ALMOST-SCHUR LEMMA

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Abstract. Schur's lemma states that every Einstein manifold of dimension $n \geq 3$ has constant scalar curvature. In this short note we ask to what extent the scalar curvature is constant if the traceless Ricci tensor is assumed to be *small* rather than identically zero. In particular, we provide an optimal L^2 estimate under suitable assumptions and show that these assumptions cannot be removed.

0. Introduction

Schur's lemma states that every Einstein manifold of dimension $n \geq 3$ has constant scalar curvature. Here (M, g) is defined to be Einstein if its traceless Ricci tensor

$$\overset{\circ}{\mathrm{Ric}} := \mathrm{Ric} - \frac{R}{n} g$$

is identically zero.

In this short note we ask to what extent the scalar curvature is constant if the traceless Ricci tensor is assumed to be *small* rather than identically zero. As it is customary, we say that M is a closed manifold if it is compact and without boundary.

Theorem 0.1. *For any integer $n \geq 3$, if (M, g) is a closed Riemannian manifold of dimension n with nonnegative Ricci curvature, then*

$$\int_M \left(R - \overline{R}\right)^2 \;\leq\; \frac{4n(n-1)}{(n-2)^2} \int_M |\overset{\circ}{Ric}|^2 \tag{0.1}$$

where $\overline{R}$ is the average value of R over M. Moreover equality holds if and only if (M, g) is Einstein.

Since

$$\left|\mathrm{Ric} - \tfrac{\overline{R}}{n}\, g\right|^2 \;=\; |\overset{\circ}{\mathrm{Ric}}|^2 + \frac{1}{n}\left(R - \overline{R}\right)^2, \tag{0.2}$$

we immediately get:

Corollary 0.2. *Under the same conditions as in the theorem,*

$$\int_M \left|Ric - \tfrac{\overline{R}}{n}\, g\right|^2 \;\leq\; \frac{n^2}{(n-2)^2} \int_M \left|Ric - \tfrac{R}{n}\, g\right|^2 , \tag{0.3}$$

and equality holds if and only if (M, g) *is Einstein.*

These estimates are sharp in the following senses.

First, the constants are the best possible because if we were to reduce either constant the inequalities would fail for certain small but high-frequency deformations of the round sphere as we discuss in Section 2. Indeed, if g_0 is the metric of the round sphere then we can take a conformal deformation $(1 + f)g_0$ where f is an eigenfunction of the Laplacian on the sphere corresponding to a suitably large eigenvalue.

Second, the curvature condition $\mathrm{Ric} \geq 0$ cannot simply be dropped, as we discuss in Section 3: For $n \geq 5$, we show that any such inequality then fails even if we restrict M to be diffeomorphic to the sphere. For example, we can find metrics g on S^n which make the ratio of the left-hand side of (0.3) to the right-hand side of (0.3) arbitrarily large. If we are able to prescribe the topology of M, then the same thing can be engineered even in dimension $n = 3$: we can find manifolds (M^3, g) which make the same ratio arbitrarily large. We leave open the possibility that inequalities of this form may hold for $n = 3$ and $n = 4$ with constants depending on the topology of M. We finally mention that Ge and Wang in [2] have followed up on the first version of this paper by demonstrating that for four dimensional manifolds our hypothesis can be weakened to nonnegative *scalar* curvature. This is surely not possible for $n \geq 5$ (as can be shown using constructions similar to the ones of Section 3), whereas the case $n = 3$ is still open.

In the context of the sectional Schur's lemma, two results which are somewhat related to ours have appeared in [4] and [3]. However, of all known inequalities which generalise a geometric rigidity statement, the closest one to our result seems to be the inequality of Müller and the first author [1], which generalises the well-known assertion that the only totally umbilic closed surfaces of the Euclidean three dimensional space are spheres. In fact our method also gives that result with the sharp constant for convex hypersurfaces of any dimension, even within more general Einstein ambient manifolds; details will appear in [5]. As with the proof in this paper, our method in that case has the advantage of being completely elementary, whereas the proof of [1] exploits deep tools from partial differential equations and its only advantage is that it holds for general smooth surfaces.

1. Proof of Theorem 0.1

1.1. **Proof of** (0.1)**.** Recall that the contracted second Bianchi identity tells us that $\delta\mathrm{Ric}+\frac{1}{2}dR=0$ (where $(\delta\mathrm{Ric})_j:=-\nabla_i R_{ij}$) and hence that

$$\delta\overset{\circ}{\mathrm{Ric}}=-\frac{n-2}{2n}dR. \tag{1.1}$$

Let $f:M\to\mathrm{R}$ be the unique solution to

$$\begin{cases} \Delta f=R-\overline{R} \\ \displaystyle\int_M f=0. \end{cases} \tag{1.2}$$

We may then compute

$$\begin{aligned} \int_M \left(R-\overline{R}\right)^2 &= \int_M \left(R-\overline{R}\right)\Delta f=-\int_M \langle dR, df\rangle \\ &= \frac{2n}{n-2}\int_M \langle \delta\overset{\circ}{\mathrm{Ric}}, df\rangle = \frac{2n}{n-2}\int_M \langle \overset{\circ}{\mathrm{Ric}}, \mathrm{Hess}f\rangle \\ &= \frac{2n}{n-2}\int_M \langle \overset{\circ}{\mathrm{Ric}}, \mathrm{Hess}f-\tfrac{\Delta f}{n}g\rangle \\ &\leq \frac{2n}{n-2}\|\overset{\circ}{\mathrm{Ric}}\|_{L^2}\|\mathrm{Hess}f-\tfrac{\Delta f}{n}g\|_{L^2}. \end{aligned} \tag{1.3}$$

Now by integration by parts (i.e. the Bochner formula) we know that

$$\int_M |\mathrm{Hess}f|^2=\int_M (\Delta f)^2-\int_M \mathrm{Ric}(\nabla f,\nabla f) \tag{1.4}$$

and therefore

$$\begin{aligned} \int_M |\mathrm{Hess}f-\tfrac{\Delta f}{n}g|^2 &= \int_M |\mathrm{Hess}f|^2-\frac{1}{n}(\Delta f)^2 \\ &= \frac{n-1}{n}\int_M (\Delta f)^2-\int_M \mathrm{Ric}(\nabla f,\nabla f) \\ &= \frac{n-1}{n}\int_M (R-\overline{R})^2-\int_M \mathrm{Ric}(\nabla f,\nabla f), \end{aligned} \tag{1.5}$$

and since the Ricci curvature is nonnegative, we have

$$\|\mathrm{Hess}f-\tfrac{\Delta f}{n}g\|_{L^2}\leq\left(\frac{n-1}{n}\int_M \left(R-\overline{R}\right)^2\right)^{\frac{1}{2}}, \tag{1.6}$$

which can be combined with (1.3) to give (0.1).

Remark 1.1. *We note that the Ricci term which we throw away in the proof does not destroy optimality because that term is 'lower order' – i.e. only involves first derivatives – and is thus insignificant for very high frequency f.*

Remark 1.2. *We only use the Ricci hypothesis in the proof in order to obtain the* L^2 *estimate*

$$\int_M |Hessf|^2 \leq \int_M (\Delta f)^2. \tag{1.7}$$

Moreover, a slight adaptation of the proof would establish an L^p *version of our results on any manifold supporting a Calderon-Zygmund inequality*

$$\int_M |Hessf|^p \leq C \int_M (\Delta f)^p. \tag{1.8}$$

1.2. **Equality.** Obviously, if (M,g) is an Einstein manifold, then both sides of (0.1) vanish. Assume next that we have equality in (0.1) for some (M,g). Then equality must hold in (1.3) and (1.6). Equality holds in the latter inequality if $\mathrm{Ric}(\nabla f, \nabla f) = 0$ (see (1.5)) and since $\mathrm{Ric} \geq 0$,

$$\mathrm{Ric}(\nabla f, \cdot) = 0. \tag{1.9}$$

Meanwhile, equality holds in (1.3) if and only if the two tensors $\overset{\circ}{\mathrm{Ric}}$ and $\mathrm{Hess}f - \frac{\Delta f}{n} g$ are linearly dependent. If one of them vanishes, then (1.3) implies that R is constant and hence, since equality in (0.1) holds, that g is Einstein. Otherwise, there is $\mu > 0$ such that $\mu \overset{\circ}{\mathrm{Ric}} = (\mathrm{Hess}f - \frac{\Delta f}{n} g)$. This, together with (1.9) and (1.1) implies that

$$-\frac{n-1}{n} d\Delta f \;=\; \delta\left(\mathrm{Hess}f - \frac{\Delta f}{n} g\right) \;=\; \mu\, \delta \overset{\circ}{\mathrm{Ric}} \;=\; -\mu \frac{n-2}{2n} dR\,. \tag{1.10}$$

Since $\Delta f = R - \bar{R}$, from (1.10) we conclude

$$\left(\frac{n-2}{2n}\mu - \frac{n-1}{n}\right) dR \;=\; 0\,. \tag{1.11}$$

Thus R is a constant (and hence g is Einstein) unless $\mu = \frac{2n-2}{n-2}$. Assuming this is the case, then

$$\mathrm{Hess}f - \frac{\Delta f}{n} g \;=\; \frac{2n-2}{n-2} \overset{\circ}{\mathrm{Ric}}\,. \tag{1.12}$$

Combining (1.9) with (1.12) and the identity $\Delta f = R - \bar{R}$ we infer

$$\mathrm{Hess}f(\nabla f, \cdot) - \frac{R - \bar{R}}{n} df = -\frac{2n-2}{(n-2)n} R\, df,$$

and we may rewrite this last identity as

$$\nabla \frac{|\nabla f|^2}{2} = -\left[\frac{\bar{R}}{n} + \frac{R}{n-2}\right] \nabla f\,. \tag{1.13}$$

Fix a point $x_0 \in M$ and let $\gamma : [0, +\infty[\to M$ be the solution of $\dot\gamma(t) = -\nabla f(\gamma(t))$ with $\gamma(0) = x_0$. Consider $\alpha(t) = f(\gamma(t))$. Then $\alpha'(t) = -|\nabla f(\gamma(t))|^2$ and, by (1.13),

$$\alpha''(t) = -2\left[\frac{\bar R}{n} + \frac{R}{n-2}\right] |\nabla f(\gamma(t))|^2 \ \leq \ 0\,.$$

Thus, α is a bounded nonincreasing concave function on $[0, +\infty[$ and therefore it must be constant. We conclude that $-|\nabla f(x_0)|^2 = \alpha'(0) = 0$. The arbitrariness of x_0 implies that f is constant which completes the proof.

2. Second variation arguments

We will show that the constants in (0.1) and (0.3) are optimal. We do this by computing the second variation formula of each side of the inequalities based at the round sphere of dimension $n \geq 3$. If the constant in either inequality were reduced at all, then we could find small, high-frequency perturbations of the round sphere which violated both estimates.

Optimality of (0.1) *and* (0.3). First of all observe that, by (0.2), the optimality of one inequality implies the optimality of the other. We next consider the standard sphere $M = (\mathbb{S}^n, \sigma)$ for which $\mathrm{Ric} = (n-1)\sigma$ and $R = n(n-1)$, and deform it through a one-parameter family of Riemannian manifolds $M_t = (\mathbb{S}^n, g_t)$ where $g_t = (1 + tf)\sigma$. We assume that $f \in C^\infty(M)$ and $\int_M f = 0$. Set

$$\begin{aligned} F(t) \ &:= \ C \int_{M_t} \left|\mathrm{Ric} - \tfrac{R}{n}\, g\right|^2 - \int_{M_t} \left|\mathrm{Ric} - \tfrac{\bar R}{n}\, g\right|^2 \\ &= \ (C-1)\int |\mathrm{Ric}|^2 - \frac{C}{n}\int \mathrm{R}^2 + \frac{1}{n\mathrm{V}}\left(\int \mathrm{R}\right)^2 \\ &=: \ (C-1)F_1(t) - \frac{C}{n}F_2(t) + \frac{1}{n}F_3(t) \qquad (2.1) \end{aligned}$$

where V is the volume of M_t. We write $d$vol for the volume element. Straightforward calculations (see for instance Section 2.3.1 of [7]) give

$$\partial_t d\mathrm{vol}\big|_0 = \frac{n}{2} f\, d\mathrm{vol} \tag{2.2}$$

$$\frac{d}{dt}\mathrm{V}\Big|_0 = 0 \tag{2.3}$$

$$\partial_t g^{ij}\big|_0 = -f\sigma^{ij} \tag{2.4}$$

$$\partial_t \mathrm{Ric}_{ij}\big|_0 = -\frac{1}{2}\left(\Delta f \sigma_{ij} + (n-2) f_{;ij}\right) \tag{2.5}$$

$$\partial_t \mathrm{R}\big|_0 = -(n-1)\Delta f - (n-1) n f \tag{2.6}$$

$$\frac{d}{dt}\int \mathrm{R}\Big|_0 = 0\,. \tag{2.7}$$

Note that $F'(0) = 0$. We next show that, for any constant $C < n^2(n-2)^{-2}$, there is a choice of f such that $F''(0) < 0$. This will imply the optimality of (0.3) as desired.

We start by remarking that

$$\begin{aligned}
F_2''(0) &= \frac{d}{dt}\left(2\int \mathrm{R}\,\partial_t \mathrm{R} + \int \mathrm{R}^2\,\partial_t d\mathrm{vol}\right) \\
&= 2n(n-1)\int \partial_t^2 \mathrm{R} + 2\int (\partial_t \mathrm{R})^2 \\
&\quad +4n(n-1)\int \partial_t \mathrm{R}\,\partial_t d\mathrm{vol} + n(n-1)\int \mathrm{R}\,\partial_t^2 d\mathrm{vol} \\
&= 2n(n-1)\frac{d^2}{dt^2}\int \mathrm{R} + 2\int (\partial_t \mathrm{R})^2 - n^2(n-1)^2\frac{d^2 V}{dt^2}\,.
\end{aligned} \tag{2.8}$$

Similarly,

$$\begin{aligned}
F_3''(0) &= \frac{d}{dt}\left(-\frac{1}{\mathrm{V}^2}\frac{d\mathrm{V}}{dt}\left(\int \mathrm{R}\right)^2 + \frac{2}{V}\int \mathrm{R}\frac{d}{dt}\int \mathrm{R}\right) \\
&= -n^2(n-1)^2\frac{d^2\mathrm{V}}{dt^2} - \underbrace{\frac{dV}{dt}\frac{d}{dt}\left(\frac{1}{\mathrm{V}^2}\left(\int \mathrm{R}\right)^2\right)}_{=0\text{ by }(2.3)}
\end{aligned}$$

$$
\begin{aligned}
& + \underbrace{\frac{d}{dt}\left(\frac{2}{V}\int \mathrm{R}\right)\frac{d}{dt}\int \mathrm{R}}_{=0 \text{ by } (2.7)} + 2n(n-1)\frac{d^2}{dt^2}\int \mathrm{R} \\
= \;& -n^2(n-1)^2\frac{d^2\mathrm{V}}{dt^2} + 2n(n-1)\frac{d^2}{dt^2}\int \mathrm{R}\,. \qquad (2.9)
\end{aligned}
$$

Finally we compute

$$
F_1''(0) \;=\; \int \partial_t^2|\mathrm{Ric}|^2 + 2\int \partial_t|\mathrm{Ric}|^2\,\partial_t d\mathrm{vol} + \int |\mathrm{Ric}|^2\,\partial_t^2 d\mathrm{vol}\,. \qquad (2.10)
$$

Note that

$$
\begin{aligned}
\partial_t|\mathrm{Ric}|^2\big|_0 &= 2\partial_t\mathrm{Ric}_{ij}\mathrm{Ric}_{kl}g^{ik}g^{jl} + 2\mathrm{Ric}_{ij}\mathrm{Ric}_{kl}\partial_t g^{ik}g^{jl} \\
&= 2(n-1)\,\partial_t R|_0\,. \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t^2|\mathrm{Ric}|^2\big|_0 &= 2\partial_t\left[\partial_t\left(\mathrm{Ric}^{ij}g_{jl}\right)\mathrm{Ric}_{i\alpha}g^{\alpha l}\right] \\
&= 2(n-1)\,\partial_t^2\mathrm{R}\big|_0 + 2\left[\partial_t\left(\mathrm{Ric}_{ij}g^{jl}\right)\partial_t\left(\mathrm{Ric}_{l\alpha}g^{\alpha i}\right)\right] \\
&= 2(n-1)\,\partial_t^2\mathrm{R}\big|_0 - 4(n-1)f\,\partial_t\mathrm{Ric}_{ij}\sigma^{ij} \\
&\quad + 2|\partial_t\mathrm{Ric}|^2 + 2n(n-1)^2f^2 \qquad (2.12)
\end{aligned}
$$

Therefore, we conclude

$$
\begin{aligned}
F_1''(0) &= 2(n-1)\int \partial_t^2\mathrm{R} + 2\int |\partial_t\mathrm{Ric}|^2 - 4(n-1)\int f\,\partial_t\mathrm{Ric}_{ij}\sigma^{ij} \\
&\quad + 2n(n-1)^2\int f^2 + 4(n-1)\int \partial_t\mathrm{R}\,\partial_t d\mathrm{vol} \\
&\quad + (n-1)\int \mathrm{R}\,\partial_t^2 d\mathrm{vol} \\
&= 2(n-1)\frac{d^2}{dt^2}\int \mathrm{R} + 2\int |\partial_t\mathrm{Ric}|^2 - 4(n-1)\int f\,\partial_t\mathrm{Ric}_{ij}\sigma^{ij} \\
&\quad + 2n(n-1)^2\int f^2 - n(n-1)^2\frac{d^2\mathrm{V}}{dt^2} \qquad (2.13)
\end{aligned}
$$

Putting together (2.13), (2.8) and (2.9) we get

$$
\begin{aligned}
F''(0) \;=\; & -\frac{2C}{n}\int (\partial_t R)^2 + 2(C-1)\int |\partial_t\mathrm{Ric}|^2 \\
& -4(C-1)(n-1)\int f\,\partial_t\mathrm{Ric}_{ij}\sigma^{ij} + 2(C-1)n(n-1)^2\int f^2\,. \qquad (2.14)
\end{aligned}
$$

Next, we have

$$\int (\partial_t R)^2 = (n-1)^2 \left(\int (\Delta f)^2 - 2n \int |df|^2 + n^2 \int f^2 \right) \quad (2.15)$$

$$\int f\, \partial_t \mathrm{Ric}_{ij} \sigma^{ij} = (n-1) \int |df|^2 \quad (2.16)$$

$$\begin{aligned} \int |\partial_t \mathrm{Ric}|^2 &= \frac{n}{4} \int (\Delta f)^2 + \frac{n-2}{2} \int (\Delta f)^2 + \frac{(n-2)^2}{4} \int |D^2 f|^2 \\ &= \frac{n(n-1)}{4} \int (\Delta f)^2 - \frac{(n-2)^2(n-1)}{4} \int |df|^2 \end{aligned} \quad (2.17)$$

(where in the last line we used the Bochner formula (1.4)). Assume now that $C = n^2(n-2)^{-2} - \varepsilon$ for some positive ε. Inserting (2.15), (2.16) and (2.17) into (2.14), we conclude

$$F''(0) \le -a(n)\varepsilon \int (\Delta f)^2 + b(n,\varepsilon) \int |df|^2 + c(n,\varepsilon) \int f^2 , \quad (2.18)$$

where the constant a is strictly positive (since $n \ge 3$). By choosing f to be an eigenfunction of the Laplacian with sufficiently large eigenvalue, we then have $F''(0) < 0$ as desired. □

3. Counterexamples without the hypothesis Ric ≥ 0.

Our results assume we are working on a manifold of nonnegative Ricci curvature. We now wish to ask when we have a hope of proving an inequality of the form

$$\int_M \left(R - \overline{R}\right)^2 \le C \int_M |\overset{\circ}{\mathrm{Ric}}|^2 \quad (3.1)$$

on more general manifolds (M, g).

Proposition 3.1. *For any $C < \infty$ and integer $n \ge 5$, there exists a metric g on the sphere S^n such that* (3.1) *fails.*

For smaller n, we know counterexamples only when the topology of M is allowed to depend on C:

Proposition 3.2. *For any $C < \infty$, there exists a closed 3-manifold (M, g) such that* (3.1) *fails.*

Proof. (Proposition 3.1.) All we have to do is to connect two round spheres of radii 1 and 2, say, by a small neck. On the two spherical parts, the traceless Ricci tensor $\overset{\circ}{\mathrm{Ric}}$ is zero. Therefore (for any C) we can make the right-hand side of (3.1) as small as desired for $n \ge 5$, since by scaling down the size of the neck, the integral of $|\overset{\circ}{\mathrm{Ric}}|^2$ over the neck will also be scaled down to as small a value as we wish. Meanwhile, the different radii of the spherical parts ensure that the scalar curvature R is different on each sphere, and thus the left-hand side of (3.1) cannot be small. $\square$

Proof. (Proposition 3.2.) This construction is loosely related to the one above. The basic building block is any hyperbolic (constant sectional curvature -1) 3-manifold (N, h) which fibres over the circle. A result of Thurston implies that if S is a closed surface of genus at least 2, then the 3-manifold arising by gluing the boundary components of $[0, 1] \times S$ using a pseudo-Anosov diffeomorphism of the fibre S must admit a hyperbolic metric ([6]).

Let us write N^m for the m-fold covering of N obtained by taking covers of the base circle, and lift the metric h to a metric $\tilde{h}$ on N^m. We also pick a point p in N and any one point $\tilde{p}$ in each N^m which projects to p under the covering. The idea then, for each $m \in \mathbb{N}$, is to attach one $(N^m, \tilde{h})$ to another scaled copy $(N^m, 2\tilde{h})$ via an m-independent neck attached to small neighbourhoods of $\tilde{p}$ in each N^m, to give a new manifold (M, g). With this construction, the right-hand side of (3.1) is independent of m, but the left-hand side will increase without bound as $m \to \infty$ at an asymptotically linear rate. $\square$

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