a_i^k = x_0$ *for* $1 \leq k \leq m$, *and*

(iii) $\{\{\lambda_i^k\}\}_{k=1}^m$ *with* $\lambda_i^k > 0$ *for* $1 \leq k \leq m$ *and any* i, *and* $\lim_{i\to\infty} \lambda_i^k = 0$ *for* $1 \leq k \leq m$,

such that

$$\frac{\lambda_i^k}{\lambda_i^j} + \frac{\lambda_i^j}{\lambda_i^k} + \frac{|a_i^k - a_i^j|^2}{\lambda_i^k \lambda_i^j} \to \infty \qquad \text{as } i \to \infty, \tag{3}$$

and

$$\lim_{i\to\infty} E_U(u(\cdot,t_i)) = E_U(u(\cdot,T)) + \sum_{k=1}^m E(\omega_k), \tag{4}$$

and moreover,

$$u(x,t_i) - \sum_{k=1}^m \left(\omega_k \left(\frac{x - a_i^k}{\lambda_i^k} \right) - \omega_k(\infty) \right) \to u(x,T)$$

strongly in $W^{1,2}(U,\mathcal{N})$ *as* $i \to \infty$, *where in the case that* $T = \infty$, *we read* u_∞ *for* $u(\cdot,T)$.

The main result of this article is that there do exist flows in which more than one bubble develops at the same point, and at different scales. In other words, nested bubble trees exist.

Theorem 5 *There exist a target manifold $\mathcal{N}$, and an initial map $u_0 : D \to \mathcal{N}$ where D is a 2-disc, such that the subsequent flow u blows up at infinite time at precisely one point, but so that upon analysing the blow-up with Theorem 4, we must have two bubbles developing at that point (in other words $m = 2$). In fact we have $a_i^1 = a_i^2 = 0$ for all $i \in \mathbb{N}$ and consequently the bubbles develop at different scales in that (by swapping the bubbles if necessary) we have $\frac{\lambda_i^1}{\lambda_i^2} \to \infty$ as $i \to \infty$.*

The example relies on the construction of a target with a warped metric. Targets of this type have been used by the author to settle a variety of questions concerning the blow-up and stability of harmonic map flows [10], and may be used to identify huge classes of flows which blow up in finite time.

The remainder of this chapter is devoted to the proof of Theorem 5.

Let the domain $\mathcal{M}$ be the flat 2-disc of radius π with polar coordinates (r, ϕ), and the target $\mathcal{N}$ be $S^2 \times S^2$ with a metric to be described shortly.

We parameterise each S^2 considered in this proof with spherical polar coordinates; the coordinates (θ, ϕ) then correspond to the point $(\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ in cartesian coordinates on $S^2 \hookrightarrow \mathbb{R}^3$. Let $h(\theta, \phi)$ be the standard metric on S^2 at the point (θ, ϕ).

In $\mathcal{N}$ we parameterise the first S^2 by (α, A) and the second by (β, B). At the point (α, A, β, B) in $\mathcal{N}$, we set the metric to be $h(\alpha, A) + f(\alpha)h(\beta, B)$, where $f(\alpha) \equiv 1$ would give the standard metric on $S^2 \times S^2$. We choose $f : [0, \pi] \to \mathbb{R}$ to be any smooth function satisfying

(i) $f(\alpha) = 1$ for $0 \le \alpha \le \frac{\pi}{2}$,

(ii) $f'(\alpha) > 0$ for $\frac{\pi}{2} < \alpha < \pi$,

(iii) $v : S^2 \to \mathbb{R}$, defined by $v(\theta, \phi) = f(\theta)$, is smooth

So with v as above, we have $\mathcal{N} = S^2 \times_v S^2$. We refer to f as well as v as a warping function.

Let us consider symmetric maps of the form

$$(r, \phi) \to (\alpha(r), \phi, \beta(r), \phi).$$

This symmetry is preserved under the heat flow. We will be using the fixed, constant boundary conditions

$$u(\pi, \phi, t) = (\pi, \phi, \pi, \phi),$$

and will start with the initial map

$$u_0(r, \phi) = (r, \phi, r, \phi). \tag{5}$$

Writing the heat flow as

$$u(r, \phi, t) = (\alpha(r, t), \phi, \beta(r, t), \phi),$$

the evolution equations for α and β according to (1) are

$$\frac{\partial\alpha}{\partial t} = \frac{\partial^2\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \frac{\sin\alpha\cos\alpha}{r^2} - \frac{f'(\alpha)}{2}\left(\frac{\partial\beta}{\partial r}\right)^2 - \frac{f'(\alpha)}{2}\frac{\sin^2\beta}{r^2}, \tag{6}$$

$$\frac{\partial\beta}{\partial t} = \frac{\partial^2\beta}{\partial r^2} + \frac{1}{r}\frac{\partial\beta}{\partial r} - \frac{\sin\beta\cos\beta}{r^2} + \frac{f'(\alpha)}{f(\alpha)}\frac{\partial\alpha}{\partial r}\frac{\partial\beta}{\partial r}. \tag{7}$$

By the symmetry imposed, and the finiteness of any blow-up points, blow-up may only occur at $r = 0$. We will argue that both α and β blow up at infinite time, and that rescaling to capture a bubble will not account for the blow-up of both α and β - they must blow up at different rates. The first step is to argue that both α and β blow up at some stage. This is evidently true since any harmonic map from a contractible surface with constant boundary values is constant according to a theorem of Lemaire [5, Theorem 3.2] and the initial map (5) is homotopically nontrivial after projections onto either S^2 of the target $\mathcal{N}$. (Therefore the projection of the flow onto either S^2 of the target $\mathcal{N}$ must change homotopy class at some point - possibly at 'infinite time'.)

The next step is to argue that α and β do not blow up at finite time. For this, we will employ the following result from [2] which Chang and Ding used to establish a global existence result for certain symmetric harmonic map flows from the disc to the 2-sphere.

Lemma 1 *Suppose that $d > 0$ and we have a smooth function $h_0 : [0, d] \to [0, \pi]$ with $h_0(0) = 0$ and $h_0(d) = \pi$. Then we may find a (unique) smooth solution $h : [0, d] \times [0, \infty) \to [0, \pi]$ of*

$$\begin{cases} \dfrac{\partial h}{\partial t} = \dfrac{\partial^2 h}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial h}{\partial r} - \dfrac{\sin h \cos h}{r^2}, \\ h(\cdot, 0) = h_0, \\ h(0, t) = 0, \ h(d, t) = \pi. \end{cases} \tag{8}$$

We will use Lemma 1 to generate supersolutions to the solutions of (6) and (7). Finite-time singularities in the heat flow will then be ruled out by what we learnt about blow-up in Theorem 3. To begin with, we set $d = \pi$, and choose h_0 to be any function as in Lemma 1 with $h_0(r) > r = \alpha(r, 0)$ when $r \in (0, \pi)$. Applying Lemma 1 to get a function h, and comparing this function to the solution α of (6) using the parabolic maximum principle (as in [2] and [6]) we see that

$$h(r, t) > \alpha(r, t) \qquad \text{for all } (r, t) \in (0, \pi) \times [0, \infty).$$

(Note that we are using the fact that the final two terms in (6) are negative.) Consequently, we see that α does not blow up in finite time.

Next we turn to β, the solution of (7). Suppose that β blows up in finite time - at time $t = T$ say. As α exists for all time, we know that $\frac{\partial \alpha}{\partial r}$ is bounded for $t \in [0, T]$. In particular, as $\alpha(0, t) = 0$ for all t, there exists $d \in (0, \pi)$ such that if $(r, t) \in [0, d] \times [0, T]$ then $\alpha(r, t) < \frac{\pi}{2}$, and hence $f'(\alpha(r, t)) = 0$ from condition (i) that we imposed on f. Consequently, β evolves for $(r, t) \in [0, d] \times [0, T]$ under the same equation as h in Lemma 1. Set $h_0 : [0, d] \to [0, \pi]$ to be $h_0(r) = \frac{\pi}{d} r$, so that $h_0(r) > r = \beta(r, 0)$, and apply Lemma 1 to get the corresponding function h. Comparing h with β using the maximum principle as before, we see that

$$h(r, t) > \beta(r, t) \qquad \text{for all } (r, t) \in (0, d) \times [0, T],$$

contradicting the fact that β blows up at time $t = T$. Hence β does not blow up in finite time.

So we have established that α and β blow up at infinite time, and not at finite time. It remains to prove that there does not exist one bubble which can account for the blow-up of both α and β.

We will use several times the following consequence of Lemaire's theorem [5, Theorem 3.2] that a harmonic map with constant boundary values from a contractible surface is constant.

Lemma 2 *For any $0 < d < \pi$, there does not exist a smooth solution $g : [0, d] \to [0, \pi]$ to*

$$\begin{cases} 0 = \dfrac{d^2 g}{d\theta^2} + \dfrac{1}{\tan \theta}\dfrac{dg}{d\theta} - \dfrac{\sin g \cos g}{\sin^2 \theta}, \\ g(0) = 0, \ g(d) = \pi. \end{cases} \tag{9}$$

Proof. The reason for this is simply because if g were a solution to (9) then we would have a nonconstant harmonic map with constant boundary values

$$(\theta, \phi) \to (g(\theta), \phi)$$

from the part of S^2 with $0 \leq \theta \leq d$ to S^2, which would contradict Lemaire's result. ∎

Suppose one bubble accounts for the blow-up of both α and β. Then the bubble would be a harmonic map from S^2 to $\mathcal{N}$ which we could write as

$$(\theta, \phi) \to (\alpha(\theta), \phi, \beta(\theta), \phi), \tag{10}$$

where α and β are smooth functions with

$$\alpha(0) = \beta(0) = 0, \qquad \alpha(\pi) = \beta(\pi) = \pi \qquad 0 \leq \alpha, \beta \leq \pi, \tag{11}$$

satisfying the system

$$0 = \frac{d^2\alpha}{d\theta^2} + \frac{1}{\tan\theta}\frac{d\alpha}{d\theta} - \frac{\sin\alpha\cos\alpha}{\sin^2\theta} - \frac{f'(\alpha)}{2}\left(\frac{d\beta}{d\theta}\right)^2 - \frac{f'(\alpha)}{2}\frac{\sin^2\beta}{\sin^2\theta}, \tag{12}$$

$$0 = \frac{d^2\beta}{d\theta^2} + \frac{1}{\tan\theta}\frac{d\beta}{d\theta} - \frac{\sin\beta\cos\beta}{\sin^2\theta} + \frac{f'(\alpha)}{f(\alpha)}\frac{d\alpha}{d\theta}\frac{d\beta}{d\theta}. \tag{13}$$

However, such a bubble cannot exist:

Lemma 3 *There do not exist any harmonic maps $S^2 \to \mathcal{N}$ of the form* (10) *satisfying* (11), (12) *and* (13).

Proof. Suppose such a map exists; we will find that (12) contains a contradiction. Let us multiply (12) by $2\sin^2\theta\frac{d\alpha}{d\theta}$, and integrate over the region $(0, \theta)$. We find that

$$\begin{aligned} \sin^2\theta\left(\frac{d\alpha}{d\theta}\right)^2 &= \sin^2\alpha + \int_0^\theta f'(\alpha(\zeta))\sin^2\zeta\left(\frac{d\beta}{d\theta}(\zeta)\right)^2\frac{d\alpha}{d\theta}(\zeta)d\zeta \\ &\quad + \int_0^\theta f'(\alpha(\zeta))\sin^2\beta(\zeta)\frac{d\alpha}{d\theta}(\zeta)d\zeta. \end{aligned} \tag{14}$$

Define

$$\theta_0 = \inf\{\theta \in (0, \pi) \mid \alpha(\theta) = \frac{\pi}{2}\} \in (0, \pi).$$

So for $\theta \in (0, \theta_0)$ we have $f'(\theta) = 0$ by condition (i) which we imposed on f.

Of course $\alpha(\theta_0) = \frac{\pi}{2}$, but we also know that $\beta(\theta_0) < \pi$. This is because otherwise we would have $\beta(\theta_0) = \pi$ and by setting $d = \theta_0$ and $g = \beta|_{[0,\theta_0]}$ and applying Lemma 2, we would have a contradiction.

Next, by setting $\theta = \theta_0$ in (14) we see that $\frac{d\alpha}{d\theta}(\theta_0) \neq 0$, and hence that

$$\frac{d\alpha}{d\theta}(\theta_0) > 0.$$

Suppose that $\frac{d\alpha}{d\theta}(\theta) > 0$ for all $\theta \in (\theta_0, \pi)$. Then by setting $\theta = \pi$ in (14) we have a contradiction - for $\theta \in (\theta_0, \pi)$ we would have $f'(\theta) > 0$, $\frac{d\alpha}{d\theta}(\theta) > 0$ and $\sin^2\beta$ not identically equal to zero, so the final term of (14) and thus the whole right-hand-side, would be strictly positive.

So we must have $\theta \in (\theta_0, \pi)$ with $\frac{d\alpha}{d\theta}(\theta) = 0$. Set

$$\theta_1 = \inf\{\theta \in (\theta_0, \pi) \mid \frac{d\alpha}{d\theta}(\theta) = 0\} \in (\theta_0, \pi).$$

Putting $\theta = \theta_1$ into (14), the left hand side is zero, and all the terms on the right are nonnegative, so they must all be zero. In particular, we must have $\alpha(\theta_1) = \pi$, and $\left(\frac{d\beta}{d\theta}\right)^2 \equiv \sin^2\beta \equiv 0$ on $\theta \in (\theta_0, \theta_1)$. But then setting $d = \theta_1$, and $g = \alpha|_{[0,\theta_1]}$ and applying Lemma 2, we have a contradiction. ■

So one bubble cannot account for the blow-up of both α and β, and we have finished the proof of Theorem 5.

Remark 1 It would be interesting to extend the fact that there is no one suitable bubble which can account for the change in homotopy - as stated in Lemma 3 - to prove that there are no harmonic spheres homotopic to the diagonal embedding

$$(\theta, \phi) \to (\theta, \phi, \theta, \phi).$$

It is not hard to see that there can be no energy minimising harmonic maps in this homotopy class.

Remark 2 The above construction would also work if the domain were the upper hemisphere of S^2 rather than the 2-disc. By connecting two such examples together, we could construct an example in which the domain was S^2 itself.

Remark 3 It is an open question as to whether nontrivial bubble trees may occur at finite time also.

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email: topping@ihes.fr