J. Eur. Math. Soc. 12, 1429–1451

DOI 10.4171/JEMS/237

Peter Topping

Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics

Received November 22, 2007 and in revised form December 1, 2008

Abstract. By exploiting Perelman's pseudolocality theorem, we prove a new compactness theorem for Ricci flows. By optimising the theory in the two-dimensional case, and invoking the theory of quasiconformal maps, we establish a new existence theorem which generates a Ricci flow starting at an arbitrary incomplete metric, with Gauss curvature bounded above, on an arbitrary surface. The criterion we assert for well-posedness is that the flow should be complete for all positive times; our discussion of uniqueness also invokes pseudolocality.

1. Introduction; Ricci flows on surfaces

Consider a smooth flow $g(t)$ of Riemannian metrics on a manifold $\mathcal{M}$, for t lying within some time interval. We call $g(t)$ a *Ricci flow*—a concept introduced by Hamilton in [11]—if it satisfies the nonlinear PDE

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) \tag{1.1}$$

where $\mathrm{Ric}(g)$ is the Ricci curvature of g at time t. One can view this equation as a type of heat equation for the metric $g(t)$, as discussed, for example, in [20]. When the manifold $\mathcal{M}$ is of dimension two, the Ricci curvature of a metric g can be written in terms of its Gauss curvature K as

$$\mathrm{Ric}(g) = Kg.$$

In particular, the Ricci flow preserves the conformal class in this dimension.

Given an initial metric $\bar{g}$ on a closed manifold $\mathcal{M}^n$, Hamilton showed in [11] that there exists a Ricci flow $g(t)$ for $t \in [0, T]$, for some $T > 0$, with $g(0) = \bar{g}$. The proof was subsequently simplified by DeTurck [6]. See [20] for further details.

Dropping the assumption that $\mathcal{M}$ is compact, Shi [17] proved that if $\bar{g}$ is a complete initial metric of bounded curvature, then a complete Ricci flow $g(t)$ exists on $\mathcal{M}$ for $t \in [0, T]$, for some $T > 0$, with $g(0) = \bar{g}$ and with bounded curvature for each $t \in [0, T]$. Here, by *complete* flow, we mean that $(\mathcal{M}, g(t))$ is complete for each $t \in [0, T]$.

In the case that $\mathcal{M}$ is two-dimensional, there is a wider literature. First, if $\mathcal{M}$ is a manifold with boundary (this is the only point in this paper that a manifold is permitted to

P. Topping: Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK;
url: http://www.warwick.ac.uk/˜maseq

have boundary) then as one might guess by viewing (1.1) as a type of heat equation, one could solve the Ricci flow equation with appropriate boundary conditions. A result in this direction, requiring that the boundary has zero geodesic curvature initially, and flowing with this condition preserved, has been given by Brendle [3]. Second, by exploiting the conformal invariance of the flow in two dimensions, and writing (1.1) as an equation for the conformal factor u of a metric $e^{2u}|dz|^2$, one recovers the so-called fast diffusion equation

$$\frac{\partial u}{\partial t} = e^{-2u}\Delta u, \tag{1.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian with respect to the local complex coordinate $z = x + iy$. (Beware that more than one equation is referred to as the fast diffusion equation in the literature.) This equation arises in a number of contexts in physics, and there is an extensive literature focussing on the case that $(\mathcal{M}, \bar{g})$ is conformally $\mathbb{C}$, which we do not attempt to survey. One highlight is the result of Daskalopoulos and del Pino [5] which, in the language of the present paper, proves the existence of solutions with arbitrary conformal initial metric on $\mathbb{C}$, the solutions being highly nonunique. Related results also appear in the paper of DiBenedetto and Diller [7], and in papers of Esteban, Rodríguez and Vázquez (see for example [16] where some of this work is surveyed).

In this paper, we prove a new compactness theorem for Ricci flows, valid in all dimensions (Theorem 2.1) by exploiting Perelman's pseudolocality theorem [14]. Roughly speaking, the theorem will give us appropriate subconvergence of a sequence of Ricci flows even when the curvature is unbounded at $t = 0$, assuming that the initial metrics of each flow satisfy some weak notion of convergence. We discuss that core result in detail in Section 2; one consequence, relevant to the present discussion, is that it can be optimised in the two-dimensional case using the better curvature pinching results one has in this dimension and the theory of conformal and quasiconformal maps, to prove a Ricci flow existence theorem for arbitrary metrics on arbitrary Riemann surfaces, with the sole hypothesis that the Gauss curvature be bounded above.

In contrast to the result of Shi, say, we permit the initial metric to be incomplete, but our solutions become complete instantaneously, and we make the case that this condition is the natural one for well-posedness.

Theorem 1.1 (2D existence theorem). *Let $\mathcal{M}$ be a smooth surface equipped with a smooth metric $\bar{g}$ which need not be complete, but has Gauss curvature K bounded above. Then there exist $T > 0$ dependent only on the supremum of $K(\bar{g})$, and a smooth Ricci flow $G(t)$ on $\mathcal{M}$, for $t \in [0, T]$, such that $G(0) = \bar{g}$, but $G(t)$ is complete for $t \in (0, T]$.*

The issue of well-posedness of Ricci flows with incomplete initial metrics constitutes a second theme of this paper (the first being our compactness result, Theorem 2.1) and the one we focus on in this section.

Remark 1.2. It will be clear from the proof of Theorem 1.1 that we may choose T to be any positive number for which $K(\bar{g}) < 1/(2T)$. This can be seen to be sharp by considering an initial surface $(\mathcal{M}, \bar{g})$ which is a round 2-sphere. If $K(\bar{g})$ is weakly negative, then the solution can be extended for all $t \in [0, \infty)$.

Remark 1.3. If one were ultimately only interested in two-dimensional Ricci flows, then one would try to exploit the conformal invariance of the Ricci flow in this dimension throughout the proof rather than passing via a compactness theorem valid in all dimensions as we do here.

Remark 1.4. The assumption that $\bar{g}$ is smooth could be relaxed if required.

One can view the flow $G(t)$ of the theorem as a smooth flow which undergoes singular behaviour in the limit $t \downarrow 0$. In particular, we propose the flows arising in this theorem as natural analogues of the *reverse bubbling* harmonic map heat flows constructed in [18] and [2].

Although Theorem 1.1 is sufficiently general to handle some rather bizarre initial metrics, it is useful to consider some rather simple examples to gain some intuition.

Example 1.5. Suppose $(\mathcal{M}, \bar{g})$ is the standard 2-disc. Then the theorem and Remark 1.2 tell us that there exists a Ricci flow $G(t)$ starting at this manifold which is instantaneously complete, and exists for all time. After some further work, one could show that after a short time t, the flow $(\mathcal{M}, G(t))$ will look in the middle roughly like a flat disc of radius $1 - r$, where r behaves like t^2, but at the edge like a metric of very negative constant curvature. Points close to the edge are shot apart very rapidly. More precisely, it will be a byproduct of the discussion in Section 5 that if h is the complete conformal metric on $\mathcal{M}$ of constant curvature -1 (the Poincaré metric) then for all $x, y \in \mathcal{M}$,

$$d_{G(t)}(x, y) \geq \sqrt{2t}\, d_h(x, y), \tag{1.3}$$

where $d_g(x, y)$ represents the geodesic distance between x and y with respect to the metric g.

Theorem 1.1 is sufficiently general that even when we work on the disc, and thus can write $\bar{g} = e^{2\bar{u}}|dz|^2$ for some global conformal factor $\bar{u}$, the behaviour of $\bar{u}$ near "infinity" can be extremely wild. For example, if one takes a compact hyperbolic surface, punctures it a few times, and blows up the metric conformally near some of the punctures to give, say, Euclidean ends, then when one lifts the resulting metric to its universal cover D, the conformal factor oscillates wildly without the Gauss curvature ever having to become very positive.

Example 1.6. Suppose that $(\mathcal{M}, \bar{g})$ is the standard flat Euclidean 2-plane with a point removed. Again, Theorem 1.1 gives us a Ricci flow starting with this incomplete initial metric, which exists for all time. After a short time t, the flow looks roughly like a plane with a disc of radius $r \sim t^2$ removed and replaced with a complete cusp of curvature decreasing to $-1/(2t)$ towards the end. Now, points close to the puncture are shot apart as t increases from zero. One could check that if A and $A_{1/2}$ are the punctured discs consisting of all points in $\mathcal{M}$ within a distance 1 and $1/2$ of the puncture (with respect to $\bar{g}$) respectively, and h is the complete conformal metric of constant curvature -1 on A, then for sufficiently small $t > 0$, and for all $x, y \in A_{1/2}$, we must have

$$d_{G(t)}(x, y) \geq \sqrt{2t}\, d_h(x, y). \tag{1.4}$$

In fact, this particular flow can be written essentially explicitly as a Ricci soliton, although this special feature would typically disappear if we perturbed the initial metric a little.

These examples may give the misleading impression that we should try to make sense of a boundary of our Riemann surfaces. In fact, the issue of boundary conditions has been directly replaced by the assertion that the flows are complete for positive times. One might also be misled that the existence theorem takes some global conformal factor and somehow sets it to infinity at "infinity". One could keep in mind the simple example in which $(\mathcal{M}, \bar{g})$ is the flat Euclidean space. The unique complete, bounded curvature Ricci flow starting at this configuration can be shown to be the stationary one.

We propose the condition of instantaneous completeness as the correct one for well-posedness. For instance, in Example 1.5, one might think that one could feed large amounts of metric in at infinity with a certain amount of freedom to give many different rotationally symmetric solutions starting at the flat disc, which are all instantaneously complete and have bounded curvature away from $t = 0$. However, there is a unique such flow:

Theorem 1.7. *Suppose that $g(t)$ is a smooth rotationally symmetric Ricci flow on the 2-disc D, for $t \in [0, T]$, with bounded curvature on $[\varepsilon, T]$ for all $\varepsilon \in (0, T]$, and such that*

(i) $g(0) = \bar{g}$, *the metric of the flat unit disc;*
(ii) $g(t)$ *is complete for* $t \in (0, T]$.

Then $g(t) = G(t)$ for all $t \in [0, T]$, where $G(t)$ is the flow discussed in Example 1.5.

Remark 1.8. Similarly, it looks *a priori* as if in Example 1.6, the metric near the puncture could blow up in a variety of ways. However, again, one can find very reasonable uniqueness classes. The analogue of Theorem 1.7, which can be proved in a similar way, is that the flow of Example 1.6 is the unique smooth rotationally symmetric Ricci flow starting at the punctured plane which is instantaneously complete and has bounded curvature for $t \geq \varepsilon$, with $\varepsilon > 0$ arbitrary.

Note that in the uniqueness statements of Theorem 1.7 and Remark 1.8, we allow any competing flow $g(t)$ to have Gauss curvature unbounded above, as well as below, as $t \downarrow 0$. To control flows with such uncontrolled curvature, we will need to invoke Perelman's pseudolocality theorem again; the proofs are simplified if one also assumes an upper curvature bound. The requirement that competitors are rotationally symmetric will only be used to get some very rudimentary control on the asymptotics of the conformal factor, and can be weakened.

Remark 1.9. Without the condition of instantaneous completeness, the solution $G(t)$ found by Theorem 1.1 is certainly not unique, as one could take, for instance, the stationary solutions in Examples 1.5 and 1.6 above. However, $G(t)$ is a maximal solution in the sense that if $g(t)$ is any other smooth Ricci flow with $g(0) = G(0)$, then $g(t) \leq G(t)$ throughout $\mathcal{M}$ for as long as the two flows both exist. This will be apparent from the proof, since the flow $G(t)$ will arise as a limit of flows $g_i(t)$, constructed in Section 4, which could be used as barriers for an arbitrary alternative flow $g(t)$.

As a result of this remark, when proving uniqueness results such as in Theorem 1.7 and Remark 1.8, the only difficulty is in proving that our solution is minimal as well as maximal. We will address this in Section 5. By inspection of the proofs there, one can see that we could extend the ideas to give a uniqueness result for fairly general initial metrics, but not yet of the generality of the existence result Theorem 1.1.

So far in this introduction we have focussed on the existence and uniqueness of Ricci flows on surfaces, with the existence part coming from a compactness result valid in all dimensions. We now briefly discuss and motivate that compactness result—Theorem 2.1—before giving details in the next section. Loosely speaking, Hamilton [13] proved that Ricci flows with uniformly bounded curvature (and uniform injectivity radius control) are compact (see Theorem B.5). To address problems such as the existence of Ricci flows with incomplete initial metrics, we require compactness not covered by this result. For example, to construct directly the flow of Example 1.6, one could imagine taking a sequence of complete metrics $\bar{g}_i$ on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ which approximate the flat metric $\bar{g}$ on $\mathbb{R}^2$ away from the puncture (for example, we might ask that $\bar{g}_i$ and $\bar{g}$ agree on $\mathbb{R}^2 \setminus B_{\bar{g}}(0, \varepsilon)$ for sufficiently large i depending on $\varepsilon > 0$) and then considering Shi's Ricci flows $g_i(t)$ with $g_i(0) = \bar{g}_i$. Given the right compactness theorem which could handle the necessarily unbounded curvature of these flows at $t = 0$ (as $i \to \infty$) one could hope to pass to a subsequence and extract a limiting Ricci flow agreeing with $\bar{g}|_{\mathbb{R}^2\setminus\{0\}}$ at $t = 0$, with the required properties. We turn now to make this precise, and prove such a compactness theorem in greater generality.

Added in proof: In the years since this paper was submitted, there have been further investigations into the instantaneously complete Ricci flows we introduced here. In particular, a uniqueness result was proved in [9] and a more refined existence result can be found in [10].

2. Pseudolocality compactness theorem

In order to state the theorem of this section, we require the notion of smooth, pointed (Cheeger–Gromov) convergence with the unusual feature that the limit is not assumed to be complete. We give a precise definition and some important basic consequences of this convergence, for manifolds and flows, in Appendix B. As we describe there, we use the notation $\dashrightarrow$ to emphasise the possibility that the limit might be incomplete, since in that situation we lose the uniqueness of limits: one can take a subdomain of any one limit to give another smaller limit.

The following theorem gives a compactness for Ricci flows $g_i(t)$ which may have unbounded curvature at $t = 0$ as $i \to \infty$, but whose initial metrics $g_i(0)$ enjoy a weak local convergence as $i \to \infty$ to a metric $\bar{g}$. Two limits, $G_{\mathcal{M}}(t)$ and $G_{\mathcal{N}}(t)$, are produced, both of which could be said to represent Ricci flows starting at $(\mathcal{M}, \bar{g})$, in some sense.

Theorem 2.1. *Suppose $(\mathcal{M}, \bar{g}, q)$ is a smooth pointed Riemannian manifold, not necessarily complete, and that $(\mathcal{M}_i, g_i(t), q_i)$ is a sequence of smooth, pointed, complete Ricci flows for $t \in [0, T]$, $T > 0$, with uniformly bounded curvature away from $t = 0$ in the sense that there exists a function M on $(0, T]$ (independent of i) such that for all*

$t_0 \in (0,T]$, $|\mathrm{Rm}(g_i(t))| \leq M(t_0)$ for all $t \in [t_0,T]$. Suppose further that

$$(\mathcal{M}_i, g_i(0), q_i) \dashrightarrow (\mathcal{M}, \bar{g}, q). \tag{2.1}$$

Then there exists a smooth Ricci flow $G_{\mathcal{M}}(t)$ on $\mathcal{M}$, for $t \in [0,T]$, with $G_{\mathcal{M}}(0) = \bar{g}$, such that after passing to a subsequence in i, we have

$$(\mathcal{M}_i, g_i(t), q_i) \dashrightarrow (\mathcal{M}, G_{\mathcal{M}}(t), q)$$

on $[0,T]$ as $i \to \infty$, where we can take the same diffeomorphisms φ_i in the definition of this convergence as in the definition of (2.1).

Moreover, there exists a smooth manifold $\mathcal{N}$, a smooth complete *Ricci flow $G_{\mathcal{N}}(t)$ on $\mathcal{N}$, for $t \in (0,T]$, and a point $Q \in \mathcal{N}$ such that*

$$(\mathcal{M}_i, g_i(t), q_i) \to (\mathcal{N}, G_{\mathcal{N}}(t), Q) \tag{2.2}$$

on the smaller time interval $(0,T]$ as $i \to \infty$, and with the property that for all $t_0 \in (0,T]$, $|\mathrm{Rm}(G_{\mathcal{N}}(t))| \leq M(t_0)$ for all $t \in [t_0,T]$.

There exists a map $I : \mathcal{M} \to \Sigma \subset \mathcal{N}$, sending q to Q, which is an isometry from $(\mathcal{M}, G_{\mathcal{M}}(t))$ to $(\Sigma, G_{\mathcal{N}}(t))$ for each $t \in (0,T]$. If ψ_i are the diffeomorphisms from the definition of the convergence (2.2)*—and φ_i are still the diffeomorphisms from the definition of the convergence* (2.1)*—then $\psi_i^{-1} \circ \varphi_i \to I$ smoothly on compact subsets of $\mathcal{M}$ as $i \to \infty$.*

In order to digest this result more easily, one could imagine $(\mathcal{M}, \bar{g})$ to be the flat unit 2-disc as in Example 1.5. If we hoped to construct a Ricci flow continuation of this metric which was complete for $t > 0$, then we might try to do so by taking a limit of complete Ricci flows $g_i(t)$ whose initial metrics $g_i(0)$ approximated $\bar{g}$ in some sense. For example, imagine $g_i(0)$ to be the metric $\bar{g}$ on most of the interior, but made complete by blowing up nearer and nearer to the edge as $i \to \infty$.

One problem which arises with this approach is that we could lose the initial condition in the limit $i \to \infty$. In other words, because the curvature of $g_i(0)$ will typically be blowing up as $i \to \infty$, the time for the flow $g_i(t)$ to move far from $\bar{g}$ on some interior region might be decreasing to zero as $i \to \infty$. It turns out that this could really happen if the approximating flows $g_i(t)$ were not assumed to be complete. However, by virtue of the completeness, we can invoke a remarkable recent "pseudolocality" result of Perelman [14] which will prevent this loss of initial condition. We briefly clarify and survey pseudolocality, and some consequences, in Appendix A.

Although Theorem 2.1 escapes the issue of "loss of initial conditions", a result of this generality cannot escape the problem of "loss of completeness". For example, in the discussion above of constructing a flow starting at the flat unit 2-disc, the initial metrics $g_i(0)$ have been made complete by blowing them up near the edge, but there is no guarantee that this stretching must diffuse into the interior sufficiently fast as t increases from zero to force any limiting flow to be complete. In the language of the theorem, the limit $G_{\mathcal{M}}(t)$ could end up as the stationary flow $G_{\mathcal{M}}(t) = \bar{g}$ for all t, or some other incomplete flow: Σ could be strictly smaller than $\mathcal{N}$.

More generally, if $G_{\mathcal{M}}(t)$ is ever complete, then we can replace $\mathcal{M}$ by a subdomain, and reapply the theorem to get the same limit $(\mathcal{N}, G_{\mathcal{N}}(t))$ but a smaller Σ. However, $(\mathcal{M}, G_{\mathcal{M}}(t))$ always arises as a part of a larger complete flow $(\mathcal{N}, G_{\mathcal{N}}(t))$ for $t > 0$, and in Section 3, we will restrict to a situation in which we can establish that $G_{\mathcal{M}}(t)$ is itself complete for $t > 0$, or equivalently that $\mathcal{N} = \Sigma$ so that $(\mathcal{M}, G_{\mathcal{M}}(t))$ and $(\mathcal{N}, G_{\mathcal{N}}(t))$ are isometric for $t > 0$. In that case, we propose $G_{\mathcal{M}}(t)$ as the natural Ricci flow continuation of $(\mathcal{M}, G_{\mathcal{M}}(t))$.

Proof of Theorem 2.1. Throughout this proof, $\varepsilon > 0$ will be the positive constant whose existence is asserted by the pseudolocality result Corollary A.5. Given a point $x_0 \in \mathcal{M}$, let $r > 0$ be sufficiently small so that $B_{\bar{g}}(x_0, 2r) \subset\subset \mathcal{M}$ and $(\varepsilon r)^2 \leq T$. After picking $k \in \mathbb{N}$, by reducing $r > 0$ further if necessary (roughly speaking so that $B_{\bar{g}}(x_0, 2r)$ looks sufficiently like a Euclidean ball) and by exploiting the definition of the convergence (2.1) (denoting the diffeomorphisms associated to that convergence by φ_i still) we may also assume, for sufficiently large i, that $|\nabla^l \mathrm{Rm}(g_i(0))| \leq r^{-2}$ on $B_{g_i(0)}(\varphi_i(x_0), r)$, for $l \in \{0, \ldots, k\}$, and that $\mathrm{Vol}(B_{g_i(0)}(\varphi_i(x_0), r)) \geq (1-\varepsilon)\omega_n r^n$, where ω_n is the volume of the unit ball in Euclidean n-space.

This puts us in a position to apply Corollary A.5 to deduce that for all $t \in [0, (\varepsilon r)^2]$ and $x \in B_{g_i(0)}(\varphi_i(x_0), \varepsilon r)$, we have $|\nabla^k \mathrm{Rm}(g_i(t))|(x) \leq C r^{-2-k}$ for sufficiently large i.

Note that strictly speaking, the statement of Corollary A.5 required the curvature of $g_i(0)$ to be bounded, which we have not assumed. However, by readjusting $t = 0$ an arbitrarily small amount, we are free to draw the same conclusions in our situation.

Let us abuse notation by denoting also by $g_i(t)$ the flows on or within $\mathcal{M}$ obtained by pulling back the $g_i(t)$ by the diffeomorphisms φ_i. By what we have seen, for all $k \in \mathbb{N}$ and $\hat{\Omega} \subset\subset \mathcal{M}$, there exists $\hat{T} \in (0, T]$ (depending on $\hat{\Omega}$) and $C < \infty$ (depending on $\hat{\Omega}$ and k) such that

$$|\nabla^k \mathrm{Rm}(g_i(t))|(x) \leq C$$

for $(x, t) \in \hat{\Omega} \times [0, \hat{T}]$ and sufficiently large i.

By virtue of this curvature control, we may work directly with the Ricci flow equation (1.1) (cf. Lemma 2.4 in [13] and the remarks following its proof) to argue that there exists a Ricci flow $\hat{G}(t)$ on $\hat{\Omega}$ for $t \in [0, \hat{T}]$ such that $\hat{G}(0) = \bar{g}$ on $\hat{\Omega}$, and after passing to a subsequence in i,

$$g_i(t) \to \hat{G}(t) \tag{2.3}$$

smoothly locally on $\hat{\Omega} \times [0, \hat{T}]$. In particular, we have the convergence

$$(\mathcal{M}_i, g_i(t), q_i) \dashrightarrow (\hat{\Omega}, \hat{G}(t), q), \tag{2.4}$$

on $[0, \hat{T}]$ as $i \to \infty$, whenever $\hat{\Omega} \ni q$, using the diffeomorphisms φ_i.

A first consequence of this convergence, arrived at by taking any $\hat{\Omega} \subset\subset \mathcal{M}$ containing q and setting $\bar{t} := \hat{T}$ is that there exists $\bar{t} \in (0, T]$ for which the injectivity radius of $g_i(\bar{t})$ at q_i is bounded below by a positive constant independent of i.

This puts us in a situation in which we can apply Hamilton's compactness of Ricci flows (Theorem B.5) with the time zero of that theorem corresponding to time $\bar{t}$ in the present situation. That theorem gives us a manifold $\mathcal{N}$, a complete Ricci flow $G_{\mathcal{N}}(t)$ on $\mathcal{N}$, for $t \in (0, T]$, and a point $Q \in \mathcal{N}$ such that

$$(\mathcal{M}_i, g_i(t), q_i) \to (\mathcal{N}, G_{\mathcal{N}}(t), Q) \tag{2.5}$$

on $(0, T]$ as $i \to \infty$. We denote the diffeomorphisms involved in this convergence by ψ_i.

By restricting the convergence statements (2.4) and (2.5) to some fixed $t \in (0, \hat{T}]$ (to give convergence in the sense of Definition B.1) and applying Lemma B.3, we find that there exists a map $\hat{I} : (\hat{\Omega}, \hat{G}(t)) \to (\mathcal{N}, G_{\mathcal{N}}(t))$ isometric onto its image, which, after taking another subsequence, arises as the smooth limit, on compact subsets of $\hat{\Omega}$, of $\psi_i^{-1} \circ \varphi_i$. Clearly, the map $\hat{I}$ can be taken to be independent of the $t \in (0, \hat{T}]$, since the diffeomorphisms ψ_i and φ_i have no t dependency. Consequently, we may extend $\hat{G}(t)$ smoothly to the whole time interval $[0, T]$, on $\hat{\Omega}$, by pulling back the metric $G_{\mathcal{N}}(t)$ under $\hat{I}$. Moreover, we then have the flow convergence (2.4) on the whole time interval $[0, T]$.

Now that $\hat{G}(t)$ is defined on a time interval independent of $\hat{\Omega}$, we may exhaust $\mathcal{M}$ by such subsets $\hat{\Omega}$, and thus extend $\hat{G}(t)$ to a Ricci flow $G_{\mathcal{M}}(t)$ on the whole of $\mathcal{M}$, with $G_{\mathcal{M}}(0) = \bar{g}$. Moreover, we may extend the maps $\hat{I}$ to give a map $I : \mathcal{M} \to \mathcal{N}$ which is an isometry from $(\mathcal{M}, G_{\mathcal{M}}(t))$ to its image in $(\mathcal{N}, G_{\mathcal{N}}(t))$ for all $t \in (0, T]$, and arises as a smooth local limit of $\psi_i^{-1} \circ \varphi_i$, after taking a diagonal subsequence. □

We mentioned after the statement of Theorem 2.1 that we cannot generally expect that $\Sigma = \mathcal{N}$ in that theorem. In this paper, we will only try to apply the theorem in the case that $\mathcal{M}_i \subset \mathcal{M}$ is a sequence of subdomains exhausting $\mathcal{M}$, and the diffeomorphisms φ_i associated to the convergence (2.1) are the identity on their domains $\Omega_i \subset \mathcal{M}$. In fact, we will always arrange that on any $\Omega \subset\subset \mathcal{M}$, we have $g_i(0) = \bar{g}$ on Ω for sufficiently large i. Even in this special situation, it can happen that $\Sigma \neq \mathcal{N}$, as we now demonstrate.

Example 2.2. Let $(\mathcal{M}, \bar{g})$ be the flat unit two-dimensional disc D, and let $\mathcal{M}_i = \mathcal{M}$. Choose g_i to be metrics on $\mathcal{M}_i$ which agree with $\bar{g}$ on $D_{1-1/i}$, the disc of radius $1 - 1/i$, but so that $(\mathcal{M}_i, g_i)$ is isometric to flat Euclidean 2-space. These metrics, being flat, are stationary Ricci flows $g_i(t)$ for $t \in [0, 1]$, say. After setting q to be the origin in D, we may apply Theorem 2.1. Clearly, $(\mathcal{N}, G_{\mathcal{N}}(t))$ is again the stationary Ricci flow which is flat Euclidean 2-space for all $t \in [0, 1]$. However, Σ is just a unit disc within this space.

More elaborate examples could have $\mathcal{M}$ and Σ simply connected, but $\mathcal{N}$ multiply connected, for example. The particular example of Example 2.2 has been chosen to contrast with our next theorem, in the next section, in which we will make further hypotheses in order to guarantee that $\Sigma = \mathcal{N}$.

3. Surface flow compactness theorem

In this section we specialise Theorem 2.1 to the case that the dimension n of $\mathcal{M}$ is two, and the metrics $g_i(0)$ are each conformally equivalent to $\bar{g}$. As we mentioned in the introduction, the flows $g_i(t)$ for $t > 0$ will then also be conformally equivalent to $\bar{g}$ because $n = 2$.

The main result in this situation is that despite the discussion in Section 2, and Example 2.2 in particular, we will then be sure that $G_{\mathcal{M}}(t)$ is complete. This will then simplify and substantially strengthen the conclusions.

One rough way of viewing this result is that complete flows $g_i(t)$ for which the initial $g_i(0)$ approximate $\bar{g}$, must experience great stretching of lengths as t increases from 0, in order for their limit $G_{\mathcal{M}}(t)$ to send any 'boundary' of $(\mathcal{M}, \bar{g})$ instantaneously to infinity, and make this 'pseudolocality limit' instantaneously complete. Some concrete estimates of this form, for concrete examples, were given in Examples 1.5 and 1.6.

Theorem 3.1. *Suppose that $\mathcal{M}$ is a smooth surface equipped with a metric $\bar{g}$, which need not be complete, and that $\mathcal{M}_i \subset \mathcal{M}$ is a sequence of subdomains exhausting $\mathcal{M}$.*

Suppose that $g_i(t)$ is a sequence of smooth complete Ricci flows on $\mathcal{M}_i$, for $t \in [0, T]$, $T > 0$, each conformally equivalent to $\bar{g}$, with uniformly bounded curvature away from $t = 0$ in the sense that there exists a function M on $(0, T]$ (independent of i) such that for all $t_0 \in (0, T]$, $|K(g_i(t))| \leq M(t_0)$ for all $t \in [t_0, T]$.

Suppose further that for all $\hat{\Omega} \subset\subset \mathcal{M}$, we have $g_i(0) = \bar{g}$ on $\hat{\Omega}$ for sufficiently large i.

Then there exists a smooth Ricci flow $G(t)$ on $\mathcal{M}$, for $t \in [0, T]$, which is complete *for $t > 0$, such that after passing to a subsequence in i, we have*

$$g_i(t) \to G(t) \tag{3.1}$$

smoothly locally in $\mathcal{M}\times[0, T]$ as $i \to \infty$. In particular, $G(0) = \bar{g}$, and for all $t_0 \in (0, T]$, $|K(G(t))| \leq M(t_0)$ when $t \in [t_0, T]$.

Of course, the local convergence (3.1) makes sense over the whole of $\mathcal{M}$, because $g_i(t)$ will eventually be defined on any given compact subset of $\mathcal{M}$.

Before starting the proof, we survey what needs to be done, and assemble some tools.

Under the hypotheses of this theorem, for any sequence of compact $\Omega_i \subset \mathcal{M}$ exhausting $\mathcal{M}$, with $q \in \Omega_i$, we can pass to a subsequence in i for the sequences in the theorem (leaving the sequence Ω_i intact) and be sure that $\Omega_i \subset \mathcal{M}_i$ and $g_i(0) = \bar{g}$ on Ω_i for each i. In particular, letting $\varphi_i : \Omega_i \to \mathcal{M}_i$ be the identity map, and setting $q_i := q$ for each i, we find that (2.1) is satisfied, and we may apply Theorem 2.1.

All we need check is that when we do this, we have $\Sigma = \mathcal{N}$, or equivalently that $G_{\mathcal{M}}(t)$ is complete for each $t \in (0, T]$. The $G(t)$ of the present theorem would then be the $G_{\mathcal{M}}(t)$ of Theorem 2.1.

The first thing to do is to reduce to the case that $\mathcal{M}$ is simply connected. Let us suppose that we have satisfied the stronger hypotheses of Theorem 3.1 but still have $\Sigma \neq \mathcal{N}$ when applying Theorem 2.1. We will show that if we take $\tilde{\mathcal{M}}$ to be the universal cover of $\mathcal{M}$ (writing $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ for the projection), pick $\tilde{q} \in \pi^{-1}(q)$, define $\hat{\mathcal{M}}_i = \pi^{-1}(\mathcal{M}_i) \subset \tilde{\mathcal{M}}$, and lift the metric $\bar{g}$ to $\tilde{\mathcal{M}}$ and the flows $g_i(t)$ to $\hat{\mathcal{M}}_i$, then when Theorem 2.1 is applied to this lifted situation to give a new pointed Ricci flow $(\hat{\mathcal{N}}, \hat{G}(t), \hat{Q})$, a new flow $\tilde{G}_{\tilde{\mathcal{M}}}(t)$, a new subset $\tilde{\Sigma} \subset \hat{\mathcal{N}}$ and a new map $\tilde{I} : \tilde{\mathcal{M}} \to \tilde{\Sigma}$, then again we must have $\tilde{\Sigma} \neq \hat{\mathcal{N}}$.

Indeed, one can construct a covering map $\hat{\mathcal{N}} \to \mathcal{N}$ which is a local isometry from $(\hat{\mathcal{N}}, \hat{G}(t)) \to (\mathcal{N}, G(t))$ for any $t \in (0, T]$, and which sends $\hat{Q}$ to Q, in the following way. Fix $t \in (0, T]$. A point $x \in \hat{\mathcal{N}}$, once written $x = \exp_{\hat{Q},\hat{G}(t)}(v)$ for some $v \in T_{\hat{Q}}\hat{\mathcal{N}}$, is sent to $\exp_{Q,G(t)}(v)$, after $T_{\hat{Q}}\hat{\mathcal{N}}$ and $T_Q\mathcal{N}$ have been identified using $I \circ \pi \circ \tilde{I}^{-1}$.

The map $\tilde{I} : \tilde{\mathcal{M}} \to \hat{\mathcal{N}}$ is the lift of $I \circ \pi : \tilde{\mathcal{M}} \to \mathcal{N}$, lifting Q to $\hat{Q}$. Therefore, the projection $\hat{\mathcal{N}} \to \mathcal{N}$ restricted to $\tilde{\Sigma}$ can be written explicitly as $I \circ \pi \circ \tilde{I}^{-1}$.

In particular, the image of $\tilde{\Sigma}$ under the projection $\hat{\mathcal{N}} \to \mathcal{N}$ is Σ, and we must have $\tilde{\Sigma} \neq \hat{\mathcal{N}}$.

The argument so far has not used the conformal equivalence of the metrics $g_i(t)$ and $\bar{g}$, which we will shortly exploit to show that $\Sigma = \mathcal{N}$, and in particular that $\hat{\mathcal{N}}$ is simply connected. Without the conformal equivalence, $\hat{\mathcal{N}}$ could be as large as the universal cover of $\mathcal{N}$, but could be as small as $\mathcal{N}$, even in the case that $\mathcal{N}$ is not simply connected. For an example of this latter phenomenon we make the following construction.

Example 3.2. Take $\mathcal{M} = \mathcal{M}_i = D^2$, the two-dimensional disc, with $q_i = q$ the origin, and $\bar{g}$ the standard flat metric. Define $g_i(0)$ to be exactly the standard flat metric on $D_{1-1/i}$, but so that metrically, $(D, g_i(0))$ is the cylinder $(-\infty, i] \times S^1$ capped on the end $\{i\} \times S^1$, with $q \in \{0\} \times S^1$. It is then possible to show that the limit $(\mathcal{N}, G_{\mathcal{N}}(t))$ is a cylinder $\mathbb{R} \times S^1$ for all $t \in (0, T]$. Since $\mathcal{M}$ is simply connected, we must have $\hat{\mathcal{N}} = \mathcal{N}$, which is not simply connected, so we have gained topology in the limit as $i \to \infty$, despite the special situation.

As desired, we have reduced to proving that $\Sigma = \mathcal{N}$, or equivalently that $G_{\mathcal{M}}(t)$ is complete, under the additional assumption that $\mathcal{M}$ is simply connected. By the uniformisation theorem, $(\mathcal{M}, \bar{g})$ is then conformally one of S^2, $\mathbb{C}$ or D.

Moreover, by the definition of the convergence (2.2), $\mathcal{N}$ must be orientable, and we may see it also as a Riemann surface with respect to the unique conformal structure arising from any of the metrics $G_{\mathcal{N}}(t)$, $t \in (0, T]$. The map $I : \mathcal{M} \to \mathcal{N}$ may then be viewed as a conformal injection between Riemann surfaces. (When we say conformal, we mean strictly conformal. That is, the differential vanishes nowhere.)

The following elementary lemma shows that the above observations dramatically restrict the situations we can be in.

Lemma 3.3. *Suppose $I : \mathcal{M} \to \mathcal{N}$ is a conformal injection from a simply connected Riemann surface $\mathcal{M}$ to a Riemann surface $\mathcal{N}$. Then we are in precisely one of the following situations:*

(i) $\mathcal{M} = \mathcal{N} = S^2$ *and* $I : S^2 \to S^2$ *is a Möbius map;*
(ii) $\mathcal{M} = \mathcal{N} = \mathbb{C}$, *and* $I : \mathbb{C} \to \mathbb{C}$ *is a Möbius map;*
(iii) $\mathcal{M} = \mathbb{C}$, $\mathcal{N} = S^2$ *and* $I : \mathbb{C} \to S^2$ *omits one point;*
(iv) $\mathcal{M} = D$.

Proof. The universal cover $\tilde{\mathcal{N}}$ of $\mathcal{N}$ must also be either S^2, $\mathbb{C}$ or D, and the map I may be lifted to a conformal injection $\tilde{I} : \mathcal{M} \to \tilde{\mathcal{N}}$. Keeping in mind Liouville's theorem, if $\mathcal{M} = S^2$ then we must have $\tilde{\mathcal{N}} = S^2$, and we find ourselves in case (i) above.

Similarly, if $\mathcal{M} = \mathbb{C}$, then Liouville tells us that we cannot have $\tilde{\mathcal{N}} = D$. Either $\mathcal{N} = S^2$—in which case by removing the singularity at infinity in $\mathcal{M}$ we end up in case (iii)—or $\tilde{\mathcal{N}} = \mathbb{C}$, and for similar reasons, we have $\tilde{I}$ Möbius and we end up in case (ii). □

Finally, we prove and recall some lemmata concerning quasiconformal maps.

Lemma 3.4. *Suppose $U : \mathcal{R} \to \mathcal{M}$ is a smooth map between Riemann surfaces, which is diffeomorphic onto its image, and preserves orientations. Let z and u be local complex coordinates on the domain and target, the notation u also being used to denote the map U in the complex coordinate. Let G and g be smooth metrics on the domain and target which are compatible with the conformal structures. Then at any point in $\mathcal{R}$ where $|u^*g - G|_G \leq 1$, we have*

$$\frac{|u_{\bar{z}}|}{|u_z|} \leq 4|u^*g - G|_G. \tag{3.2}$$

We note that the left-hand side of (3.2) is invariantly defined. The quantity $|u^*g - G|_G$ is the norm of the tensor $u^*g - G$ with respect to the metric G.

Proof. Locally, let us write $G = \sigma^2|dz|^2$, where $|dz|^2 := dx^2 + dy^2$, $z = x + iy$, and write $g = \rho^2|du|^2$. The pull-back u^*g can be written in terms of the Hopf differential and energy density as

$$u^*g = \rho^2 u_z u_{\bar{z}} dz^2 + \rho^2(|u_z|^2 + |u_{\bar{z}}|^2)|dz|^2 + \rho^2 \bar{u}_z \bar{u}_{\bar{z}} d\bar{z}^2,$$

and so

$$|u^*g - G|_G^2 = \frac{8\rho^4}{\sigma^4}|u_z|^2|u_{\bar{z}}|^2 + 2\left[\frac{\rho^2}{\sigma^2}(|u_z|^2 + |u_{\bar{z}}|^2) - 1\right]^2.$$

Therefore, if $|u^*g - G|_G \leq 1$, we have

$$\left|\frac{\rho^2}{\sigma^2}(|u_z|^2 + |u_{\bar{z}}|^2) - 1\right| \leq \frac{1}{\sqrt{2}},$$

and in particular,

$$\frac{\rho^2}{\sigma^2}(|u_z|^2 + |u_{\bar{z}}|^2) \geq 1 - \frac{1}{\sqrt{2}} > \frac{1}{4},$$

say. Consequently, in this case, we have

$$\begin{aligned} |u^*g - G|_G &\geq \frac{\sqrt{8}\rho^2}{\sigma^2}|u_z|\,|u_{\bar{z}}| \geq \frac{\sqrt{8}(\rho^2/\sigma^2)|u_z|\,|u_{\bar{z}}|}{4(\rho^2/\sigma^2)(|u_z|^2 + |u_{\bar{z}}|^2)} \\ &\geq \frac{1}{2}\frac{|u_z|\,|u_{\bar{z}}|}{|u_z|^2 + |u_{\bar{z}}|^2} \geq \frac{1}{4}\frac{|u_{\bar{z}}|}{|u_z|}, \end{aligned} \tag{3.3}$$

since by hypothesis, the Riemann surfaces are oriented such that $|u_{\bar{z}}| < |u_z|$ rather than the other way round. ◻

We also need a Schwarz-type lemma for quasiconformal maps, which is a form of Mori's Theorem.

Lemma 3.5. *Let $u : D \to D$ be a smooth map from the unit disc in $\mathbb{C}$ to itself, diffeomorphic onto its image, orientation preserving, and with quasiconformal constant less than $K \geq 1$, that is, for which*

$$\frac{|u_{\bar{z}}|}{|u_z|} \leq \frac{K-1}{K+1}.$$

If $u(0) = 0$, then

$$|u(z)| \leq 16|z|^{1/K} \quad \textit{for all } z \in D.$$

Proof. By the Riemann mapping theorem, we can write $u = \xi \circ U$, where $U : D \to D$ is a smooth bijection with quasiconformal constant less than K, with $U(0) = 0$, and $\xi : D \to D$ is a univalent (holomorphic, injective) map, with $\xi(0) = 0$. By Mori's Theorem [1, p. 47], we have

$$|U(z)| \leq 16|z|^{1/K} \quad \text{for all } z \in D,$$

and by the Schwarz–Pick lemma,

$$|\xi(z)| \leq |z| \quad \text{for all } z \in D.$$

Composition then gives the lemma. □

Proof of Theorem 3.1. By the discussion following the statement of Theorem 3.1, our goal is to show that $\Sigma = \mathcal{N}$, that is, $I : \mathcal{M} \to \mathcal{N}$ is onto, and we need only consider the case that $\mathcal{M}$ is simply connected.

By Lemma 3.3, we need only consider the case that $\mathcal{M} = D$, and the case that both $\mathcal{M} = \mathbb{C}$ and $\mathcal{N} = S^2$. This latter case may be discounted immediately since when $\mathcal{N}$ arising as a limit in (2.2) is compact, we must have $\mathcal{M}_i = \mathcal{N}$ for sufficiently large i by the definition of convergence, and hence $\mathcal{M} = \mathcal{N}$: the theorem is obvious for compact $\mathcal{M}$.

We may then assume for the remainder of the proof that $\mathcal{M} = D$. Let us assume that $\mathcal{N} \neq \Sigma$ and try to arrive at a contradiction. In this case, it must be possible to pick $y \in \Sigma$ and a simply connected neighbourhood of y (compactly contained in $\mathcal{N}$) which we may view conformally as the unit disc $\mathcal{D} \subset \mathbb{C}$ (with y at its origin) such that the disc $\mathcal{D}_{1/32}$ of radius $1/32$ about y (with respect to the standard flat metric on $\mathcal{D}$) in that same conformal chart intersects $\mathcal{N} \setminus \Sigma$.

Within any compact subdomain of Σ, the maps ψ_i given by Theorem 2.1 are converging to I^{-1} smoothly, so they are quasiconformal maps with quasiconformal constants converging to 1. However, by definition of the convergence (2.2), we may invoke Lemma 3.4 to see that even the maps ψ_i restricted to $\mathcal{D}$ are quasiconformal maps into $\mathcal{M}_i \subset \mathcal{M} = D$, with quasiconformal constants tending to one. In other words, for $K > 1$ arbitrarily close to 1, after a change in orientation if necessary, we have

$$\sup_{\mathcal{D}} \frac{|\bar{\partial}\psi_i|}{|\partial\psi_i|} \leq \frac{K-1}{K+1} \quad \text{for sufficiently large } i.$$

Moreover, because y was chosen to lie within Σ, we must have $\psi_i(y) \to I^{-1}(y) \in \mathcal{M} = D$, and by making a conformal reparametrisation of $\mathcal{M}$, we may assume that $I^{-1}(y) = O \in D$, the origin.

When this fact is combined with the quasiconformality and Lemma 3.5, we find that for sufficiently large i, we have $\psi_i(\mathcal{D}_{1/32}) \subset D_{3/4} \subset D = \mathcal{M}$, say. In particular, $\psi_i^{-1}(D_{3/4})$ intersects $\mathcal{N} \setminus \Sigma$ for all sufficiently large i. However, on any $\hat{\Omega} \subset\subset \mathcal{M} = D$, we have $\psi_i^{-1} \to I$ uniformly as $i \to \infty$, and so $\psi_i^{-1}(D_{3/4}) \subset \Sigma$ for sufficiently large i, a contradiction. □

4. Ricci flows with incomplete initial metrics

We now wish to use our compactness theorems to prove Theorem 1.1, giving the existence of a Ricci flow starting at an arbitrary metric of Gauss curvature bounded above. Given Theorem 3.1, we are left with the problem of finding appropriate Ricci flows $g_i(t)$ (on an appropriate exhaustion $\mathcal{M}_i$) which approximate the initial metric $\bar{g}$ at time zero.

Proof of Theorem 1.1. In the case that $\mathcal{M}$ is compact, the theorem follows from Hamilton's classical existence theorem, with the control of the maximal existence time in terms of the supremum of K following from the maximum principle. (See [20] for a discussion of these issues.) Therefore, for the remainder of the proof, we assume that $\mathcal{M}$ is noncompact.

Let us define

$$\bar{K} = \max\{\sup_{\mathcal{M}} K(\bar{g}), 0\},$$

a weakly positive upper bound for the Gauss curvature of $\bar{g}$. The first step will be to approximate $\bar{g}$ by complete, conformally equivalent initial metrics $\bar{g}_i$ on appropriate $\mathcal{M}_i \subset \mathcal{M}$ exhausting $\mathcal{M}$ as $i \to \infty$, where $\bar{g}_i$ agree with $\bar{g}$ over any compact subset of $\mathcal{M}$ for sufficiently large i, and where each has Gauss curvature bounded above by $\bar{K}$, and below by some number which may depend on i.

To do this, we take any sequence of subdomains $\mathcal{M}_i \subset \mathcal{M}$ with smooth (one-dimensional) boundaries, such that $\mathcal{M}_i \subset\subset \mathcal{M}_{i+1}$ for all i, and $\mathcal{M}_i$ exhausts $\mathcal{M}$ as $i \to \infty$. Next we take the unique complete conformal metric on $\mathcal{M}_i$ of curvature -1 on the surface $\mathcal{M}_i$, and shrink it homothetically to give a metric h_i of perhaps very negative curvature, so that

$$h_i \leq e^{-2}\bar{g} \quad \text{on } \mathcal{M}_{i-1} \text{ for } i > 1. \tag{4.1}$$

We will define the metrics $\bar{g}_i$ on $\mathcal{M}_i$ as interpolations between h_i and $\bar{g}|_{\mathcal{M}_i}$ as we now describe. Define a function $w_i : \mathcal{M}_i \to \mathbb{R}$ by the relation

$$h_i = e^{2w_i}\bar{g}.$$

By (4.1), we have $w_i \leq -1$ on $\mathcal{M}_{i-1}$.

We observe that $w_i(x) \to \infty$ as $x \in \mathcal{M}_i$ tends to any point on the boundary $\partial\mathcal{M}_i$. In fact, although we do not need finer asymptotics, we observe that $w_i(x)$ will blow up like (minus) the logarithm of the distance of x to the boundary measured with respect to $\bar{g}$. To see this, consider the conformal map $\varphi_i : D \to \tilde{\mathcal{M}}_i$ from the unit disc to the universal cover of $\mathcal{M}_i$, given by the Uniformisation Theorem. This map pulls back the complete conformal metric on $\tilde{\mathcal{M}}_i$ of curvature -1 to the Poincaré metric on D, whose conformal factor blows up logarithmically in the distance to the boundary, measured with respect to the flat metric $|dz|^2$. By exploiting the Kellogg–Warschawski Theorem [15], one finds that φ_i extends smoothly (and without degeneration) to a map from the closure $\bar{D}$ to the universal cover of the closure $\bar{\mathcal{M}}_i$, and hence the pull-back under φ_i of the lift of $\bar{g}$ is metrically equivalent to $|dz|^2$. It follows that $w_i(x)$ blows up logarithmically with respect to $\bar{g}$ as claimed.

One consequence is that

$$\{w_i \leq 1\} = \{h_i \leq e^2 \bar{g}\} \subset \mathcal{M}_i \text{ is compact.} \tag{4.2}$$

Choose a smooth cut-off function $\Psi : \mathbb{R} \to \mathbb{R}$ with the properties that $\Psi(s) = 0$ for $s \leq -1$, $\Psi(s) = s$ for $s \geq 1$, and $\Psi''(s) \geq 0$ for all s. Automatically, we have $0 \leq \Psi' \leq 1$ and $\Psi(s) \geq s$ for all s. We then define

$$\bar{g}_i = e^{2\Psi(w_i)} \bar{g}.$$

Throughout $\mathcal{M}_i$, we have $\bar{g}_i \geq e^{2w_i}\bar{g} = h_i$, so for each i, $\bar{g}_i$ inherits the completeness of h_i.

Where $\bar{g}$ has a significantly larger conformal factor than the metric h_i, in the sense that $w_i \leq -1$, we have $\bar{g}_i = \bar{g}$. In particular, $\bar{g}_i = \bar{g}$ on $\mathcal{M}_{i-1}$.

On the other hand, where $w_i \geq 1$, we have $\bar{g}_i = e^{2w_i}\bar{g} = h_i$. In particular, we have $\bar{g}_i = h_i$ off the subset $\{w_i \leq 1\} \subset \mathcal{M}_i$, which is compact by (4.2).

These considerations show that outside the compact set where $-1 \leq w_i \leq 1$, we have $K(\bar{g}_i) \leq \bar{K}$, and $K(\bar{g}_i)$ bounded below by some i-dependent constant. We would now like to check that this is also true where $-1 \leq w_i \leq 1$, the region where the two metrics h_i and $\bar{g}|_{\mathcal{M}_i}$ are interpolated.

Let us work locally with respect to a local complex coordinate $z = x + iy$, and write $\bar{g} = e^{2u}|dz|^2$ and $h_i = e^{2v_i}|dz|^2$, so that $w_i = v_i - u$. We also write Δ_z and ∇_z for the Laplacian and gradient with respect to the local flat metric $|dz|^2 := dx^2 + dy^2$—for example, $\Delta_z = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The Gauss curvature of a metric $e^{2a}|dz|^2$ is $-e^{-2a}\Delta a$, so the fact that h_i has negative curvature implies $\Delta_z v_i \geq 0$, or equivalently $-\Delta_z w_i \leq \Delta_z u$. Because $\Psi'' \geq 0$ and $\Psi' \in [0, 1]$, we may then bound the Gauss curvature of the metrics $\bar{g}_i$ according to

$$\begin{aligned} K(\bar{g}_i) &= -e^{-2(\Psi(w_i)+u)}\Delta_z(\Psi(w_i) + u) \\ &= -e^{-2(\Psi(w_i)+u)}(\Psi''(w_i)|\nabla_z w_i|^2 + \Psi'(w_i)\Delta_z w_i + \Delta_z u) \\ &\leq -e^{-2(\Psi(w_i)+u)}(1 - \Psi'(w_i))\Delta_z u = e^{-2\Psi(w_i)}(1 - \Psi'(w_i))K(\bar{g}) \\ &\leq e^{-2\Psi(w_i)}(1 - \Psi'(w_i))\bar{K} \leq \bar{K}. \end{aligned} \tag{4.3}$$

An i-dependent lower bound on the set $\{-1 \leq w_i \leq 1\} \subset \mathcal{M}_i$ is automatic by compactness.

As desired, we have constructed complete, conformal initial metrics $\bar{g}_i$ on our subdomains $\mathcal{M}_i \subset \mathcal{M}$, with $\mathcal{M}_i \subset\subset \mathcal{M}_{i+1}$, and with $\mathcal{M}_i$ exhausting $\mathcal{M}$ as $i \to \infty$, so that $\bar{g}_i = \bar{g}$ on $\mathcal{M}_{i-1}$ ($i > 1$) and in particular so that $\bar{g}_i = \bar{g}$ on any compact subset of $\mathcal{M}$ for sufficiently large i, and where each metric $\bar{g}_i$ has Gauss curvature bounded above by $\bar{K}$, and below by some number which may depend on i.

We now argue that there exists $T > 0$ dependent only on $\bar{K}$ such that for each i, there exists a Ricci flow $g_i(t)$ on $\mathcal{M}_i$ for $t \in [0, T]$ with $g_i(0) = \bar{g}_i$, and each flow has bounded Gauss curvature (above and below) uniformly in i, over any time interval compactly contained in $(0, T]$. (Certainly, the curvature of $g_i(0)$ will typically be unbounded below as $i \to \infty$.)

The claimed existence of $G(t)$ will then follow from Theorem 3.1.

By Shi's Ricci flow existence theorem and Shi's derivative estimates (see [17] and [20]) there exists, for each i, a Ricci flow $g_i(t)$ with $g_i(0) = \bar{g}_i$ on a maximal time interval $[0, T_i)$ (for some $T_i \in (0, \infty]$) with bounded curvature on any time interval compactly contained in $[0, T_i)$, but, if $T_i < \infty$, with unbounded curvature as $t \uparrow T_i$.

Under Ricci flow, one may compute that the Gauss curvature obeys the equation

$$\frac{\partial K}{\partial t} = \Delta K + 2K^2.$$

(More generally, in any dimension, the scalar curvature satisfies a nonlinear heat equation —see [20].) When the curvature of a Ricci flow is bounded, one may apply the maximum principle (more precisely, the comparison principle) to such equations governing bounded functions (cf. [8]—generally for Ricci flow we require the boundedness of the full sectional curvature rather than just the Ricci curvature). Indeed, we may compare solutions of this PDE with solutions of the ODE $dk/dt = 2k^2$ to get upper and lower bounds on the solution K. Because the curvature of our flows $g_i(t)$ is bounded before we approach time T_i, we are thus able to deduce that for all $t \in [0, T_i)$, if $\bar{K} = 0$, then

$$-\frac{1}{2t} \leq K(g_i(t)) \leq 0,$$

whilst if $\bar{K} > 0$, then

$$-\frac{1}{2t} \leq K(g_i(t)) \leq \frac{1}{\bar{K}^{-1} - 2t}.$$

In particular, choosing any $T > 0$ in the case that $\bar{K} = 0$, or choosing any $T \in (0, \frac{1}{2}\bar{K}^{-1})$ in the case that $\bar{K} > 0$, we can be sure that the curvature is bounded within any compact time interval in $(0, T]$, and in particular that $T_i > T$ for all i.

We have finally established enough about the flows $g_i(t)$ to conclude the existence of $G(t)$ with an application of Theorem 3.1. □

Remark 4.1. As an alternative to the approach in this section, one could lift to the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$—either D or $\mathbb{C}$—and prove existence there. The construction of the metrics $\bar{g}_i$ can then be made somewhat more explicit. In this case, one would need to make precise a uniqueness theorem for maximal solutions in order to be sure that the flow $G(t)$ on $\tilde{\mathcal{M}}$ retains the symmetry of $G(0)$ which allows it to be quotiented to give a flow down on $\mathcal{M}$. Note that the flows $g_i(t)$ on $\tilde{\mathcal{M}}$ would not enjoy this symmetry in general.

5. Uniqueness of surface flows

In this section, we give a proof of Theorem 1.7 and comment on the analogous claim of Remark 1.8. We already mentioned in Remark 1.9 that any Ricci flow $G(t)$ produced by Theorem 1.1 must be maximal, and so we only need to show that any competing flow $g(t)$ cannot drop below $G(t)$.

Proof of Theorem 1.7. Our basic wish is to use $G(t)$ itself as a barrier from below for $g(t)$. To state this more precisely, let $z = x + iy$ be the standard global complex coordinate on the unit disc D in $\mathbb{C}$, and write $g(t) = e^{2u}|dz|^2$ and $G(t) = e^{2v}|dz|^2$. Then $u(\cdot, 0) = v(\cdot, 0) = 0$ on D, and we need to show only that

$$u(\cdot, t) \geq v(\cdot, t) \tag{5.1}$$

on D for later times t. Let us observe now that because $g(t)$ is rotationally symmetric, complete, and has bounded curvature for $t > 0$, we must have

$$u(x, t) \to \infty \quad \text{as } x \to \partial D, \tag{5.2}$$

for any $t > 0$. (This is the only time in the proof that we use the rotational symmetry of $g(t)$.)

It is easy to check that for $\lambda \in (0, 1)$, the function $v_\lambda : D_{1/\lambda} \times [0, T] \to \mathbb{R}$ defined by $v_\lambda(x, t) = v(\lambda x, t) + \ln \lambda$ gives rise to a Ricci flow $e^{2v_\lambda}|dz|^2$ on $D_{1/\lambda}$. Incidentally, this flow is isometric to $G(t)$; here it has been written with respect to dilated coordinates. We will show that for $\lambda < 1$ arbitrarily close to 1, and for $t > 0$ sufficiently small, depending on λ, we must have

$$v_\lambda(x, t) \leq 0 \leq u(x, t) \quad \text{for all } x \in D, \tag{5.3}$$

and hence by virtue of the comparison principle applied to the equation (1.2) satisfied by u and v_λ (keeping in mind (5.2)) we would then have $v_\lambda(x, t) \leq u(x, t)$ for all $x \in D$, and all $t \in [0, T]$. Since λ can be made arbitrarily close to 1, we would then be able to obtain (5.1). Note that by taking λ a little *larger* than 1, so that v_λ lies above u, we could show, for this particular example, the general fact from Remark 1.9 that $G(t)$ is a maximal solution.

We are therefore reduced to proving (5.3), and since the first inequality there is obvious (because $v(\cdot, t) \to 0$ uniformly on D_λ as $t \downarrow 0$) we need only show that

$$u(x, t) \geq 0 \tag{5.4}$$

for $t > 0$ sufficiently small, on the whole of D, or equivalently that $g(t) \geq \bar{g}$. A priori, the function u could even be unbounded below in the limit $t \downarrow 0$, and a first step will be to prove that u is bounded below uniformly in x and t.

After defining $h : D \times (0, T] \to \mathbb{R}$ by

$$h(x, t) = \ln \frac{2}{1 - |x|^2} + \frac{1}{2} \ln(2t),$$

the metric $e^{2h}|dz|^2$ is a Ricci flow which at time t is the Poincaré metric scaled to have constant curvature $-1/2t$. We will now show that this is a barrier from below for $g(t)$—in other words, that

$$h(x, t) \leq u(x, t) \quad \text{for all } t \in (0, T] \text{ and } x \in D. \tag{5.5}$$

To see this, we again rewrite the flow with respect to the adjusted conformal factor $h_\lambda(x,t) = h(\lambda x, t) + \ln\lambda$, now defined on $D_{1/\lambda}$. For arbitrarily small $\varepsilon \in (0, T]$, $u(\cdot, \varepsilon)$ must be bounded below (by virtue of (5.2)) and thus there exists $\delta > 0$ (small) such that $h_\lambda(\cdot, \delta) \le u(\cdot, \varepsilon)$ on D. By applying the comparison principle again, and using the fact that for each $x \in D$, $h_\lambda(x,t)$ is an increasing function of t, we deduce that

$$u(\cdot, \varepsilon + t) \ge h_\lambda(\cdot, \delta + t) \ge h_\lambda(\cdot, t).$$

Allowing ε to decrease to zero, and then λ to increase to one, we establish (5.5).

The lower bound (5.5) gives us good control on u from below near the boundary of D. To get better control nearer the interior, we will exploit the pseudolocality of Theorem A.1. For all $x_0 \in D$, we may apply that theorem to $g(t)$ with $r_0 > 0$ less than, but arbitrarily close to $r := 1 - |x_0|$ to deduce that

$$|K|(x_0, t) \le (\varepsilon r)^{-2} \quad \text{for all } t \in [0, (\varepsilon r)^2]. \tag{5.6}$$

Note here that by reducing the $\varepsilon > 0$ of that theorem if necessary, we may assume that $g(t)$ is defined on the whole time interval $[0, \varepsilon^2] \supset [0, (\varepsilon r)^2]$. Also, as in our previous application of pseudolocality, we should adjust time zero forwards an arbitrarily small amount so that the curvature of $g(t)$ may be assumed to be bounded.

By rewriting the Ricci flow equation (1.1) or (1.2) as

$$\frac{\partial u}{\partial t} = -K,$$

we deduce from (5.6) that

$$|u(x_0, t)| \le t(\varepsilon r)^{-2} \le 1 \quad \text{for } t \in [0, (\varepsilon r)^2]. \tag{5.7}$$

For later times, when $t \ge (\varepsilon r)^2$, we can switch to the lower bound (5.5) to deduce that

$$\begin{aligned} u(x_0, t) \ge h(x_0, t) &= \ln \frac{2}{1 - (1-r)^2} + \frac{1}{2}\ln(2t) \\ &\ge \ln\frac{1}{r} + \frac{1}{2}\ln(2(\varepsilon r)^2) = \frac{1}{2}\ln 2 + \ln\varepsilon. \end{aligned} \tag{5.8}$$

By combining (5.7) and (5.8), we see that $u(x,t)$ is bounded below by the constant $\min\{-1, \frac{1}{2}\ln 2 + \ln\varepsilon\}$ for all $x \in D$ and $t \ge 0$.

It remains to convert this lower bound into the stronger bound (5.4), and for this we use integral estimates with respect to the flat metric $\bar{g}$. Since $u(x,t) \to 0$ for each $x \in D$ as $t \downarrow 0$, and u is bounded below, we must have

$$\int_D [-u(\cdot, t)]_+ \to 0 \quad \text{as } t \downarrow 0, \tag{5.9}$$

where $[s]_+ := \max\{s, 0\}$. Let $\varphi \in C^\infty(\mathbb{R}, [0,1])$ satisfy $\varphi(s) = 0$ for $s \le 0$ and $\varphi' \ge 0$, and define $\psi : \mathbb{R} \to [0, \infty)$ by

$$\psi(s) = \int_{-\infty}^{s} \varphi.$$

It may help to think of φ as being a smooth approximation to the Heaviside function. Since $0 \leq \psi(s) \leq [s]_+$, we have

$$\int_D \psi(-u(\cdot, t)) \to 0 \quad \text{as } t \downarrow 0. \tag{5.10}$$

By virtue of (5.2), for each $t > 0$, we know that $\psi(-u(\cdot, t)) = 0$ in a neighbourhood of the boundary, so we may compute, using (1.2),

$$\begin{aligned} \frac{d}{dt} \int_D \psi(-u) &= -\int \psi'(-u) u_t = -\int \psi'(-u) e^{-2u} \Delta u \\ &= -\int e^{-2u} |\nabla u|^2 (\psi''(-u) + 2\psi'(-u)) \leq 0. \end{aligned} \tag{5.11}$$

Returning to (5.10) we deduce that

$$\int_D \psi(-u(\cdot, t)) = 0 \quad \text{for all } t,$$

and since we can make ψ as close to $[\cdot]_+$, uniformly, as we like (by choosing φ to lie only a little below the Heaviside function) we finally deduce that $u(\cdot, t) \geq 0$ for all t as desired. ◻

We end this section by making some comments on the uniqueness of the Ricci flow starting at the punctured plane, as claimed in Remark 1.8. Essentially the same argument as we have just employed will show that there too, any competing flow $g(t)$ must satisfy $g(t) \geq \bar{g}$ for $t \geq 0$, and as part of this, the direct analogue of (5.5) is to compare the conformal factor u, near the puncture and for small $t > 0$, to that of a hyperbolic cusp, expanding from nothing under the Ricci flow. (This comparison is best done now with a simple ODE argument rather than the comparison principle, as we could have done also in the proof of Theorem 1.7.) The concluding argument that $G(t)$ can be seen as a lower bound for $g(t)$ (as well as an upper bound) is in fact a little easier in this case since it is possible to construct a sequence of incomplete Ricci solitons on $\mathbb{R}^2 \setminus \{0\}$ increasing to the flow $G(t)$, which can all easily be seen to serve as lower barriers for any $g(t)$, given that $g(t) \geq \bar{g}$ and that $g(t)$ blows up with respect to $\bar{g}$ near the puncture. It is not clear that the additional details will be of use to address the general uniqueness problem, so we omit them.

A. Appendix: Pseudolocality

One of the basic tools at the heart of our compactness results in Sections 2 and 3, and the uniqueness of Section 5, is Perelman's pseudolocality theorem [14, Theorem 10.3], which we briefly survey in this appendix. We use ω_n to denote the volume of the unit ball in Euclidean n-space, and Rm to denote the curvature tensor.

Theorem A.1. *For each $n \in \mathbb{N}$, there exists $\varepsilon > 0$ depending on n such that for all $r_0 > 0$, if $g(t)$ is a complete, bounded curvature Ricci flow for $t \in [0, T]$, where $0 < T \leq (\varepsilon r_0)^2$, on an n-dimensional manifold $\mathcal{M}$ containing some point x_0, and if, with respect to $g(0)$,*

(i) $|\mathrm{Rm}| \leq r_0^{-2}$ *on* $B(x_0, r_0)$,
(ii) $\mathrm{Vol}(B(x_0, r_0))/r_0^n \geq (1 - \varepsilon)\omega_n$,

then

$$|\mathrm{Rm}|(x, t) \leq (\varepsilon r_0)^{-2} \quad \text{for } t \in [0, T] \text{ and } \mathrm{dist}_{g(t)}(x, x_0) < \varepsilon r_0.$$

Remark A.2. The original statement of this theorem in [14] did not make clear that completeness is required. In fact, the result is false without that assumption. In particular, this theorem is not a local result (see [21]). Immediately after the statement of the result in [14], Perelman asks whether the hypothesis (ii) could be dropped. The answer is no, as we demonstrate in [21].

Remark A.3. Theorem A.1 gives curvature control at points in the ball $B_{g(t)}(x_0, \varepsilon r_0)$, at each time. However, by exploiting the most basic control on the stretching of lengths under Ricci flow (see [20] or [12]) we know that the length of a path on which the curvature $|\mathrm{Rm}|$ is bounded by M over a time interval of order $1/M$, can increase only by a factor F depending only on n. In particular, in Theorem A.1, for each $t \in [0, T] \subset [0, (\varepsilon r_0)^2]$, we have $B_{g(0)}(x_0, \varepsilon r_0/F) \subset B_{g(t)}(x_0, \varepsilon r_0)$. One consequence is that by reducing ε, the conclusion of the theorem remains valid also for points $x \in \mathcal{M}$ with $\mathrm{dist}_{g(0)}(x, x_0) < \varepsilon r_0$.

Once one has control on the curvature, one can control its higher covariant derivatives also. Indeed, simpler versions of the arguments used to prove Shi's local derivative estimates (see for example [4]) yield the following.

Lemma A.4. *Suppose that $(\mathcal{M}^n, g(t))$ is a Ricci flow for $t \in [0, T]$, not necessarily complete, and that $x \in \mathcal{M}$, $r > 0$ and $B_{g(0)}(x, r) \subset\subset \mathcal{M}$. Suppose that $|\mathrm{Rm}(g(t))| \leq r^{-2}$ in $B_{g(0)}(x, r) \times [0, T]$, and for $k \in \mathbb{N}$, that $|\nabla^l \mathrm{Rm}(g(0))| \leq r^{-2-l}$ in $B_{g(0)}(x, r)$ for all $l \in \{1, \dots, k\}$. Then for any $\eta \in (0, 1)$, there exists $C < \infty$ depending on k, n, η and an upper bound for T/r^2, such that*

$$|\nabla^l \mathrm{Rm}(g(t))| \leq C r^{-2-l} \quad \text{in } B_{g(0)}(x, \eta r) \times [0, T] \quad \text{for } l \in \{1, \dots, k\}.$$

Combining Theorem A.1, Remark A.3 and Lemma A.4 gives the following.

Corollary A.5. *For each $k, n \in \mathbb{N}$, there exists $\varepsilon > 0$ depending on n and $C < \infty$ depending on n and k, such that for all $r_0 > 0$, if $g(t)$ is a complete, bounded curvature Ricci flow for $t \in [0, T]$ with $0 < T \leq (\varepsilon r_0)^2$, on an n-dimensional manifold $\mathcal{M}$ containing some point x_0, and if, with respect to $g(0)$,*

(i) $|\nabla^l \mathrm{Rm}| \leq r_0^{-2-l}$ *on* $B(x_0, r_0)$ *for* $l \in \{0, \dots, k\}$,
(ii) $\mathrm{Vol}(B(x_0, r_0))/r_0^n \geq (1 - \varepsilon)\omega_n$,

then

$$|\nabla^k \mathrm{Rm}(g(t))|(x) \leq C r_0^{-2-k} \quad \text{for } t \in [0, T] \text{ and } x \in B_{g(0)}(x_0, \varepsilon r_0).$$

One use of this result, in Section 2, is to obtain a local compactness of Ricci flows which are similar in some local region initially. However wild a Ricci flow begins outside this local region, the interior of the region is protected in a uniform manner. This contrasts sharply with the situation for more classical parabolic equations such as the linear heat equation.

B. Appendix: Convergence and compactness of manifolds and flows

Let us clarify the notion of convergence of pointed Riemannian manifolds we will be using; we will take the unusual step of avoiding the assumption that the limit is complete, and will use an unusual notation $\dashrightarrow$ to emphasise this. Otherwise, the definition follows Cheeger and Gromov.

Definition B.1 (Smooth pointed convergence of manifolds). A sequence $(\mathcal{M}_i, g_i, q_i)$ of smooth, complete, *pointed* Riemannian manifolds (that is, Riemannian manifolds $(\mathcal{M}_i, g_i)$ and points $q_i \in \mathcal{M}_i$) is said to *converge* smoothly to the smooth, pointed manifold $(\mathcal{N}, G, Q)$, written

$$(\mathcal{M}_i, g_i, q_i) \dashrightarrow (\mathcal{N}, G, Q) \quad \text{as } i \to \infty,$$

if there exist

(i) a sequence of compact sets $\Omega_i \subset \mathcal{N}$ exhausting $\mathcal{N}$ (that is, so that any compact set $K \subset \mathcal{N}$ satisfies $K \subset \Omega_i$ for sufficiently large i) with $Q \in \mathrm{int}(\Omega_i)$ for each i;
(ii) a sequence of smooth maps $\varphi_i : \Omega_i \to \mathcal{M}_i$ which are diffeomorphic onto their image and satisfy $\varphi_i(Q) = q_i$ for all i,

such that

$$\varphi_i^* g_i \to G \quad \text{smoothly locally on } \mathcal{N} \text{ as } i \to \infty. \tag{B.1}$$

In the case that $(\mathcal{N}, G)$ is complete, we write

$$(\mathcal{M}_i, g_i, q_i) \to (\mathcal{N}, G, Q).$$

It is sometimes important to keep in mind that if we have the convergence in (B.1), then the norm over compact subsets of $\mathcal{N}$ of $\varphi_i^* g_i - G$, and all its covariant derivatives, must converge to zero as $i \to \infty$, when we compute norms and Levi-Civita connections with respect to any fixed background metric, or indeed with respect to $\varphi_i^* g_i$.

Remark B.2. The notation $\dashrightarrow$ is used to emphasise the unfamiliar situation that the limit is not assumed to be complete. Working within this generality has the consequence that the limit is not unique. One could always take a subdomain of the limit (containing Q) to give a new limit. One must also be aware that the curvature of $(\mathcal{M}_i, g_i)$ in a ball of fixed radius centred at q_i might not be bounded uniformly in i.

The following lemma is the generalisation of the standard fact that there can be at most one complete limit of a sequence of complete pointed manifolds. (We emphasise that manifolds, for us, are assumed always to be connected.)

Lemma B.3. *Suppose that $(\mathcal{M}_i, g_i, q_i)$ is a sequence of smooth, complete, pointed Riemannian manifolds such that*

$$(\mathcal{M}_i, g_i, q_i) \to (\mathcal{N}, G, Q) \tag{B.2}$$

(with the limit complete) and

$$(\mathcal{M}_i, g_i, q_i) \dashrightarrow (\Omega, g, q) \tag{B.3}$$

(a possibly incomplete limit) as $i \to \infty$. Then there exists a map $I : (\Omega, g) \to (\mathcal{N}, G)$ which is an isometry onto its image, and sends q to Q. Moreover, if ψ_i are the diffeomorphisms in the definition of the convergence (B.2)*, and φ_i are the diffeomorphisms in the definition of the convergence* (B.3)*, then after taking a subsequence, we may assume that*

$$\psi_i^{-1} \circ \varphi_i \to I \quad \text{smoothly as } i \to \infty$$

on arbitrary compact subsets of Ω.

Proof. Let us adopt the shorthand $J_i := \psi_i^{-1} \circ \varphi_i$. Note that $J_i(q) = Q$. We must take a subsequence in the lemma precisely to ensure that the linear map $(J_i)_* : T_q\Omega \to T_Q\mathcal{N}$ converges as $i \to \infty$. The limit then provides an identification of $T_q\Omega$ and $T_Q\mathcal{N}$ which we will use implicitly in some of what follows.

By definition of the convergence (B.2) and (B.3), we have

$$J_i^*(G) \to g \quad \text{smoothly locally on } \Omega \text{ as } i \to \infty. \tag{B.4}$$

By tautologically rewriting

$$J_i = \exp_{J_i(q),(J_i)_* g} \circ \exp_{q,g}^{-1}$$

near q, for sufficiently large i (where $T_q\Omega$ and $T_Q\mathcal{N}$ have been identified using J_i) and exploiting the convergence

$$\exp_{J_i(q),(J_i)_* g} \to \exp_{Q,G}$$

near the 'origin', we find that

$$J_i \to \exp_{Q,G} \circ \exp_{q,g}^{-1}$$

smoothly near q, and the limit is then necessarily an isometry.

We have shown that q belongs to the set $A \subset \Omega$ defined by

$$A := \{p \in \Omega \mid J_i \text{ converges smoothly to an isometry in some neighbourhood of } p\},$$

where the isometry is with respect to g on Ω and G on $\mathcal{N}$.

A minor adaptation of the argument above shows that whenever a point p is in A, then J_i converges to an isometry in any geodesic ball in (Ω, g) centred at p which is compactly contained in Ω and has radius less than the injectivity radius at p. This entire geodesic ball must then lie within A. In particular, since Ω is connected (being a manifold) we have $A = \Omega$, and we have established the smooth local convergence of J_i to a local isometry $I : \Omega \to \mathcal{N}$. Since J_i is a diffeomorphism, the map I must be injective, and hence a global isometry onto its image. □

We will use the following notion of convergence of pointed flows.

Definition B.4 (Convergence of flows). Let $(\mathcal{M}_i, g_i(t))$ be a sequence of smooth, complete flows for t in some time interval $\mathcal{I} \subset \mathbb{R}$. Let $q_i \in \mathcal{M}_i$ for each i. Let $(\mathcal{N}, G(t))$ be a smooth flow for $t \in \mathcal{I}$ and let $Q \in \mathcal{N}$. We say that

$$(\mathcal{M}_i, g_i(t), q_i) \dashrightarrow (\mathcal{N}, G(t), Q) \quad \text{on } \mathcal{I} \text{ as } i \to \infty$$

if there exist

(i) a sequence of compact $\Omega_i \subset \mathcal{N}$ exhausting $\mathcal{N}$ and satisfying $Q \in \operatorname{int}(\Omega_i)$ for each i;
(ii) a sequence of smooth maps $\varphi_i : \Omega_i \to \mathcal{M}_i$, diffeomorphic onto their image, and with $\varphi_i(Q) = q_i$,

such that

$$\varphi_i^* g_i(t) \to G(t) \quad \text{smoothly locally on } \mathcal{N} \times \mathcal{I} \text{ as } i \to \infty.$$

In the familiar situation that $(\mathcal{N}, G(t))$ is complete for every t, we shall use the standard notation

$$(\mathcal{M}_i, g_i(t), q_i) \to (\mathcal{N}, G(t), Q).$$

Finally, we record that in [13]—see also [12]—Hamilton proved the following compactness theorem for Ricci flows with controlled curvature and injectivity radius, which we use in Section 2.

Theorem B.5 (Hamilton's compactness theorem for Ricci flows). *Let $\mathcal{M}_i$ be a sequence of manifolds of dimension n, and let $q_i \in \mathcal{M}_i$ for each i. Suppose that $g_i(t)$ is a sequence of complete Ricci flows on $\mathcal{M}_i$ for $t \in \mathcal{I}$, where either $\mathcal{I} = (a, b)$ and $-\infty \le a < 0 < b \le \infty$, or $\mathcal{I} = (a, b]$ and $-\infty \le a < 0 \le b < \infty$. Suppose that*

1. *for every compact subset $\Gamma \subset \mathcal{I}$ and $s > 0$, there exists $C = C(\Gamma, s)$ such that $|\mathrm{Rm}(g_i(t))|(x) \le C$ for all $i \in \mathbb{N}$, $t \in \Gamma$ and $x \in B_{g_i(t)}(q_i, s)$;*
2. $\inf_i \operatorname{inj}(\mathcal{M}_i, g_i(0), q_i) > 0$.

Then there exist a manifold $\mathcal{N}$ of dimension n, a smooth complete Ricci flow $G(t)$ on $\mathcal{N}$ for $t \in \mathcal{I}$, and a point $Q \in \mathcal{N}$ such that, after passing to a subsequence in i,

$$(\mathcal{M}_i, g_i(t), q_i) \to (\mathcal{N}, G(t), Q) \quad \text{on } \mathcal{I} \text{ as } i \to \infty.$$

Acknowledgements. Part of this work was written up at the Albert Einstein – Max Planck Institute in Golm, and the Freie Universität in Berlin. The author would like to thank these institutions for their hospitality, and Gerhard Huisken and Klaus Ecker for their invitations.

This research was partly supported by an EPSRC Advanced Research Fellowship, and The Leverhulme Trust.

References

[1] Ahlfors, L. V.: Lectures on Quasiconformal Mappings. Van Nostrand Math. Stud. 10, Van Nostrand (1966) Zbl 0138.06002 MR 0200442

[2] Bertsch, M., Dal Passo, R., van der Hout, R.: Nonuniqueness for the heat flow of harmonic maps on the disk. Arch. Ration. Mech. Anal. **161**, 93–112 (2002) Zbl 1006.35050 MR 1870959

[3] Brendle, S.: Curvature flows on surfaces with boundary. Math. Ann. **324**, 491–519 (2002) Zbl 1024.53045 MR 1938456

[4] Chow, B., Lu, P., Ni, L.: Hamilton's Ricci Flow. Grad. Stud. Math. 77, Amer. Math. Soc. (2006) Zbl 1118.53001 MR 2274812

[5] Daskalopoulos, P., del Pino, M. A.: On a singular diffusion equation. Comm. Anal. Geom. **3**, 523–542 (1995) Zbl 0851.35072 MR 1371208

[6] DeTurck, D.: Deforming metrics in the direction of their Ricci tensors. In: Collected papers on Ricci flow, H. D. Cao et al. (eds.), Ser. Geom. Topol. 37, Int. Press, 163–165 (2003) Zbl 1108.53002 MR 2145154

[7] DiBenedetto, E., Diller, D.: About a singular parabolic equation arising in thin film dynamics and in the Ricci flow for complete $\mathbb{R}^2$. In: Partial Differential Equations and Applications, Lecture Notes in Pure Appl. Math. 177, Dekker, 103–119 (1996) Zbl 0846.35070 MR 1371583

[8] Dodziuk, J.: Maximum principle for parabolic inequalities and the heat flow on open manifolds. Indiana Univ. Math. J. **32**, 703–716 (1983) Zbl 0526.58047 MR 0711862

[9] Giesen, G., Topping, P. M.: Ricci flow of negatively curved incomplete surfaces. Calc. Var. Partial Differential Equations **38**, 357–367 (2010) Zbl pre05730931

[10] Giesen, G., Topping, P. M.: Existence of Ricci flows of incomplete surfaces. arXiv:1007.3146 (2010)

[11] Hamilton, R. S.: Three-manifolds with positive Ricci curvature. J. Differential Geom. **17**, 255–306 (1982) Zbl 0504.53034 MR 0664497

[12] Hamilton, R. S.: The formation of singularities in the Ricci flow. In: Surveys in Differential Geometry, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 7–136 (1995) Zbl 0867.53030 MR 1375255

[13] Hamilton, R. S.: A compactness property for solutions of the Ricci flow. Amer. J. Math. **117**, 545–572 (1995) Zbl 0840.53029 MR 1333936

[14] Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:math.DG/0211159v1 (2002) Zbl 1130.53001

[15] Pommerenke, Ch.: Boundary Behaviour of Conformal Maps. Springer (1992) Zbl 0762.30001 MR 1217706

[16] Rodríguez, A., Vázquez, J. L., Esteban, J. R.: The maximal solution of the logarithmic fast diffusion equation in two space dimensions. Adv. Differential Equations **2**, 867–894 (1997) Zbl 1023.35515 MR 1606339

[17] Shi, W.-X.: Deforming the metric on complete Riemannian manifolds. J. Differential Geom. **30**, 223–301 (1989) Zbl 0676.53044 MR 1001277

[18] Topping, P. M.: Reverse bubbling and nonuniqueness in the harmonic map flow. Int. Math. Res. Notices **2002**, no. 10, 505–520 Zbl 1003.58014 MR 1883901

[19] Topping, P. M.: Diameter control under Ricci flow. Comm. Anal. Geom. **13**, 1039–1055 (2005) Zbl 1111.53032 MR 2216151

[20] Topping, P. M.: Lectures on the Ricci Flow. London Math. Soc. Lecture Note Ser. 325, Cambridge Univ. Press (2006); http://www.warwick.ac.uk/~maseq/RFnotes.html Zbl 1105.58013 MR 2265040

[21] Topping, P. M.: Entropy, heat kernels and pseudolocality for Ricci flow. Lecture notes, in preparation