"Non-traditional" Coriolis terms, inertial waves and mixing in the deep ocean

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The Motivations

The Broad Picture

The global warming predictions are based primarily upon climate models. The key element of all such models are ocean circulation models. The ocean circulation models, as the name suggests, are concerned only with the large-scale motions, while all other motions of scales, less than, say, 10^2 km are not resolved. They are merely parameterised, very often on quite flimsy grounds.

It is widely believed (after Munk 66) that the abyssal basin-average vertical diffusivity K_v is about $10^{-4} m^2/s$. However, this view is being challenged now. The current state of knowledge (or ignorance) is such that opinions ranging from the key importance of mixing in the stratified ocean to its total insignificance are equally legitimate.

There exist hand waiving arguments that the mixing is provided by small scale turbulence caused by internal wave breaking, however, specific mechanisms (apart from the critical reflection by bottom relief) have not been even identified.

In this we work we suggest a plausible mechanism of wave breaking in the abyssal ocean and develop its quantitative description.

The "Traditional Approximation"

Internal waves, as well as all oceanic motions of scales small compared to the Earth's radius are commonly described as if the Earth were locally flat, i.e. the motions are considered in a plane tangent to the Earth's surface, attached at the location under consideration. This plane is co-rotating with the Earth's angular velocity $\vec{\Omega}$.

In the equations of motions, written in a Cartesian frame fixed to this plane at latitude ϕ, the Coriolis vector has two components: a horizontal (meridional) one $\tilde{f} = 2\Omega\cos\phi$, and a vertical one $f = 2\Omega\sin\phi$ ($\Omega = |\vec{\Omega}|$). At mid-latitudes the two are comparable.

The neglect of the terms involving the horizontal component represents, the so-called, ***"traditional approximation"***.

The Coriolis force acts in a direction perpendicular to fluid velocity; hence the horizontal component $\tilde{f}$ can act in only two ways: in the presence of a zonal motion it creates a vertical acceleration, and in the presence of a vertical velocity it creates a zonal acceleration.

The commonly used intuitive argument to justify the traditional approximation is: since low-frequency motions are predominantly horizontal, neither vertical accelerations nor vertical velocities play a significant role; hence the effect of $\tilde{f}$ must be negligible.

The fundamental flaw of this argument lies in the implicit assumption that the effect due to the horizontal component of rotation represents a *regular perturbation*.

We will show that:

Simultaneous taking into account of

(i) *"non-traditional" effects*,

(ii) the fact that the fluid is vertically confined, and

(iii) β-effect (or any other horizontal inhomogeneity)
leads to profound implications for ocean dynamics.

A new generic mechanism enhansing inertial wave breaking and intensification of mixing emerges.

The near-inertial waves are by far the most energetic part of the IW spectrum in the ocean and, therefore, whatever happens with them -matters.

Basic equations and properties

The Earth's rotation angular velocity is $\vec{\Omega}$ ($|\vec{\Omega}| = 7.29 \cdot 10^{-5} rad/s$).

The Coriolis vector has two components:
a a vertical one

$$f = 2\Omega \sin\phi$$

($\Omega = |\vec{\Omega}|$, ϕ is latitude; and a horizontal (strictly meridional) one:

$$\tilde{f} = 2\Omega \cos\phi$$

.

We begin with the standard linear equations on the non-traditional f-plane under the Boussinesq approximation,

$$
\begin{aligned}
u_t - fv + \tilde{f}w &= -p_x && (1)\\
v_t + fu &= -p_y && (2)\\
w_t - \tilde{f}u &= -p_z + b && (3)\\
u_x + v_y + w_z &= 0 && (4)\\
b_t + N^2 w &= 0\,, && (5)
\end{aligned}
$$

where p is the departure of pressure from its hydrostatic value (divided by a constant reference density), b the buoyancy. The Cartesian frame with the coordinates : x (west-east), y (south-north); z (vertical, positive upward, with the origin at the unperturbed ocean surface) is used; while u, v and w are the corresponding velocity components.

The eqs of motion can be reduced to a single equation for vertical velocity w

$$\nabla^2 w_{tt} + (\vec{f} \cdot \nabla)^2 w + N^2 \nabla_h^2 w = 0 \tag{6}$$

$\vec{f} = (0, \tilde{f}, f)$, and ∇_h^2 denotes the horizontal Laplacian. We allow the buoyancy frequency N to depend on z.

In the traditional approximation one would take $\tilde{f} = 0$.

For plane monochromatic ($w = W\exp(i\sigma t)$) waves travelling in the $\chi = x\cos\alpha + y\sin\alpha$ direction, we find

$$(N^2 - \sigma^2 + f_s^2)W_{\chi\chi} + 2ff_sW_{\chi z} - (\sigma^2 - f^2)W_{zz} = 0\,, \tag{7}$$

where $f_s = \tilde{f}\sin\alpha$. Equation (7) is the starting point of our study.

Shorthand presentation

$$AW_{\chi\chi} + 2BW_{\chi z} + CW_{zz} = 0\,, \tag{8}$$

If we consider unbounded fluid and set $N = const$ the the eq-n admits plane wave solutions for obliquely propagating waves

$$\exp i(kx + ly + mz - \sigma t)$$

we recover a well-known dispersion relation

$$\sigma^2 = N^2 \cos^2\theta + (\vec{f} \cdot \vec{k})^2 / |\vec{k}|^2$$

The limiting cases:

(i) *Gravity internal waves:*

$$\sigma^2 = N^2 \cos^2\theta$$

(ii) *Gyroscopic or inertial waves*

$$\sigma^2 = (\vec{f} \cdot \vec{k})^2 / |\vec{k}|^2$$

"Non-traditional" boundary-value problem

On employing the standard boundary conditions at the surface and the bottom, the substitution of

$$W = \psi(z) \exp i(k\chi + \delta z)$$

with

$$\delta = -k f f_s/(\sigma^2 - f^2)$$

leads to the following BVP for ψ

$$\psi'' + k^2 \left[\frac{N^2(z) - \sigma^2}{\sigma^2 - f^2} + \left(\frac{\sigma f_s}{\sigma^2 - f^2} \right)^2 \right] \psi = 0 \,, \qquad \psi(0) = \psi(H) = 0, \quad (9)$$

Note, that the "non-traditional term" containing f_s yields a higher order singularity, but does not pose any extra difficulty.

The nontraditional term becomes dominant when $\sigma \approx f$, that is, in the vicinity of the inertial frequency. To the leading order the solution becomes independent on $N(z)$ and can be readily found for the $n-$th mode

$$(\sigma^2 - f^2)/\sigma = \pm \frac{f_s H}{2\pi n} k, \qquad k > 0 \tag{10}$$

Thus, in contrast to the "traditional" boundary-value problem where for all eigenmodes $\sigma^2 - f^2 > 0$, we, in addition, have got another family of *sub-inertial* modes with $\sigma^2 - f^2 < 0$.

Frequencies of the modes of both families tend to inertial frequency in the longwave limit. For super-inertial waves this looks similar to the modes behaviour under the traditional approximation, however, there is a qualitative difference: ***group velocity does not vanish at the inertial frequency***:

$$\frac{d\sigma}{dk} = \frac{f_s H}{2\pi n}$$

For sub-inertial waves the expression for σ_k ***differs only in sign.***

The range of allowed sub-inertial frequencies is $(O((\tilde{f}/N)^2))$ narrow :

$$\sigma_{min}^2 = \frac{1}{2}\left(\lambda - \sqrt{\lambda^2 - (2fN)^2}\right) < \sigma^2 < f^2 \qquad (\lambda = N^2 + f^2 + f_s^2)\,.$$

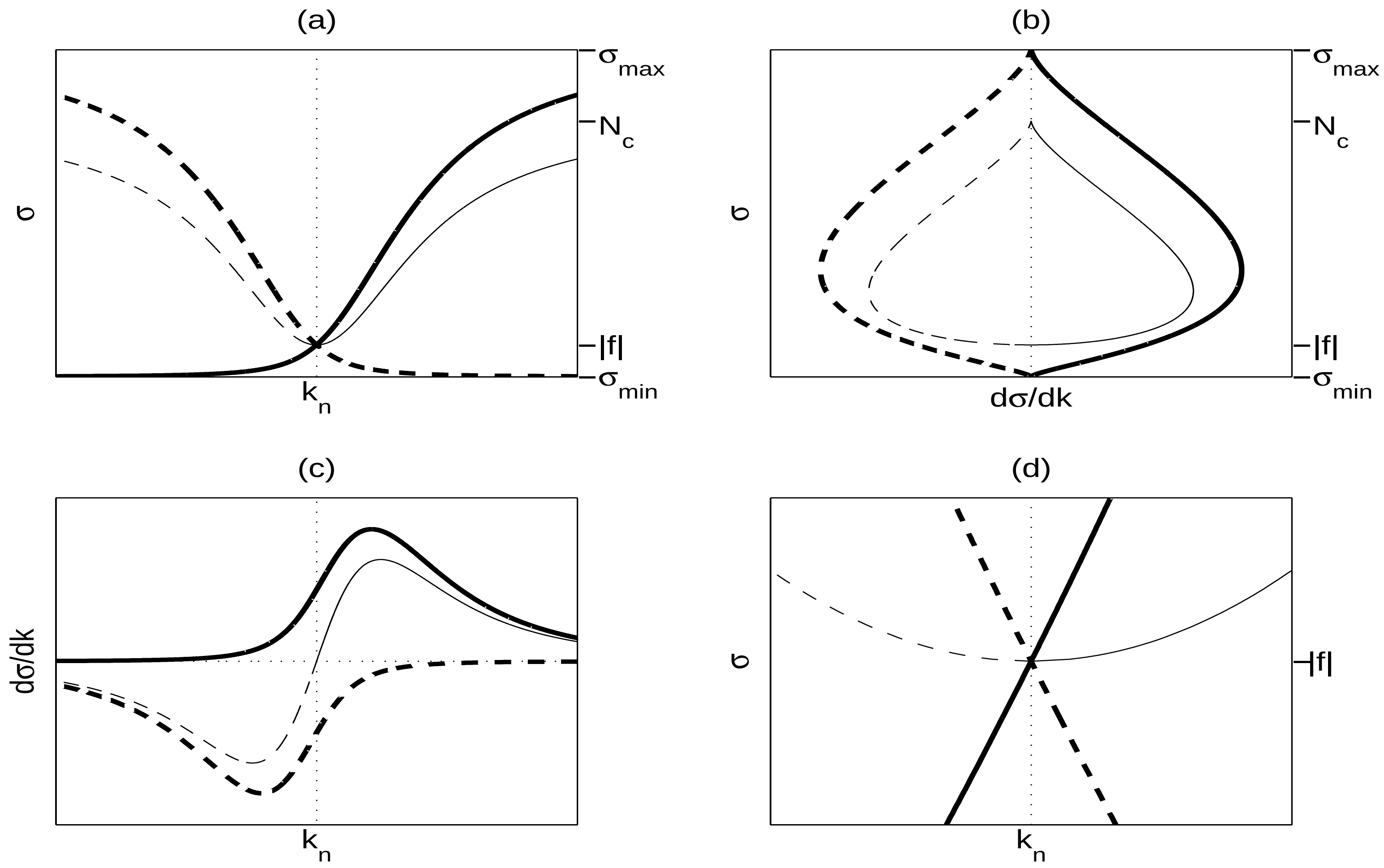


Figure 1: Dispersion relation $\sigma(k_1)$

Trapped sub-inertial modes

It is convenient to re-write (9) in the equivalent more explicit form

$$\psi'' + \widetilde{k}^2 \Big[- N^2(z) + \alpha^2(\sigma) \Big] \psi = 0, \qquad \psi(0) = \psi(H) = 0, \qquad (11)$$

where

$$\widetilde{k}^2 = \frac{k^2}{|\sigma^2 - f^2|} \qquad \text{and} \qquad \alpha^2(\sigma) = -\frac{\sigma^2 f_s^2 - \sigma^2(\sigma^2 - f^2)}{(\sigma^2 - f^2)} > 0$$

For a sub-inertial wave motion to exist the square brackets must be positive, i.e. $N^2(z) < \alpha^2(\sigma)$, which implies that for the most of σ in the allowed range the sub-inertial wave motion will be localised near ***minima*** of $N^2(z)$ and prohibited near the maxima.

Modal structure for stratification $N(z)$ typical of ocean

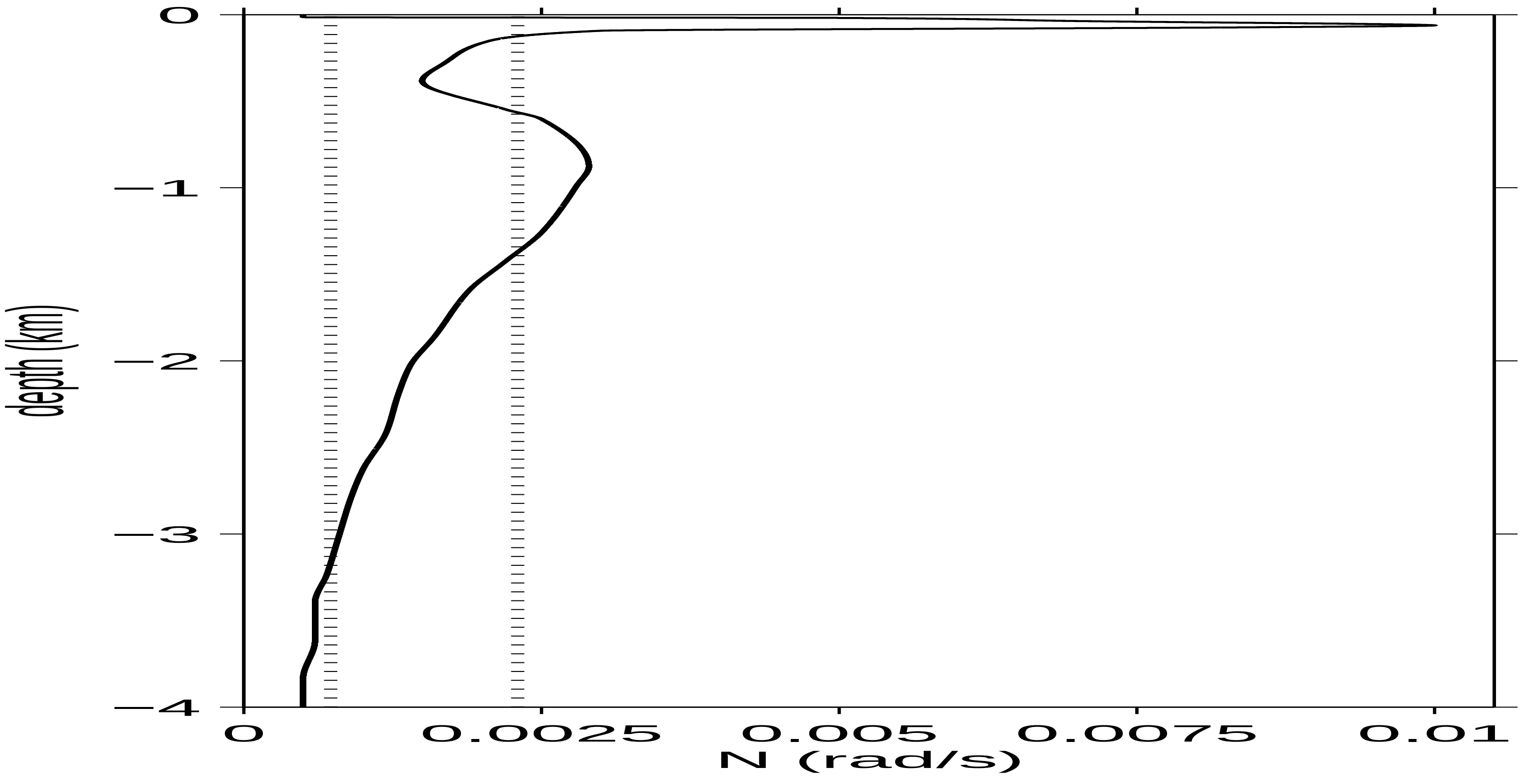


Figure 2: An empirical profile of the buoyancy frequency N, from the Bay of Biscay (summer). The black dashes denote the regions where sub-inertial waves, within the range $0.9785f$–$0.999f$, can propagate

The bottom waveguide

For isolated bottom-trapped sub-inertial waves in the abyssal ocean the weak dependence of N^2 with depth is well approximated by a linear dependence: $N^2(z) = N_0^2 + \gamma z$. Eq. (9) then becomes

$$\psi'' - l^2 \left[z - \frac{\Delta}{\gamma C} \right] \psi = 0 \,. \tag{12}$$

where $l^2 = k^2\gamma/C$ and $\Delta = B^2 - A_0 C > 0$, with $A_0 = N_0^2 - \sigma^2 + f^2$, $C = -\sigma^2 + f^2 > 0$, $B = f f_s$. The transformation

$$\hat{z} = l^{2/3}[z - \frac{\Delta}{\gamma C}]$$

now reduces the equation to the standard Airy-equation

$$\frac{d^2\psi}{d\hat{z}^2} - \hat{z}\psi = 0 \,.$$

We shall consider a domain that extends indefinitely upwards; hence the solution in terms of z reads

$$\psi = \mathrm{Ai}\left(l^{2/3}\left[z - \frac{\Delta}{\gamma C}\right]\right). \tag{13}$$

We require that ψ be zero at the bottom, here defined at $z = 0$. Let the zeros of Ai be given by $-s_n$ ($s_n > 0$); one then finds for the wavenumbers l_n: $l_n^{2/3} = s_n \gamma C/\Delta$, so that

$$k_n = \pm\gamma C^2 \left(\frac{s_n}{\Delta}\right)^{3/2}; \qquad \delta_n = \mp\gamma B C \left(\frac{s_n}{\Delta}\right)^{3/2}. \tag{14}$$

The complete solution thus becomes

$$W = \sum_{n=1}^{\infty} \mathrm{Ai}\left(\frac{s_n \gamma C}{\Delta}\, z - s_n\right) \left\{a_n \sin(k_n y + \delta_n z) + b_n \cos(k_n y + \delta_n z)\right\}. \tag{15}$$

An example of $|W|$ is shown in Fig. 3.

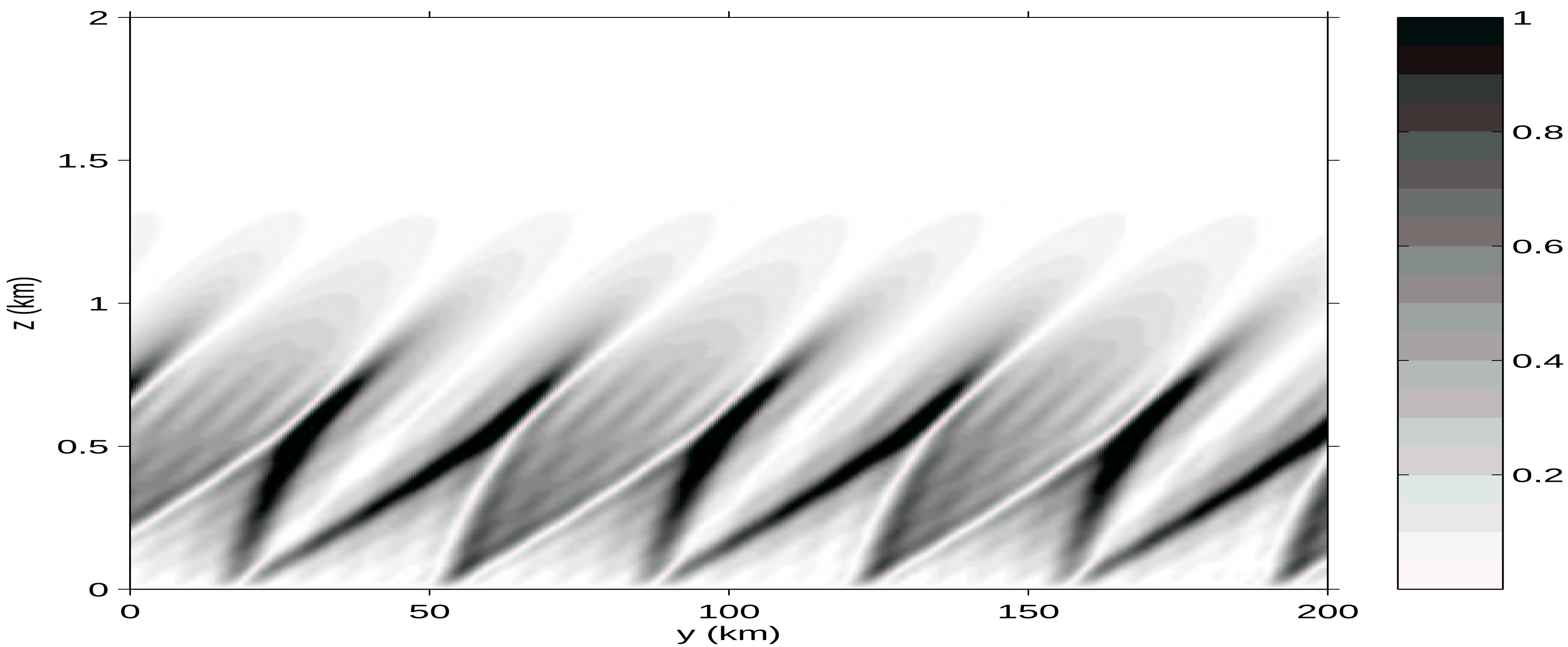


Figure 3: An example of a bottom-trapped sub-inertial wave ($\sigma = 0.99f$), given by a superposition of the first five modes in (15), with $a_n = b_n = n^{-1}$. The stratification is given by the realistic values $N_0 = 5 \times 10^{-4} \mathrm{rad\,s^{-1}}$ $\gamma = 4 \times 10^{-10} \mathrm{rad^2\,m^{-1}s^{-2}}$; latitude $\phi = 45^o$ and propagation in the meridional direction: $\alpha = \pi/2$ (poleward to the right).

Normal modes

Even though the functions ψ_n are normal modes with respect to the reduced BVP, they **are not normal *vertical* modes of the full problem, because they carry only part of the vertical dependence.**

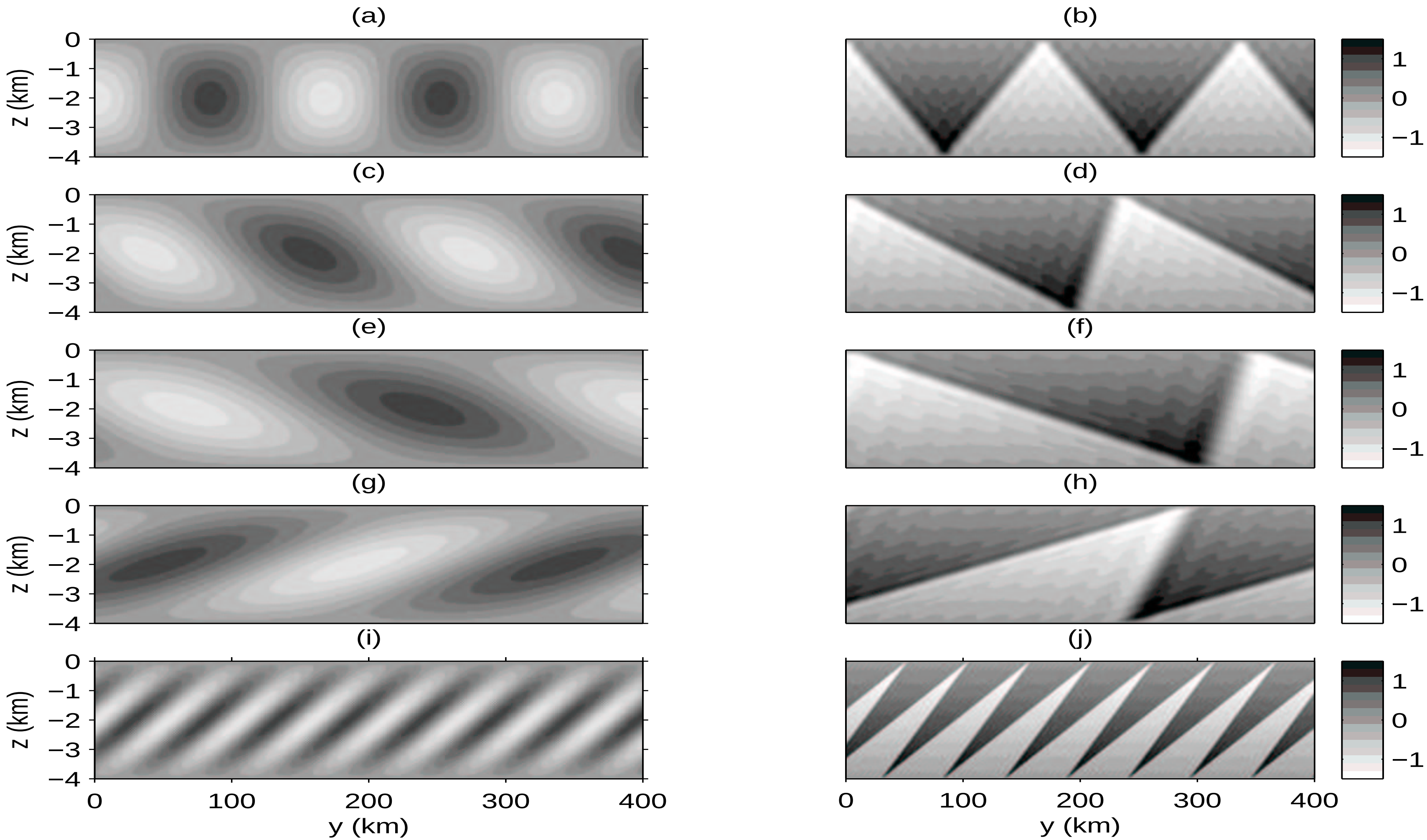


Figure 4: Modal solutions for constant N, at mid-latitude.

$\psi_n(z)$ are not the normal modes of the full problem, but ***unseparable 2D eigenfunctions*** $W_n(z,\chi)$ ***are***.

W_n ***are orthogonal and the set is complete.***

The general solution of (7) thus is

$$W = \sum_{n=1}^{\infty} \psi_n(z) \Big\{ a_n \sin(k_n\chi + \delta_n z) + b_n \cos(k_n\chi + \delta_n z) \Big\} \quad (16)$$

where a_n and b_n are arbitrary constants. (The time-dependence can be introduced by replacing $k_n\chi$ by $k_n\chi \pm \sigma t$ for right- or leftward propagating waves).

Transformation into small scales and "breaking"

Since the sub-inertial dispersion branches represent smooth continuation of the corresponding super-inertial branches, the horizontal variations of either intrinsic frequency or the inertial frequency f strongly affects propagation of near-inertial waves by allowing ***transformation of waves of the super-inertial branch into sub-inertial ones*** or vice versa.

On the $\beta-$plane the sub-inertial waveguides turn into wedges (because of dependence on y through f).

Below we describe the wave evolution in the bottom "wedge".

Evolution on the $\beta-$plane

On the $\beta-$plane

$$f = f_0 + \beta y, \qquad (\beta = 2|\mathbf{\Omega}|\cos\phi/R)\,,$$

ϕ denotes a fixed latitude, $\mathbf{\Omega}$ the Earth's angular velocity, and R the Earth's radius.

For strictly meridional propagation in terms of the stream=function Ψ

$$v = \Psi_z \qquad w = -\Psi_y \tag{17}$$

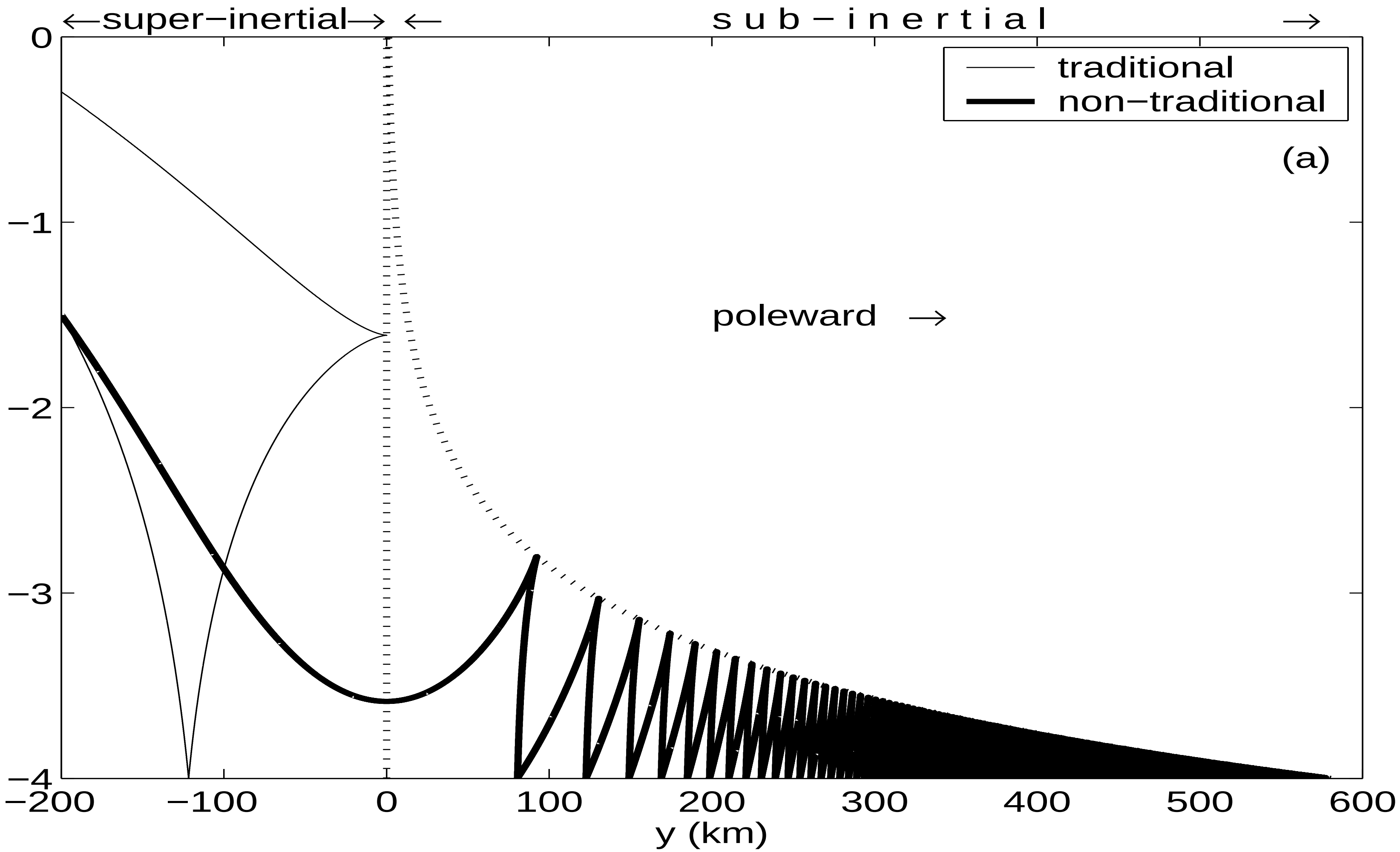
the problem reduces to the single equation

$$\Delta\Psi_{tt} + N^2\Psi_{yy} + f^2\Psi_{zz} + \tilde{f}^2\Psi_{yy} + 2f\tilde{f}\Psi_{yz} + \beta\tilde{f}\Psi_z = 0 \tag{18}$$

with the boundary conditions

$$\Psi_y(0) = \Psi_y(H) = 0\,. \tag{19}$$

This is a Tricomi type problem for which no method exists.


←super−inertial→ ← s u b − i n e r t i a l →
0
−1
−2
−3
−4
z (km)
−200
−100
0
100
200
300
400
500
600
y (km)
traditional
non−traditional
(a)
poleward →

The challenge is to find out evolution of the wave field itself, not just characteristics.

WKB

Remarkably, for the most interesting situations, in particular, for the bottom waveguides, the WKB description works *up to the critical latitude.* The wavelength not only decreases, *it decreases faster than the distance to the singularity*, which a priori guarantees that the discrepancy between the WKB and exact solutions will be exponentially small, with the exponent tending to minus infinity as the wave approaches the singularity.

Consider poleward propagation in the bottom waveguide of a narrow sub-inertial wave packet with central frequency σ and assume initial separation of scales

$$\varepsilon = |\mathbf{k}_0| R \ll 1. \tag{20}$$

Then we look for the solution in the form

$$\Psi = \Phi_{n(0)}(z, \varepsilon y) \exp i[\int_{y_i}^{y} l_n(\varepsilon y) dy - \sigma t] + \sum_{m=1, s=1} \varepsilon^m \Psi_{s(m)}(z, \varepsilon y)\,. \tag{21}$$

The leading order the shortwave asymptotics take a very simple form

$$l_n(y) \simeq const \cdot y^{-3/2}\,, \qquad (const = \pm\gamma_1 (f_0^2 - \sigma^2)^2 \Big(\frac{s_n}{(\beta f_0)}\Big)^{3/2} y^{-3/2})\,, \tag{22}$$

while the vertical scale of $\psi_n(z)$ decreases as $l^{-2/3}$ or as $|y|$, where $|y|$ is the distance to the critical point.

Evolution of wave amplitude

$$a_{(\psi)} \sim l^{1/12} \sim y^{-1/8} \tag{23}$$

$$a_{(w)} \sim a_{(v)} \sim a_{(u)} \sim a_{(b)} \sim l^{13/12} \sim y^{-13/8}\,. \tag{24}$$

of prime interest for us, is the Richardson number (which evolves much more rapidly)

$$Ri = \frac{N^2}{u_z^2 + v_z^2} \sim l^{-25/6} \sim y^{25/4} \tag{25}$$

and the nonlinearity parameter ε_N, or wave steepness,

$$\varepsilon_N \sim l^{25/12} \sim y^{-25/8}\,. \tag{26}$$

A posteriori check of of the WKB validity In the process of evolution the wavelength should remain small compared to the distance $|y|$ to the singular point, that is

$$l|y| \ll 1 \tag{27}$$

has to be satisfied.

Since for the bottom waveguide

$$ly \sim y^{1/3}$$

the condition is satisfied with the increasing accuracy as $y \to 0$.

Evolution of a finite bandwidth packet

The concept of a monochromatic wave packet is an idealisation, useful provided one is careful in drawing conclusions on what happens in reality. The real wave packets all have finite bandwidth.

For each Fourier component of the packet its singularity is located in a different place, and, therefore, the outcome of the competition between the wave packet focussing at the critical latitude and its spreading due to its finite bandwidth, is impossible to quantify a priori.

Consider a packet with a central frequency σ_0 and a Gaussian spectrum of characteristic bandwidth $\Delta\sigma$, assuming $\Delta\sigma \ll \sigma_0$

$$\Psi(y,z,t) = \frac{1}{\sqrt{\pi}\Delta\sigma}\int_{-\infty}^{\infty}\hat{\Psi}(y,z,\sigma)\exp[-i\sigma t + i\int_{y_i}^{y} l(y_1,\sigma)dy_1 - (\frac{\sigma-\sigma_0}{\Delta\sigma})^2]. \tag{28}$$

Making use of the smallness of $\Delta\sigma/\sigma_0$ we expand the exponent in the brackets to the second order and evaluate the integral

$$\Psi(y,z,t) = \hat{\Psi}(y,z,\sigma_0)e^{[-i\sigma_0 t + i\int_{y_i}^{y} l(y_1,\sigma_0)dy_1]}\left\{\frac{e^{-\kappa^2(1+i\mu)/(1+\mu^2)}}{\sqrt{1-i\mu}}\right\} \tag{29}$$

where

$$\mu = (\frac{\Delta\sigma)}{2})^2\int_{y_i}^{y}\partial^2_{\sigma\sigma}l(y_1,\sigma_0)dy_1 \qquad \kappa = \frac{\Delta\sigma}{2}\int_{y}^{y_c(t)}\frac{dy_1}{c_g(y_1,\sigma_0)}\,. \tag{30}$$

Thus, meridional scale of the packet L_y defined by the natural condition $\kappa^2/(1+\mu^2)=1$, shrinks as $y_c \to 0$

$$L_y = |y - y_c(t)|_{\kappa^2/(1+\mu^2)=1} \sim y_c^2 \tag{31}$$

The amplitude growth is somewhat moderated

$$a_{(w)} \sim a_{(v)} \sim a_{(u)} \sim a_{(b)} \sim y_c^{-3/8}\,, \tag{32}$$

but the growth of $1/Ri$ and ε_N remains quite robust

$$Ri = \frac{N^2}{u_z^2 + v_z^2} \sim y_c^{15/4}\,, \qquad \varepsilon_N \sim y_c^{-15/8}\,. \tag{33}$$

Viscous effects

Since the group velocity rapidly decreases, the packet effectively stalls, then even a very small (compared to the wave frequency) viscosity might have very strong effect.

Indeed, consideration of the viscous problem yields an additional exponential factor to all amplitudes dependencies

$$\exp[-(|y|/y_{viscous})^{-5/4}] \qquad (y_{viscous} \quad \text{is the viscous length scale}$$

which arrests wave growth,
if the wave have not been destroyed before reaching the viscous scale.

Note, that for $|y| \geq y_{viscous}$ the effect of viscosity is negligible; for $|y| \leq y_{viscous}$ the viscous effects become dominant.

Concluding remarks

We have developed complete linear theory to describe vertical focussing of sub-inertial finite width wave packets approaching the singularity.

There is no reflection. The singularity acts as a ‘black hole’

Given the initial wave amplitude, we can predict when this wave will break, thus contributing to mixing. Ri can become less than $1/4$ while the wave is still linear.

Note, that even when the initial wave amplitude is very small and there is no breaking, the wave seemingly just dissipates, anyway, its energy is likely to go into enhancement of small scale turbulence.

What we don't know yet:

(i) initial amplitudes (there are exist just rough estimates)

(ii) leakage of wave energy through nonlinear interactions.

(iii) overall impact on deep ocean mixing

(iv) is it possible to infer the intensity of wave induced mixing just by analysing the asymmetry between the properties North and South propagating near-inertial waves?