Interaction of turbulence and large-scale vortices in incompressible 2D fluids

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Abstract

It is commonplace in 2D fluid dynamics that intense large-scale vortices arise during turbulence decay. Both the large-scale vortex and the turbulent components will be important in the further evolution: the vortices will modify the turbulent dynamics and, in turn, will be modified by the turbulence. Using Wigner functions, we derive a new two-component model to describe interaction of turbulence and intense large-scale vortices in incompressible inviscid 2D fluids. We apply this model to study the dynamics of a vortex dipole propagating through turbulence, a problem that allows an elegant analytic solution.

1 Introduction

It has been long realized that the evolution of 2D turbulence in incompressible fluids leads to formation of strong vortices with radii of order the size of the system [1, 2, 3, 4, 5]. The energy is pumped into the large-scale vortices via the inverse cascade. The circulation of such vortices grows to high values if dissipation in the large scales is weak, so that they start producing a significant feedback on turbulence. On the other hand, small-scale turbulence is still important because it mediates the energy transfer from the source at small scales to large-scale vortices.

The subsequent evolution will be determined by the dynamics of equally important and coupled components: turbulence and the large-scale vortices. In this paper we will study evolution and interaction of these two components under the following assumptions. First, we will assume that the characteristic scale of turbulence is much less than the size of the large-scale vortices. This assumption is natural if turbulence is produced at the scales much less than the size of the system. Second, we assume that the interaction of the small turbulent scales among themselves is much weaker than their interaction with/via the large vortex scales. This assumption is reasonable for the late evolution if the large-scale dissipation is weak. In this case large amount of energy condensed in the large-scale vortices will be responsible for the strongly nonlocal interaction of scales that we assume. The described above approach was first suggested in [7] in the context of the β-plane dynamics (see also [8]) and was latter used to study the vortex-acoustic interaction in [9]. In this paper we will study the vortex-turbulence interaction in incompressible inviscid 2D fluids.

2 Derivation of the dynamical equations

2.1 Description of the two-scale analysis

Consider an incompressible inviscid fluid described by the 2D Euler equations,

$$\begin{aligned} \partial_t \omega + \operatorname{div}(\mathbf{v}\omega) &= 0, \\ \operatorname{div}\mathbf{v} &= 0, \end{aligned} \tag{1}$$

where $\omega = \omega(\mathbf{r}, t) = \operatorname{curl}\mathbf{v}$ is the vorticity.

We consider a situation where the velocity field $\mathbf{v}(\mathbf{x}, t)$ consists of a strong large-scale component (large vortices) and a weak small-scale turbulent component:

$$\mathbf{v} = \mathbf{v}_L + \mathbf{v}_S, \tag{2}$$

$$\frac{v_S}{v_L} = \eta \ll 1. \tag{3}$$

such that the characteristic scale of $\mathbf{v}_L$ is much greater than that of $\mathbf{v}_S$,

$$\epsilon = \frac{\lambda_S}{\lambda_L} \ll 1. \tag{4}$$

The scale separation (4), as well as condition (3) can result from energy condensation in the largest scale or it can be due to the corresponding initial conditions. Note that for energy spectra steeper than k^{-3}, assumption (4) is automatically consistent with (3). At late stage of nearly inviscid 2D turbulence evolution, Dritschel [6] observes a k^{-5} spectrum at large scales, and a k^{-3} spectrum at small scales. This suggests that small/large scale interaction is non-local.

Similarly, the vorticity field $\omega(\mathbf{x}, t)$ consists of a large and small scale components:

$$\omega = \omega_L + \omega_S, \tag{5}$$

where $\omega_L \sim v_L/\lambda_L$ and $\omega_S \sim v_S/\lambda_S$.

Condition (3) means that the evolution of the small-scale field is determined by the large-scale advection and that the interaction of the small scales among themselves can be neglected. Under this condition the characteristic small-scale time is λ_S/v_L which we will assume to be $O(1)$. Then the large-scale time is of order $\lambda_L/v_L \sim O(\epsilon^{-1})$. Thus, we can use a two scale analysis to derive the dynamical equations at large scale. We introduce fast and slow variables, denoted respectively $(\mathbf{x}, t)$ and $(\mathbf{X}, T)$, with $(\mathbf{X}, T) = \epsilon(\mathbf{x}, t)$. The fast and slow variable are considered independent [10, 11] so that we have the decomposition rule:

$$\partial_t \to \partial_t + \epsilon\partial_T, \quad \partial_x \to \partial_x + \epsilon\nabla, \tag{6}$$

where ∇ denotes the partial derivative with respect to $\mathbf{X}$. With this decomposition, we can write any differential operator as a multiscale operator by applying the decomposition rule. For instance, the divergence operator can be written div $+\,\epsilon$Div where div and Div act respectively on the fast and slow variable. Similarly, the multiscale curl operator will be denoted curl $+\,\epsilon$Curl .

We write the velocity field as series in ϵ:

$$\mathbf{v} = \mathbf{v}^{(0)}(\mathbf{x}, t, \mathbf{X}, T) + \epsilon\mathbf{v}^{(1)}(\mathbf{x}, t, \mathbf{X}, T) + \epsilon^2\mathbf{v}^{(2)}(\mathbf{x}, t, \mathbf{X}, T) + ... \tag{7}$$

The corresponding expansion for the vorticity is obtained by applying the multiscale curl operator on (7). It is:

$$\begin{aligned} \omega &= \text{curl } \mathbf{v}^{(0)} + \epsilon(\text{Curl } \mathbf{v}^{(0)} + \text{curl } \mathbf{v}^{(1)}) + ..., \\ &\equiv \omega^{(0)} + \epsilon\omega^{(1)} + ... \end{aligned} \tag{8}$$

We finally introduce the average over fast variables of a function f($\mathbf{x}$,t):

$$< f(t, \mathbf{x}) >= \frac{1}{XT}\int_x^{\mathbf{x}+\mathbf{X}}\int_t^{t+T} f(t', \mathbf{x}')\, dt' d\mathbf{x}'. \tag{9}$$

This average can be defined equivalently via f_k the spatial Fourier transform of f according to:

$$< f(t, \mathbf{x}) >= \frac{1}{XT}\int_t^{t+T} dt' \int e^{i\mathbf{k}\mathbf{x}} H_\epsilon(\mathbf{k}) f_k(t')\, d\mathbf{k}, \tag{10}$$

where H_ϵ is a smoothing kernel filtering out wavenumbers with magnitude larger than $K \sim 2\pi/\lambda_L = O(\epsilon)$. This averaging procedure is such that any average of the type $< \partial_t f >$ or $< \partial_x f >$ is zero, while it commutes with partial derivatives taken on slow variable $< \partial_T f >= \partial_T < f >$. It also leaves unchanged any function of the slow variables only.

Using this average, we may decompose $\mathbf{v}^{(0)}$ into a pure large-scale component and a small-scale component with zero average. They can be respectively identified with the large vortices and the small scale turbulent velocity field:

$$\begin{aligned} \mathbf{v}_L(\mathbf{X},T) &\equiv < \mathbf{v}^{(0)} >, \\ \mathbf{v}_S(\mathbf{x},t,\mathbf{X},T) &\equiv \mathbf{v}^{(0)} - < \mathbf{v}^{(0)} >, \quad < v_S >= 0. \end{aligned} \tag{11}$$

The basic large scale flow only depends on the slow variable, while the small scale flow depends on both fast and slow variables. Note that this situation is reversed with respect to the 3D dynamics considered in [11], where the small-scale component was assumed to be stronger than the large-scale one. We shall limit our investigation to the case of small η but shall assume that η is larger than the scale separation ϵ. Technically, this means that we shall always take the limit $\epsilon \to 0$ before taking the limit $\eta \to 0$. This is a standard procedure used in WKB theory. For techniques of computing interaction of small and large scales in the opposite case, $\eta \ll \epsilon$, the reader is referred to Refs. [11, 12].

Following the notation (11), we also write:

$$\begin{aligned} \omega_S &\equiv \operatorname{curl} \mathbf{v}^{(0)} = \operatorname{curl} v_S, \\ \omega_L &\equiv \operatorname{Curl} v_L =< \omega^{(1)} > . \end{aligned} \tag{12}$$

Due to the properties of the averaging, $< \omega_S >= 0$. The function ω_S depends on both fast and slow variables, while ω_L only depends on slow variable.

To obtain the dynamical equations at large scales (i.e. at slow time and position), it is then sufficient to apply the decomposition rule (6) to equations like (1), using the expansions (7)-(8), and then average over fast variables. This procedure is used below.

2.2 Turbulence on the background of a large-scale vortex flow

We start from the Euler equation written in terms of the Fourier transform,

$$\partial_t \omega_k + i \int \mathbf{k} \cdot \mathbf{v}_q \, \omega_{k-q} \, \frac{d\mathbf{q}}{(2\pi)^2} = 0, \tag{13}$$

where

$$\omega_k = \int \omega \, e^{\mathbf{k}\cdot\mathbf{x}} \, d\mathbf{x}, \qquad \mathbf{v}_k = \int \mathbf{v} \, e^{\mathbf{k}\cdot\mathbf{x}} \, d\mathbf{x}. \tag{14}$$

Adding the equation for ω_{p+k} multiplied by ω_{p-k} to the equation for ω_{p-k} multiplied by ω_{p+k}, and acting on the resulting equation with $\int \cdot e^{2i\mathbf{p}\cdot\mathbf{x}} \frac{d\mathbf{p}}{(2\pi)^2}$, we get

$$\begin{aligned} &\partial_t \int \omega_{p+k} \omega_{p-k} \, e^{2i\mathbf{p}\cdot\mathbf{x}} \frac{d\mathbf{p}}{(2\pi)^2} \\ + \quad & i \int \mathbf{v}_q \cdot [(\mathbf{p}+\mathbf{k})\omega_{p-k}\omega_{p+k-q} + (\mathbf{p}-\mathbf{k})\omega_{p+k}\omega_{p-k-q}] \, e^{2i\mathbf{p}\cdot\mathbf{x}} \frac{d\mathbf{p}}{(2\pi)^2} \frac{d\mathbf{q}}{(2\pi)^2} = 0. \end{aligned} \tag{15}$$

This equation can be written in terms of the Wigner function,

$$W_{\mathbf{k}}(\mathbf{x}, t) = \int \omega_{p+k} \omega_{p-k} \, e^{2i\mathbf{p}\cdot\mathbf{x}} \frac{dp}{(2\pi)^2}. \tag{16}$$

Indeed, introducing new variable

$$\mathbf{p}' = \mathbf{p} - \mathbf{q}/2, \tag{17}$$

equation (15) can also be written as:

$$\begin{aligned} \partial_t W_{\mathbf{k}} \quad &+ \quad i \int \mathbf{k} \cdot \mathbf{v}_q [W_{k-q/2} - W_{k+q/2}] e^{i\mathbf{q}\cdot\mathbf{x}} \, \frac{d\mathbf{q}}{(2\pi)^2} \\ &+ \quad \frac{1}{2} \mathrm{div} \int \mathbf{v}_q [W_{k-q/2} + W_{k+q/2}] e^{i\mathbf{q}\cdot\mathbf{x}} \, \frac{d\mathbf{q}}{(2\pi)^2} = 0. \end{aligned} \tag{18}$$

To obtain the dynamical equations at large scale, we must first write the expansion of (18) using the decomposition rule (6), the expansions (7) and (8c), and then average over the fast variables. The first step requires an expansion of the function W. Using the expansion of the vorticity, written in the Fourier space, we get the following expansion for W_k

$$\begin{aligned} W_k &= \int e^{2i\mathbf{p}\cdot\mathbf{x}} \frac{d\mathbf{p}}{(2\pi)^2} \Big[\omega^{(0)}_{p+k}\omega^{(0)}_{p-k} + \epsilon(\omega^{(0)}_{p+k}\omega^{(1)}_{p-k} + \omega^{(0)}_{p-k}\omega^{(1)}_{p+k}) + ... \Big], \\ &\equiv W^{(0)}_k + \epsilon W^{(1)}_k + ... \end{aligned} \tag{19}$$

Moreover, we note that for wavenumbers $q \sim \epsilon Q$, where Q is order one, the Wigner function can be expanded :

$$\begin{aligned} W_{k-q/2} &= W_k - \frac{\epsilon}{2}\mathbf{Q}\cdot\partial_k W_k, \\ W_{k+q/2} &= W_k + \frac{\epsilon}{2}\mathbf{Q}\cdot\partial_k W_k. \end{aligned} \tag{20}$$

This situation occurs when W is multiplied by the large-scale velocity field v_L, which has only Fourier components of the order ϵ and thus, plays the role of a filter in the $\mathbf{q}$ space.

After expansion and average, the equation (18) then becomes at the order 1:

$$\begin{aligned} < \partial_t W^{(0)}_k > &+ i\int \mathbf{k}\cdot[\mathbf{v}^{(\mathbf{0})}_{\mathbf{q}} - < \mathbf{v}^{(\mathbf{0})}_{\mathbf{q}} >][W^{(0)}_{k-q/2} - W^{(0)}_{k+q/2}] e^{i\mathbf{q}\cdot\mathbf{x}} \frac{d\mathbf{q}}{(2\pi)^2} \\ &+ \frac{1}{2}\mathrm{div}\int \mathbf{v}^{(\mathbf{0})}_{\mathbf{q}}[W^{(0)}_{k-q/2} + W^{(0)}_{k+q/2}] e^{i\mathbf{q}\cdot\mathbf{x}} \frac{d\mathbf{q}}{(2\pi)^2} = 0. \end{aligned} \tag{21}$$

The first and the third term of this equation are zero because of the properties of the averaging. The second term is $O(\eta)$ and disappears upon taking the limit $\eta \to 0$. The relevant equation appears at the next order, which is $O(\epsilon)$:

$$\begin{aligned} & < \partial_T W^{(0)}_k > + < \mathrm{Div}\int < \mathbf{v}^{(0)}_Q > W^{(0)}_k e^{iQ\mathbf{X}} \frac{dQ}{(2\pi)^2} > \\ + & < i\int \mathbf{k}\cdot < \mathbf{v}^{(\mathbf{0})}_{\mathbf{Q}} > Q\partial_k W^{(0)}_k e^{iQ\mathbf{X}} \frac{dQ}{(2\pi)^2} >= \\ - & < \partial_t W^{(1)}_k > - < \mathrm{div}\int [\mathbf{v}^{(0)}_q W^{(1)}_k + \mathbf{v}^{(1)}_q W^{(0)}_k] e^{iq\cdot\mathbf{x}} \frac{d\mathbf{q}}{(2\pi)^2} > + O(\eta). \end{aligned} \tag{22}$$

Using the properties of the averaging and taking the limit $\eta \to 0$, we see that the right hand side of eq. (22) is zero. Taking into account the incompressibility condition $< \mathbf{v}_Q > \cdot\mathbf{Q} = 0$ and the explicit expansion of W_k, we finally have the following evolution equation for the small-scale turbulence,

$$\partial_T n + (\mathbf{v}_L\cdot\nabla)n - (\nabla(\mathbf{v}_L\cdot\mathbf{k}))\cdot\partial_k n = 0, \tag{23}$$

where $n = n(\mathbf{k}, \mathbf{X}, T)$ is defined in terms of the small scale vorticity as

$$n = \int < \omega_{S,p+k} \omega_{S,p-k} > e^{2i\mathbf{p}\cdot\mathbf{X}} \frac{d\mathbf{p}}{(2\pi)^2}. \tag{24}$$

Here, we have used the notation $\overline{f}$ to denote the time average over the fast time. We will call the function n the turbulence action, to emphasize analogy with the wave action arising in the WKB theory, as will be discussed later. The function n has the physical meaning of an enstrophy density in $(\mathbf{k}, \mathbf{X})$ space. This equation can be written as a conservation law for the turbulence action,

$$D_T n = 0, \tag{25}$$

where

$$\begin{aligned} D_T &= \partial_T + \dot{\mathbf{X}} \cdot \nabla + \dot{\mathbf{k}} \cdot \partial_k, \\ \dot{\mathbf{X}} &= \partial_k \Omega, & (26) \\ \dot{\mathbf{k}} &= -\nabla \Omega, & (27) \\ \Omega &= \mathbf{v}_L \cdot \mathbf{k}. & (28) \end{aligned}$$

In the derivation of the equation (23) we have used the assumptions about the scale separation and the small amplitude of the small-scale component. Similar assumptions are made in the WKB theory of sound and waves of other types [13], [14]. Not surprisingly, equation (23) resembles the WKB equation for the wave action [13], [14]. On the other hand, there is one important difference with the WKB theory, related to the fact there is no propagating waves in incompressible fluids. This difference is in relation of the action n to the energy density e in $(\mathbf{k}, \mathbf{X})$ space. For waves, the relation is $n = e/\Omega_c$, where Ω_c is the wave frequency in the frame co-moving with the large-scale component. For incompressible fluids, according to (1), we have $n = k^2 e$, that is, n is the enstrophy density.

By analogy with WKB, we will call (26) and (27) the ray equations. These equations are Hamiltonian with Hamiltonian function (28). One of the consequences of this is that these equations are integrable if Ω is independent of time (i.e. $\mathbf{v}_L$ is time independent) and there is another motion integral, such as the momentum (if $\mathbf{v}_L$ is translationaly invariant) or the angular momentum (if $\mathbf{v}_L$ is axially symmetric). These cases correspond to the large-scale flows having the form of a stationary shear flow and a stationary circular vortex respectively.

2.3 Dynamics of the large vortices forced by turbulence

We now consider the Euler equation (1). Before applying the two scale analysis (i.e. expansion in ϵ), we note that the nonlinear term can be written:

$$\begin{aligned} \mathbf{v}\cdot\nabla\omega &= \int \frac{(\mathbf{k}\times\mathbf{p})_z}{p^2}\ \omega_p\omega_{k-p}\ \exp(i\mathbf{k}\cdot\mathbf{X})\ \frac{d\mathbf{p}}{(2\pi)^2}\frac{d\mathbf{k}}{(2\pi)^2} \\ &= 2\int \frac{(\mathbf{p}'\times\mathbf{k}')_z}{(\mathbf{p}'-\mathbf{k}')^2}\ \omega_{p'-k'}\omega_{p'+k'}\ \exp(2i\mathbf{p}'\cdot\mathbf{X})\ \frac{d\mathbf{k}'}{(2\pi)^2}\frac{2d\mathbf{p}'}{(2\pi)^2}, \end{aligned} \tag{29}$$

where

$$\mathbf{p}' = \mathbf{k}/2, \tag{30}$$

$$\mathbf{k}' = \mathbf{k}/2 - \mathbf{p}, \tag{31}$$

and $(\mathbf{a}\times\mathbf{b})_z$ denotes the z-component of the vector product of vectors $\mathbf{a}$ and $\mathbf{b}$. When p' is of order ϵ (i.e. when the divergence is taken over the slow variables), one can perform the following expansion in $\mathbf{p}'$

$$\frac{1}{(\mathbf{p}'-\mathbf{k}')^2} \approx \frac{1}{k'^2}\left(1+2\frac{(\mathbf{p}'\cdot\mathbf{k}')}{k'^2}\right). \tag{32}$$

The first term in the expansion is equal to zero, because of the symmetry properties of the integrand with respect to $k\to -k$. The second term gives the following contribution

$$2\int (\mathbf{k}\times\nabla)_z \frac{(\mathbf{k}\cdot\nabla)}{k^4}\, W_k\, \frac{d\mathbf{k}}{(2\pi)^2} = O(\epsilon^2 W_k). \tag{33}$$

Taking this into account, and performing the expansion in ϵ, we find that the first non trivial equation is obtained at order ϵ^2 and reads:

$$\partial_T \omega_L + (\mathbf{v}_L \cdot \nabla)\omega_L + 2\int (\mathbf{k}\times\nabla)_z \frac{(\mathbf{k}\cdot\nabla)}{k^4}\, n\, \frac{d\mathbf{k}}{(2\pi)^2} = 0. \tag{34}$$

Because $n \sim k^2 \mathbf{v}_S^2$, the last term in this equation is $O(\eta^2)$ and is usually smaller than the second one. It takes time $T = (1/\eta^2)$ for the small scales to produce a significant change in the large-scale dynamics.

2.4 Momentum Conservation

Integrating ("uncurling") equation (34) one can get an equation for the large scales in terms of the large-scale velocity,

$$\partial_T v_{Li} + (\mathbf{v}_L \cdot \nabla) v_{Li} = -\partial_{Xj}\sigma_{ij}, \tag{35}$$

where

$$\sigma_{ij} = -2\int \frac{k_i k_j}{k^4}\, n\, \frac{d\mathbf{k}}{(2\pi)^2}. \tag{36}$$

Integrating (35) over the coordinate space we arrive at the conservation law for the momentum,

$$\mathbf{P} = \int \mathbf{v_L}\, d\mathbf{X} = \text{const}. \tag{37}$$

Because $n(\mathbf{k}) = n(-\mathbf{k})$, there is no momentum associated with the small-scale component, $\int \mathbf{k} n(\mathbf{k})\, d\mathbf{k} = 0$. Therefore, it is natural that conservation of the total momentum results in conservation of the large-scale momentum (37). One can see from (35) that the small-scale component does not produce any large-scale momentum (even locally), but it does modify its flux.

2.5 Energy Conservation

Let us introduce introduce the stream function of the large-scale flow, $\psi(\mathbf{r}, t)$, such that

$$\mathbf{v}_L = \mathrm{curl}\,(e_z\, \psi), \tag{38}$$

where e_z is the unit vector in z direction. Let us multiply (34) by ψ and integrate over the entire coordinate domain. Assuming $\mathbf{v}_L = 0$ at infinity (or periodic if the domain is finite) and integrating by parts, we have

$$-\frac{1}{2}\partial_T \int \mathbf{v}_L^2 \, d\mathbf{X} + 2 \int n \, \frac{(\mathbf{k} \cdot \nabla)}{k^4} \, (\mathbf{k} \times \nabla \psi)_z \, \frac{d\mathbf{k}}{(2\pi)^2} \, d\mathbf{X} = 0. \tag{39}$$

Assuming that $n|_{k=\infty} = 0$, and taking the second integral by parts with respect to $\mathbf{k}$, we obtain

$$-\frac{1}{2}\partial_T \int \mathbf{v}_L^2 \, d\mathbf{X} + \int \frac{1}{k^2} \, \partial_k (n \, \nabla (\mathbf{k} \cdot \mathbf{v}_L)) \, \frac{d\mathbf{k}}{(2\pi)^2} \, d\mathbf{X} = 0. \tag{40}$$

By virtue of (23), the second term here can be written as $\int \frac{1}{k^2} \, \partial_T n \, \frac{d\mathbf{k}}{(2\pi)^2} \, d\mathbf{X}$. Taking this into account, we finally obtain the energy conservation law,

$$\partial_T (E_L + E_S) = 0, \tag{41}$$

where E_L and E_S are the energies of the large vortices and turbulence correspondingly,

$$E_L = \frac{1}{2} \int \mathbf{v}_L^2 \, d\mathbf{X}, \tag{42}$$

$$E_S = \int \frac{n}{k^2} \, \frac{d\mathbf{k}}{(2\pi)^2} \, d\mathbf{X}. \tag{43}$$

Thus, the total energy of the system of the large vortices and turbulence is conserved. According to the applicability condition $\mathbf{v}_L \gg \mathbf{v}_S$, the large-scale energy is greater than the small-scale one, $E_L \gg E_S$. On the other hand, the exchange of energy between the large vortices and turbulence can change significantly the energy of the large vortices if there is a constant supply of the small scales into the system. Considering a particular example in the next section, we will see that the results strongly depends on the isotropy properties of the small scale turbulence.

Note that $\mathbf{v}_L \gg \mathbf{v}_S$ does not imply $\omega_L \gg \omega_S$, so that the enstrophy of the large vortices, $\int \omega^2 \, d\mathbf{X}$, is not necessarily larger than the small-scale enstrophy $\int n \, \frac{dk}{(2\pi)^2} d\mathbf{X}$. In our model, the small-scale enstrophy is conserved by virtue of (25). The exchange of enstrophy with the large vortices corresponds to the higher order corrections in the equation for n. This exchange is much slower than the enstrophy flux in the $(\mathbf{k}, \mathbf{X})$ space and, therefore, neglected in our model.

3 Vortex dipole propagating through small-scale turbulence

3.1 General properties

In the case of a time independent large scale flow (e.g., an circular vortex or a shear flow) the ray equations (26) and (27) are integrable. In these cases, the turbulence wavenumbers will asymptotically (for large time) increase in magnitude proportionally to time and the shear on the corresponding streamline of the large-scale flow. Therefore, according to (41), all the turbulent energy will be transfered to the large scales. If there is no source of turbulence, the transfered energy is small compared to the large-scale energy, and corresponding changes in the large-scale flow are small. On the other hand, if the large vortices propagate through the turbulence, they "collect" greater turbulent energy from large areas, which can lead to significant modifications of these vortices. Therefore, the interaction of turbulence with moving vortices is more interesting than with stationary ones, and below we will consider a problem of this kind, in which the large-scale field is represented by a stationary vortex dipole.

Suppose that the vortex dipole propagates with a constant velocity u_0 through turbulence, which is stationary and homogeneous ahead of the vortex, see Figure 1. First, we will consider general properties of such a system, which are common for any distribution of vorticity in the dipole vortex. Let the characteristic turbulent scale be less than the dipole size,

$$kR \gg 1, \tag{44}$$

and let the circulation $\Gamma = \oint \mathbf{v} \cdot d\mathbf{l}$ be large enough so that the distance from the dipole, at which the condition (44) is satisfied, would be much greater than R. This is so if

$$\Gamma/R \gg \sqrt{n}. \tag{45}$$

Then, the system of equations (26) and (27), can be used to describe the dynamics of turbulence and the vortex dipole. Note that condition (44) is equivalent to (45) even in the case when the vorticity is concentrated in a pair of point vortices, and corresponding velocity field is singular (this special case is considered in Sec. 3.2). This is so because, according to (26), the turbulence is convected by the velocity field of the dipole and can never approach the vortices (in the vicinity of which the gradients are high) closer than the separatrix.

As we will see later, condition (45) also implies that the characteristic time of the changes in the vortex dipole is much greater than the time needed by turbulence to pass the vortex (i.e. R^2/Γ). Thus, one can neglect the the dipole's time dependence when solving the equations for the small scales (26) and (27).

3.1.1 Solution for the small-scale component

According to (25), the turbulence action is conserved along the rays, so that the dynamics of turbulence is determined by the solutions of the ray equations (26) and (27). Because velocity field $\mathbf{v_L}$ is time independent in the frame co-moving with the dipole, the ray equations possess the following motion integral,

$$\Omega - pu_0 = p(u - u_0) + qv = -p_0 u_0 = \text{const}, \tag{46}$$

where $\mathbf{v}_L = (u, v)$, $\mathbf{k} = (p, q)$, and the subscript 0 denotes the initial value, i.e. the value at $T = -\infty$. Solving (46) for p, we have

$$p = \frac{p_0 u_0 + qv}{u_0 - u}. \tag{47}$$

Substituting (47) into the y-component of (27), we obtain

$$\frac{dq}{dT} = -p\partial_y u - q\partial_y v = \frac{p_0 u_0}{u - u_0}\,\partial_y u + q\frac{v\partial_y u - (u - u_0)\partial_y v}{u - u_0}. \tag{48}$$

Taking into account the incompressibility condition $\partial_y v = -\partial_x u$ and that outside the separatrix $u_0 > u$, one can rewrite (48) as follows,

$$\frac{dq}{dT} = \frac{d\ln(u_0 - u)}{dT} q + \frac{p_0 u_0}{u - u_0} \partial_y u. \tag{49}$$

In this equation, $\mathbf{X} = (x, y)$ are functions of time satisfying (26) with initial conditions $x = \infty, \quad y = y_0$ at $T = -\infty$. In other words, solutions $x = x(T)$ and $y = y(T)$ have to be substituted into $u(\mathbf{X})$ and $\partial_y u(\mathbf{X})$. Exact solution to (49) with initial condition $q(-\infty) = q_0$ is

$$q = \frac{u_0 - u}{u_0} \left(-u_0^2 p_0 \int_{-\infty}^{t} \frac{\partial_y u}{(u_0 - u)^2} \, dt + q_0 \right). \tag{50}$$

Together with (47), this expression represents the solution of the ray equations with initial coordinate (∞, y_0) and initial wavenumber (p_0, q_0).

3.1.2 Energy exchange between the vortex dipole and turbulence

The total energy in the problem under consideration is infinite, and the energy conservation law (41) has to be modified as follows,

$$\partial_T E = P\,|_{x=+\infty} - P\,|_{x=-\infty}, \tag{51}$$

where P is the flux of the turbulent energy. The energy of the large-scale component is much greater than the energy of the small scales in the region of the vortex-turbulence interaction. Therefore $\partial_T E \approx \partial_T E_L$, that is the right hand side of (51) describes the rate of change of the large-scale energy due to the interaction with the small-scale turbulence. Using the fact that at $x = \pm\infty$ the turbulence propagates with the speed u_0 (in the vortex rest frame) to find P, we have

$$\partial_t E_L = u_0 \int n_0(\mathbf{k}_0) \, (k_0^{-2} - k_f^{-2}) \, \frac{d\mathbf{k}_0}{(2\pi)^2} \, dy_0, \tag{52}$$

where $n_0(\mathbf{k})$ is the initial spectrum of (homogeneous) turbulence ahead of the dipole (i.e. at $x = \infty$) and $k_f = k_f(y_0, p_0, q_0)$ is the absolute final value of the wavevector $\mathbf{k} = (p, q)$.

It is obtained by taking the solutions (47) and (50) at $t = \infty$,

$$
\begin{aligned}
k_f &= \sqrt{p_f^2 + q_f^2}, &(53)\\
p_f &= p_0, &(54)\\
q_f &= p_0 B(y_0) + q_0, &(55)
\end{aligned}
$$

where the function $B(y_0)$ is given by

$$
B(y_0) = -u_0^2 \int_{-\infty}^{\infty} \frac{\partial_y u}{(u - u_0)^2}\, dt. \tag{56}
$$

Important properties of $B(y_0)$ are discussed in the Appendix. The rate of energy exchange can also be written as follows,

$$
\partial_T E_L = u_0 \int n_0(\mathbf{k}_0)\, k_0^{-2}\, f\, d\mathbf{k}_0, \tag{57}
$$

where $f = f(\alpha)$ is a function of only the angle α between $\mathbf{k}_0$ and the x-axis which is defined as

$$
f(\alpha) = \int_{-\infty}^{\infty} \left(1 - \frac{k_0^2}{k_f^2}\right) dy_0. \tag{58}
$$

¿From the expression for k_f, we may also write $f(\alpha)$ as

$$
f(\alpha) = \int_{-\infty}^{\infty} dy_0 \left(1 - \frac{\cos^{-2}\alpha}{1 + (B(y_0) + \tan\alpha)^2}\right). \tag{59}
$$

Some properties of $f(\alpha)$ are discussed in the Appendix.

In the case of isotropic turbulence, the energy transfer is equal to zero because $\int_0^{2\pi} f(\alpha)\, d\alpha = 0$. Indeed,

$$
\begin{aligned}
\int_0^{2\pi} f(\alpha)\, d\alpha &= 2\int_{-\infty}^{\infty} dy_0 \int_{-\pi/2}^{\pi/2} \left(1 - \frac{\cos^{-2}\alpha}{1 + (B(y_0) + \tan\alpha)^2}\right) d\alpha \\
&= 2\int_{-\infty}^{\infty} dy_0 (\alpha - \arctan(B + \tan\alpha)|_{-\pi/2}^{\pi/2} = 0. &(60)
\end{aligned}
$$

Actually, the energy transfer is equal to zero at any streamline of the vortex because the integrand in the last integral of (57) is zero for any y_0. Therefore, a local anisotropy in the small scales is essential for energy exchanges between small and large scales. Such a local anisotropy is usually present even when the turbulent spectrum is globally isotropic (i.e., when averegad over the entire coordinate space). For example, a vortex dipole passing through originally isotropic turbulence leaves an essentially anisotropic turbulence in its wake.

In this case, according to (51) and (52), the contours of constant $n(\mathbf{k})$ the k-space will change from circles (for $n(\mathbf{k}_0)$) to ellipses (for $n(\mathbf{k}_f)$) with the aspect ratio being a function of y_0. The ellipse area is equal to the area of the original circle. The aspect ratio tends to infinity as y_0 tends to zero because of the property $\lim_{y_0\to 0} B(y_0) = \infty$ explained in the Appendix. For large aspect ratios, most of the turbulent energy is concentrated in small part of the ellipse near the origin in k-space. Actually, if $\mathbf{k}$ becomes so small that the scale separation is violated, one should consider this part of the turbulent spectrum (and all its energy) as being transferred to the large-scale component. By virtue of, (25) the final turbulence spectrum (i.e., far behind the vortex) is

$$n_f(\mathbf{k}, \mathbf{r}) = n_0(p, q - pB(y)). \tag{61}$$

3.2 Special case: Point-vortex dipole

The results obtained in the previous section can be illustrated on a simple example. Let the large-scale field be represented by a pair of vortices with circulations $+\Gamma$ and $-\Gamma$ and located at $x = 0, \quad y = \pm R$ in the frame co-moving with the dipole. This vortex dipole is moving along the x-axis with the speed

$$u_0 = \frac{\Gamma}{4\pi R}, \tag{62}$$

and generates velocity field $\mathbf{v_L} = \mathrm{curl}\,(e_z\,\psi)$ with the stream function

$$\psi = \frac{\Gamma}{4\pi} \ln \frac{x^2 + (y+R)^2}{x^2 + (y-R)^2}. \tag{63}$$

There is only one parameter responsible for the changes in the point-vortex dipole, the distance between the point vortices, R. Given a solution for the small scales, the solution for $R(t)$ can be found from energy conservation (41), instead of solving the equations for the large scales (34). Now we can use the above solution for the small-scale component and the energy conservation to find the changes in the large-scale vortex, namely, to obtain the solution for the dipole size $R(t)$. Evolution equation for $R(t)$ can be obtained by substitution in (57) the expression for the energy of the point-vortex dipole,

$$E_L = -\frac{\Gamma^2}{8\pi^2} \ln(2R). \tag{64}$$

We obtain:

$$-\frac{\Gamma^2}{8\pi^2}\partial_T \ln(2R) = u_0 \int n_0 k_0^{-2} f \, \frac{d\mathbf{k}_0}{(2\pi)^2}, \tag{65}$$

where $n_0(\mathbf{k})$ is the initial spectrum of (homogeneous) turbulence ahead of the dipole (i.e. at $x = \infty$) and f is the function defined in (59). Taking into account that $u_0 = \Gamma/4\pi R$ and that, according to (55) and (63), the quantity k_f depends on y_0 only in combination y_0/R, one can see that the right hand side of (65) is independent of R. Therefore, the rate of the energy exchange between turbulence and the vortex dipole is independent of time. Integrating (65) we find

$$R(T) = R(0) \, \exp\left(-\frac{T}{2\pi\Gamma} \int n_0(\mathbf{k}_0)\, k_0^{-2}\, f \, d\mathbf{k}_0\right). \tag{66}$$

As we showed in Sec. 3.2, for isotropic turbulence $f = 0$, and R is independent of time. For anisotropic turbulence, the distance between vortices will exponentially decrease or increase depending on the shape of the spectrum.

In order to calculate the final state of the turbulent spectrum, (61), one has to know the function $B(y_0)$. The function B can be computed numerically by noting that, according to (55), $B(y_0) = q_f$ when $\alpha = 0$. It is then sufficient to integrate the ray equations (26) and (27) with $q_0 = 0$ and $p_0 = 1$, starting at $x = \infty$ and for various initial y_0. This was done using a fourth-order Runge Kunta method. The resulting $B(y_0)$ is displayed in Fig. 2. It behaves like $(y_0/R)^{-1}$ near the origin, and like $(y_0/R)^{-3}$ at large y_0/R in accordance with the properties of B considered in Appendix. A good approximation of $B(y_0)$ valid for y_0/R between 0 and 10, is given by:

$$B(y_0) \approx \frac{2.28R^3}{y_0(R^2 + 0.19y_0^2)}. \tag{67}$$

Using the function $B(y_0)$, one gets the function $f(\alpha)$ by numerical integration of (59). Note that owing to the symmetries $f(\alpha) = f(-\alpha)$ and $f(\alpha) = f(\pi - \alpha)$ (see Appendix), it is sufficient to compute $f(\alpha)$ for α between 0 and $\pi/2$. The result is shown Fig. 3. One can see that f is positive for $0 < \alpha < 0.95$ which means that for such range of the wavenumber directions the turbulent spectrum transfers its energy to the vortex dipole, whereas at $0.95 < \alpha < \pi/2$ the turbulence gains energy in expense of the vortex because $f < 0$. In accordance with the general properties of $f(\alpha)$ described in Appendix, $f(\pi/2) = 0$. On the other hand, the data in Fig. 3 suggests that $\lim_{\alpha\to\pi/2}\ f \approx 7R$. We will see in Appendix, that the value of f for $\alpha \approx \pi/2$ is determined by the behavior of $B(y_0)$ at small y_0, and that the theory predicts $\lim_{\alpha\to\pi/2}\ f \approx 7.16R$. The numerical integration of f over all alphas is equal to zero with accuracy of about 1%.

4 Conclusion

In this paper, we applied a two-scale analysis to the incompressible 2D Euler equations, and derived equations for interacting turbulent and large-scale vortex components. Equation for the small-scale turbulence are written in terms of the Wigner function for vorticity (the turbulence action) having the physical meaning of the enstrophy density. Similarly to the WKB equations, this equation describes conservation of the turbulence action in (k, x)-space. The Wigner function formalism used in the present paper is a convenient tool, but it is not the only way to describe the small scale turbulence. For example, partial Fourier transforms or other wavelet-type of transform may be used instead.

Equation for the large-scale vortices has a form of the original Euler equations with added source term describing interaction with the small scales. The total energy of the system is shown to be conserved, although an exchange of energy between the turbulence and the large vortices is possible. We solve equations for the small-scale turbulence on the background of a propagating vortex dipole, and show that for any shape of the vortex dipole the transfer of energy from the turbulence to the vortex is equal to zero if the turbulent spectrum is isotropic ahead of the vortex. The main effect in this case is redistribution and anisotropisation of the turbulent spectrum, whereas the state of the vortex dipole remains unchanged. If initially the turbulence is anisotropic, then the vortex dipole will gain or loose its energy depending on the shape of the turbulent spectrum. We find the solution for the large-scale vortex in the special case of the point-vortex dipole.

The equations derived in this paper may be used for a numerical modeling of the interaction of widely separated scales in 2D incompressible fluids. One can analyze, for example, interaction of small-scale turbulence with multiple large vortices involved in a non stationary and, possibly, chaotic motion. In some sense, such an analysis would be similar to the sub-grid scale modeling because our equations are averaged over the small (sub-grid) scales. On the other hand, the traditional sub-grid scale models are different because they do not compute evolution of the small-scale component, but rather postulate its structure.

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5 Appendix

In this appendix, we derive some properties of the functions $B(y_0)$ and $f(\alpha)$ defined by (56) and (56) correspondingly.

The ray equations are invariant under the symmetries:

$$x \to x;\ y \to -y;\ u \to u;\ v \to -v,$$
$$x \to -x;\ y \to y;\ u \to -u;\ v \to v. \tag{68}$$

Under these symmetries, function $B(y_0)$ defined by (56) is an odd function, $B(y_0) = -B(-y_0)$. Taking into account that $(u - u_0)\,dt = dx$ and the above symmetries, we can rewrite expression (53) for $B(y_0)$ as

$$B(y_0) = -2u_0^2 \int_0^\infty \frac{\partial_y u}{(u-u_0)^3}\, dx. \tag{69}$$

The integrand is understood to be a function of only x, that is, we have to substitute for y its value on the streamline, $y = y(x, y_0)$. The function $y(x, y_0)$ from the condition that the the stream function is constant on the streamline in the co-moving frame,

$$\Psi - u_0 y = \text{const} = u_0 y_0, \tag{70}$$

where Ψ is the rest-frame stream function. For the point-vortex dipole the rest-frame stream function is given by (63). In this case

$$u = \frac{\Gamma}{2\pi}\left[\frac{y+R}{(y+R)^2+x^2} - \frac{y-R}{(y-R)^2+x^2}\right], \tag{71}$$

5.1 Behavior of $B(y_0)$ at small y_0.

Consider the case of the point-vortex dipole. For small y_0, the main contribution into the integral (69) gives the vicinity of the stagnation point $x = \sqrt{3}$, $y = 0$ where $u = u_0$. Expanding (71) in series of $(x - \sqrt{3})$ and y we get in the leading order

$$u - u_0 = -\frac{\sqrt{3}u_0}{2R}(x - \sqrt{3}) - \frac{u_0}{2R^2}y^2, \tag{72}$$

and, therefore,

$$\partial_y u = -u_0 y/R. \tag{73}$$

It follows from (72) that the major contribution to the integral (69) gives the range where y^2 scales as $(x - \sqrt{3})R$. Taking into account this fact when expanding (70), we find that in the leading order (70) transforms into the following cubic equation for y,

$$y^3 + 3R\sqrt{3}(x - \sqrt{3})y - y_0 R^2 = 0. \tag{74}$$

In the range $0 < x < +\infty$ the number of the real solutions of this equation changes from three to one. When the number of the real solutions is three, two of them correspond to a closed streamline inside the separatrix, which is of no interest for us. Therefore, we have to choose the solution that remains real everywhere in the range $0 < x < +\infty$, and that corresponds to an open streamline, and along which the turbulence is approaching the vortex dipole. This solution is

$$y = R(y_0/2R + i\gamma)^{1/3} + R(y_0/2R - i\gamma)^{1/3}, \tag{75}$$

where

$$\gamma = \sqrt{3\sqrt{3}(x-\sqrt{3})^3/R^3 + y_0^2/4R^2}. \tag{76}$$

Substituting (70), (71), (73) and (74) into (67), and making the substitution $z = y_0^{-2/3} R^{-1/3}(x-\sqrt{3})$, we finally obtain the following expression for $B(y_0)$ (for small y_0),

$$B(y_0) = \lambda \frac{R}{y_0}, \tag{77}$$

where λ is the following dimensionless constant

$$\begin{aligned} \lambda &= -16\int_{-\infty}^{+\infty} \frac{\left(1/2 + i\sqrt{3\sqrt{3}z^3+1/4}\right)^{1/3} + \left(1/2 - i\sqrt{3\sqrt{3}z^3+1/4}\right)^{1/3}}{\left(\sqrt{3}z + \left[\left(1/2 + i\sqrt{3\sqrt{3}z^3+1/4}\right)^{1/3} + \left(1/2 - i\sqrt{3\sqrt{3}z^3+1/4}\right)^{1/3}\right]^2\right)^3} dz \\ &\approx 2.28 \end{aligned}$$

As we see, the function $B(y_0)$ is singular at $y_0 = 0$. This fact is related with the intense distortion of the small-scale turbulent vortices in the vicinity of the stagnation point, which corresponds to the generation of a high-wavenumber component in turbulence. Because the stagnation points are present in the vortex dipoles with any shape of the vorticity distribution, for all of such vortices the function $B(y_0)$ will have a singularity at $y_0 = 0$. It is likely that in the general case formula (74) will still be valid, although the specific value of constant λ will vary.

5.1.1 Behaviour of $B(y_0)$ at large y_0.

Expanding (67) and (68) in small R/y_0, we get in the leading order

$$y(x) = y_0,$$

$$
\begin{aligned}
u - u_0 &= -\frac{\Gamma}{4\pi R}, \\
\partial_y u &= \frac{\Gamma}{2\pi} \frac{4R^2(y_0^2 - 3x^2)}{(x^2 + y_0^2)^3} \frac{y_0}{R}.
\end{aligned}
$$

Substituting these expressions into (66) and setting $x = sy_0$, we get

$$
B(y_0) = -16\frac{R^2}{y_0^2} \int_0^\infty \frac{3s^2 - 1}{(1 + s^2)^3} ds = 0. \tag{78}
$$

This shows that $B(y_0)$ is at least of order $(R/y_0)^3$. Numerically, we find that $B(y_0) \approx 12(R/y_0)^3$.

5.2 Properties of $f(\alpha)$.

Using the definition (59) and taking into account that $B(y_0)$ is an odd function, one can derive that the function $f(\alpha)$ possesses the following symmetries,

$$
f(\pi - \alpha) = f(-\alpha) = f(\alpha). \tag{79}
$$

It follows also from the definition that $f(\pi/2) = 0$. Remarkably, $f(\alpha)$ has a finite jump at $\alpha = \pi/2$. To see this, we note that for $\alpha \approx \pi/2$, the major contribution to the integral (56) is given by the small vicinity of the point y^* such that $B(y^*) = -\tan\alpha$. Because $\tan\alpha$ is large, one can use (74). Expanding $B(y_0)$ in small $(y_0 - y^*)$, we have

$$
B(y_0) \approx -\tan\alpha - \frac{\tan^2\alpha}{\lambda R}(y_0 - y^*). \tag{80}
$$

Substituting this expression into (56) we have

$$
f(\alpha) = \int_{-\infty}^{+\infty} \frac{\tan^2\alpha}{1 + \tan^4\alpha (y_0 - y^*)^2/\lambda^2 R^2}\, dy_0 = -\pi\lambda R. \tag{81}
$$

In the case of a point-vortex dipole $\lambda \approx 2.28$, so that $\lim_{\alpha\to\pi/2} f(\alpha) \approx 7.16R$.