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PHYSICS LETTERS A
Physics Letters A 332 (2004) 230–238
www.elsevier.com/locate/pla

Probability densities and preservation of randomness in wave turbulence

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Received 9 June 2004; received in revised form 15 September 2004; accepted 15 September 2004

Available online 6 October 2004

Communicated by C.R. Doering

Abstract

Turbulence closure for the weakly nonlinear stochastic waves requires, besides weak nonlinearity, randomness in both the phases and the amplitudes of the Fourier modes. This randomness, once present initially, must remain over the nonlinear evolution time. Finding out to what extent is this true is the main goal of the present Letter. For this analysis we derive an evolution equation for the full probability density function (PDF) of the wave field. We will show that, for any statistics of the amplitudes, phases tend to stay random if they were random initially. If in addition the initial amplitudes are independent variables they will remain independent in a coarse-grained sense, i.e., when considered in small subsets which are much less than the total set of modes.

1. Introduction

The theory of stochastic wavefields in weakly nonlinear dispersive media has a long and exciting history which started in 1929 when Peierls derived his kinetic equation for phonons in solids [1]. Applications of these ideas appeared in the physics of the ocean and atmosphere [2–8], laboratory and astrophysical plasmas [10–12], Bose condensates and nonlinear optics [14], anharmonic crystals [1,15,16]. Any attempt to give a fair historical review would be doomed in such a short Letter and we refer an interested reader for further references to the book [17] and a more recent review [18]. The common name that has arisen for all these approaches is wave turbulence (WT).

WT closure requires, besides weak nonlinearity, randomness in both the phases and the amplitudes of the Fourier modes. Namely, all the phases and all the amplitudes must be statistically independent of each other, in some sense, and the phases must be uniformly distributed. Such an approach was recently formulated in [19,21] as

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2004.09.062

a generalization of the random phase approximation (RPA) much loved by the physicists which, in its traditional form, ignores the amplitude randomness [17]. We even kept the same acronym RPA but now read it as "Random Phases and Amplitudes". Below, in Section 2.1, we define explicitly what we mean by RPA. RPA does not fix the shape of the probability densities of the individual mode amplitudes and, therefore, it allows one to consider wavefields with non-decaying correlations which is helpful because such long correlations tend to arise naturally in WT systems. In [19], we used RPA to describe the arbitrary-order moments of the wave amplitude, and in [21] we extended this approach to describing the one-mode probability density function (PDF) and considered solutions for this PDF corresponding to intermittency. In these works, however, RPA was assumed (but not proven) to hold over the nonlinear time.

Such a proof is the main goal of the present Letter. We shall consider initial fields of the RPA type, and we will prove that the RPA properties are preserved (i.e., no phase or amplitude correlations are generated with accuracy sufficient for the WT closure) over the nonlinear evolution time. In order to do this we shall derive an evolution equation for the full multi-mode PDF which will turn out to be the Zaslavskii–Sagdeev (ZS) equation [13] (a WT cousin of the Brout–Prigogine equation for anharmonic crystals [15,16]). We will show that, for any statistics of the amplitudes, phases tend to stay random if they were so initially. If, in addition, the initial amplitudes are independent variables they will remain independent in a coarse-grained sense, i.e., when considered in small subsets which are much smaller than the total set of modes.

The original paper by ZS [13] was also devoted to the study of the applicability of the WT closure and, therefore, it is appropriate here to mention in which way our approach is different. First, ZS consider the nonlinear interaction arising from the potential energy only (i.e., the interaction Hamiltonian involves coordinates but not momenta). This restriction leaves out the capillary water waves, Alfvén, internal and Rossby waves, as well as many other interesting WT systems. In our work we remove this restriction by considering the most general three-wave Hamiltonian equation (11) and we show that the multi-mode PDF still obeys the ZS equation in this case. Secondly, ZS studied the phase statistics only, whereas our work considers both the phases and the amplitudes because the amplitude statistics is as important for the RPA closure as the phase statistics. Thirdly, ZS presented an argument that the nonlinear frequency correction removes the need for the initial phase randomness, whereas we only state the preservation of the initial phase randomness. However, the ZS criterion for phase randomization was obtained from a rather non-rigorous (although highly intuitive) physical argument whereas our results follow from a systematic asymptotic expansion outlined in this Letter and the details of which will be published in a more extended paper [20].

The validation of the RPA properties gives this technique the status of a well-justified approach which, due to the simplicity of its premises, is a winning tool for the future theory of non-Gaussianity of WT, its intermittency and interactions with coherent structures.

2. Statistical setup

Let us consider a wavefield $a(\mathbf{x}, t)$ in a periodic cube of with side L and let the Fourier transform of this field be $a_l(t)$ where index $l \in \mathcal{Z}^d$ marks the mode with wavenumber $k_l = 2\pi l/L$ on the grid in the d-dimensional Fourier space. For simplicity let us assume that there is a maximum wavenumber $k_{\max}$ (fixed, e.g., by dissipation) so that no modes with wave numbers greater than this maximum value can be excited. In this case, the total number of modes is $N = (k_{\max}/\pi L)^d$. Correspondingly, index l will only take values in a finite box, $l \in \mathcal{B}_N \subset \mathcal{Z}^d$ which is centered at 0 and all sides of which are equal to $k_{\max}/\pi L = N^{1/3}$. To consider homogeneous turbulence, the large box limit $N \to \infty$ will have to be taken.[1]

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[1] It is easy to extend the analysis to the infinite Fourier space, $k_{\max} = \infty$. In this case, the full joint PDF would still have to be defined as a $N \to \infty$ limit of an N-particle PDF, but this limit would have to be taken in such a way that both $k_{\max}$ and the density of the Fourier modes tend to infinity simultaneously.

Let us write the complex a_l as $a_l = A_l\psi_l$ where A_l is a real positive amplitude and ψ_l is a phase factor which takes values on $\mathcal{S}^1$, a unit circle centered at zero in the complex plane. Let us define the N-particle joint PDF $\mathcal{P}^{(N)}$ as the probability for the wave intensities A_l^2 to be in the range $(s_l, s_l + ds_l)$ and for the phase factors ψ_l to be on the unit-circle segment between ξ_l and $\xi_l + d\xi_l$ for all $l \in \mathcal{B}_N$. In terms of this PDF, taking the averages will involve integration over all the real positive s_l's and along all the complex unit circles of all ξ_l's,

$$\left\langle f\left\{A^2,\psi\right\}\right\rangle = \left(\prod_{l\in\mathcal{B}_N}\int_{\mathcal{R}^+} ds_l \oint_{\mathcal{S}^1} |d\xi_l|\right)\mathcal{P}^{(N)}\{s,\xi\} f\{s,\xi\}, \tag{1}$$

where the notation $f\{A^2,\psi\}$ means that f depends on all A_l^2's and all ψ_l's in the set $\{A_l^2, \psi_l; l \in \mathcal{B}_N\}$ (similarly, $\{s,\xi\}$ means $\{s_l, \psi_l; l \in \mathcal{B}_N\}$, etc.). The full PDF that contains the complete statistical information about the wavefield $a(\mathbf{x},t)$ in the infinite x-space can be understood as a large-box limit

$$\mathcal{P}\{s_k,\xi_k\} = \lim_{N\to\infty}\mathcal{P}^{(N)}\{s,\xi\},$$

i.e., it is a functional acting on the continuous functions of the wavenumber, s_k and ξ_k. In the large-box limit there is a path-integral version of (1),

$$\left\langle f\left\{A^2,\psi\right\}\right\rangle = \int \mathcal{D}s \oint |\mathcal{D}\xi| \mathcal{P}\{s,\xi\} f\{s,\xi\}. \tag{2}$$

The full PDF defined above involves all N modes (for either finite N or in the $N\to\infty$ limit). By integrating out all the arguments except for chosen few, one can have reduced statistical distributions. For example, by integrating over all the angles and over all but M amplitudes, we have an "M-particle" amplitude PDF,

$$\mathcal{P}_{j_1,j_2,\ldots,j_M} = \left(\prod_{l\neq j_1,j_2,\ldots,j_M}\int_{\mathcal{R}^+} ds_l \prod_{m\in\mathcal{B}_N}\oint_{\mathcal{S}^1}|d\xi_m|\right)\mathcal{P}^{(N)}\{s,\xi\}, \tag{3}$$

which depends only on the M amplitudes marked by labels $j_1, j_2, \ldots, j_M \in \mathcal{B}_N$.

Statistical derivations are greatly facilitated by the introduction of a generating functional

$$Z^{(N)}\{\lambda,\mu\} = \frac{1}{(2\pi)^N}\left\langle \prod_{l\in\mathcal{B}_N} e^{\lambda_l A_l^2}\psi_l^{\mu_l}\right\rangle, \tag{4}$$

where $\{\lambda,\mu\} \equiv \{\lambda_l, \mu_l; l \in \mathcal{B}_N\}$ is a set of parameters, $\lambda_l \in \mathcal{R}$ and $\mu_l \in \mathcal{Z}$.

$$\mathcal{P}^{(N)}\{s,\xi\} = \frac{1}{(2\pi)^N}\sum_{\{\mu\}}\left\langle \prod_{l\in\mathcal{B}_N}\delta\left(s_l - A_l^2\right)\psi_l^{\mu_l}\xi_l^{-\mu_l}\right\rangle = \hat{\mathcal{L}}_\lambda^{-1}\sum_{\{\mu\}}\left(Z^{(N)}\{\lambda,\mu\}\prod_{l\in\mathcal{B}_N}\xi_l^{-\mu_l}\right), \tag{5}$$

where $\{\mu\} \equiv \{\mu_l \in \mathcal{Z}; l \in \mathcal{B}_N\}$ is a set of indices enumerating the angular harmonics and $\hat{\mathcal{L}}_\lambda^{-1}$ stands for the inverse Laplace transform with respect to all λ_l.

2.1. *Definition of an essentially RPA field*

A *pure* RPA fields can be defined as one in which all the phases and amplitudes of the Fourier modes make a set of $2N$ statistically independent variables and in which all phase factors ψ are uniformly distributed on their respective unit circles. In such pure form RPA never survives except for in the uninteresting state of complete thermodynamic equilibrium. However, WT closure only requires an approximate RPA which holds up to certain order in small ϵ and $1/N$ and only in a coarse-grained sense, i.e., for the reduced M-particle objects with $M \ll N$. Below we give a relaxed definition of an (essentially) RPA property which, on one hand, is sufficient for the WT closure and, on the other hand, is preserved over the nonlinear time.

Definition. We will say that the field a is of an *essentially RPA* type if:

(1) The phase factors are statistically independent and uniformly distributed variables up to $O(\epsilon^2)$ corrections, i.e.,

$$\mathcal{P}^{(N)}\{s,\xi\} = \frac{1}{(2\pi)^N}\mathcal{P}^{(N,a)}\{s\}\big[1+O\big(\epsilon^2\big)\big], \tag{6}$$

where

$$\mathcal{P}^{(N,a)}\{s\} = \left(\prod_{l\in\mathcal{B}_N}\oint_{\mathcal{S}^1}|d\xi_l|\right)\mathcal{P}^{(N)}\{s,\xi\}, \tag{7}$$

is the N-particle *amplitude* PDF. In terms of the generating functional

$$Z^{(N)}\{\lambda,\mu\} = Z^{(N,a)}\{\lambda\}\prod_{l\in\mathcal{B}_N}\delta(\mu_l)\big[1+O\big(\epsilon^2\big)\big], \tag{8}$$

where

$$Z^{(N,a)}\{\lambda\} = \left\langle \prod_{l\in\mathcal{B}_N} e^{\lambda_l A_l^2}\right\rangle = Z^{(N)}\{\lambda,\mu\}\big|_{\mu=0} \tag{9}$$

is an N-particle generating function for the amplitude statistics.

(2) The amplitude variables are independent in a *coarse-grained* sense, i.e., for each $M \ll N$ modes the M-particle amplitude PDF is equal to the product of the one-particle PDF's up to $O(M/N)$ and $O(\epsilon^2)$ corrections,

$$\mathcal{P}^{(M,a)}_{j_1,j_2,\ldots,j_M} = P^{(a)}_{j_1}P^{(a)}_{j_2}\cdots P^{(a)}_{j_M}\big[1+O(M/N)+O\big(\epsilon^2\big)\big]. \tag{10}$$

As a first step in validating the RPA property we will have to prove that the generating functional remains of the form (8) over the nonlinear time provided it has this form at $t=0$.

3. Weak-nonlinearity expansion

Consider weakly nonlinear dispersive waves in a periodic box with a dispersion relation ω_k which allow three-wave interactions. Example of such systems includes surface capillary waves [2,7], Rossby waves [9] and internal waves in the ocean [8]. In Fourier space, we have the following Hamiltonian equations,

$$i\dot{a}_l = \epsilon\sum_{m,n=1}^{\infty}\big(V^l_{mn}a_m a_n e^{i\omega^l_{mn}t}\delta^l_{m+n} + 2\bar{V}^m_{ln}\bar{a}_n a_m e^{-i\omega^m_{ln}t}\delta^m_{l+n}\big), \tag{11}$$

where $a_l = a(k_l)$ is the complex wave amplitude in the interaction representation, $k_l = 2\pi l/L$ is the wavevector, L is the box side length, $\omega^l_{mn} \equiv \omega_{k_l} - \omega_{k_m} - \omega_{k_m}$, $\omega_l = \omega_{k_l}$ is the wave frequency, $\epsilon \ll 1$ is a formal nonlinearity parameter. Here, the interaction coefficient V^l_{mn} is obviously symmetric with respect to m and n but we do not assume any further symmetries.[2]

[2] Some additional symmetries involving permutations of the upper and lower indices arise, e.g., in solids due to the fact that nonlinearity is purely due to the potential energy which is a function of the displacement but not the rate of the displacement. Refs. [13,15,16] imposed such symmetries which immediately rule out the capillary, internal and other waves in fluids for which such properties do not hold. Additional symmetries also arise if the action variable is a Fourier transform of a real quantity, e.g., in the Rossby waves [9].

In order to filter out fast oscillations at the wave period, let us seek for the solution at time T such that $2\pi/\omega \ll T \ll 1/\omega\epsilon^2$. The second condition ensures that T is a lot less than the nonlinear evolution time. Now let us use a perturbation expansion in small ϵ,

$$a_l(T) = a_l^{(0)} + \epsilon a_l^{(1)} + \epsilon^2 a_l^{(2)}. \tag{12}$$

Substituting this expansion in (11) we get in the zeroth order $a_l^{(0)}(T) = a_l(0)$, i.e., the zeroth order term is time independent. This corresponds to the fact that in the interaction representation, wave amplitudes are constant in the linear approximation. For simplicity, we will write $a_l^{(0)}(0) = a_l$, understanding that a quantity is taken at $T = 0$ if its time argument is not mentioned explicitly. The first order is given by

$$a_l^{(1)}(T) = -i \sum_{m,n=1}^{\infty} \left(V_{mn}^l a_m a_n \Delta_{mn}^l \delta_{m+n}^l + 2\bar{V}_{ln}^m a_m \bar{a}_n \bar{\Delta}_{ln}^m \delta_{l+n}^m\right), \tag{13}$$

where $\Delta_{mn}^l = \int_0^T e^{i\omega_{mn}^l t}\,dt = (e^{i\omega_{mn}^l T} - 1)/i\omega_{mn}^l$. Iterating one more time we get

$$\begin{aligned} a_l^{(2)}(T) = \sum_{m,n,\mu,\nu}^{\infty} \Big[& 2V_{mn}^l\big(-V_{\mu\nu}^m a_n a_\mu a_\nu E\big[\omega_{n\mu\nu}^l, \omega_{mn}^l\big]\delta_{\mu+\nu}^m - 2\bar{V}_{m\nu}^\mu a_n a_\mu \bar{a}_\nu \bar{E}\big[\omega_{n\mu}^{l\nu}, \omega_{mn}^l\big]\delta_{m+\nu}^\mu\big)\delta_{m+n}^l \\ & + 2\bar{V}_{ln}^m\big(-V_{\mu\nu}^m \bar{a}_n a_\mu a_\nu E\big[\omega_{\mu\nu}^{ln}, -\omega_{ln}^m\big]\delta_{\mu+\nu}^m - 2\bar{V}_{m\nu}^\mu \bar{a}_n a_\mu \bar{a}_\nu E\big[-\omega_{n\nu l}^\mu, -\omega_{ln}^m\big]\delta_{m+\nu}^\mu\big)\delta_{l+n}^m \\ & + 2\bar{V}_{ln}^m\big(\bar{V}_{\mu\nu}^n a_m \bar{a}_\mu \bar{a}_\nu \delta_{\mu+\nu}^n E\big[-\omega_{l\nu\mu}^m, -\omega_{ln}^m\big] + 2V_{n\nu}^\mu a_m \bar{a}_\mu a_\nu E\big[\omega_{\nu m}^{\mu l}, -\omega_{ln}^m\big]\delta_{n+\nu}^\mu\big)\delta_{l+n}^m \Big], \end{aligned} \tag{14}$$

where we introduced $E(x, y) = \int_0^T \Delta(x - y)e^{iyt}\,dt$.

4. Evolution of the generating functional and multi-particle PDF

Let us first derive an evolution equation for the generating functional $Z\{\lambda, \mu\}$ exploiting the separation of the linear and nonlinear time scales.[3] To do this, we have to calculate Z at the intermediate time $t = T$ based on its value at $t = 0$. The derivation, although standard for WT, is quite lengthy and will have to be published in a longer paper. Here, we will only outline the main steps and give the result. First, we need to substitute the ϵ-expansion of a from (12) into the expressions $e^{\lambda_j |a_j|^2}$ and $\psi_j^{\mu_j} = \frac{1}{2}(\ln \frac{a_j}{\bar{a}_j})^{\mu_j}$. Second, the phase averaging should be done. Note that, because, we assume that initial phase factors are independent at $t = 0$ with required accuracy, we can do such phase averaging independently of the amplitude averaging (which we do not do yet). Thirdly, we take $N \to \infty$ limit followed by $T \sim 1/\epsilon \to \infty$ (this order of the limits is essential!). Taking into account that $\lim_{T\to\infty} E(0, x) = T(\pi\delta(x) + iP(\frac{1}{x}))$, and $\lim_{T\to\infty} |\Delta(x)|^2 = 2\pi T\delta(x)$ and, replacing $(Z(T) - Z(0))/T$ by $\dot{Z}$ (because the nonlinear time $\sim 1/\epsilon^2 \gg T$) we have

$$\begin{aligned} \dot{Z} = 4\pi\epsilon^2 \int \Bigg\{ & \left(\lambda_j + \lambda_j^2 \frac{\delta}{\delta\lambda_j}\right)\big[\big|V_{mn}^j\big|^2 \delta\big(\omega_{mn}^j\big)\delta_{m+n}^j + 2\big|V_{jn}^m\big|^2 \delta\big(\omega_{jn}^m\big)\delta_{j+n}^m\big] \frac{\delta^2 Z}{\delta\lambda_m \delta\lambda_n} \\ & + 2\lambda_j \left[-\big|V_{mn}^j\big|^2 \delta\big(\omega_{mn}^j\big)\delta_{m+n}^j \frac{\delta}{\delta\lambda_n} + \big|V_{jn}^m\big|^2 \delta\big(\omega_{jn}^m\big)\delta_{j+n}^m \left(\frac{\delta}{\delta\lambda_m} - \frac{\delta}{\delta\lambda_n}\right)\right] \frac{\delta Z}{\delta\lambda_j} \\ & + 2\lambda_j\lambda_m\big[-2\big|V_{mn}^j\big|^2 \delta_{m+n}^j \delta\big(\omega_{mn}^j\big) + \big|V_{jm}^n\big|^2 \delta_{j+m}^n \delta\big(\omega_{jm}^n\big)\big] \frac{\delta^3 Z}{\delta\lambda_j \delta\lambda_n \delta\lambda_m} \Bigg\} dk_j\,dk_m\,dk_n. \end{aligned} \tag{15}$$

[3] Hereafter we omit superscript (N) in the N-particle objects if it does not lead to a confusion.

Here variational derivatives appeared instead of partial derivatives because of the $N \to \infty$ limit. This expression is valid up to the $[1 + O(\epsilon^2)]$ factor. Eq. (15) does not contain μ dependence which means that these variables separate from λ's and the solution is a purely-amplitude Z times an arbitrary function of μ's which is going to be stationary in time. The latter corresponds to preservation of the initial $\prod \delta(\mu_l)$ dependence by Eq. (15) which means that no angular harmonics of the PDF higher than zeroth will be excited. In the other words, all the phases will remain statistically independent and uniformly distributed on S^1 with the accuracy of Eq. (15) integrated over the nonlinear time $1/\epsilon^2$, i.e., with the $O(\epsilon^2)$ accuracy. This proves the first of the "essential RPA" properties. In fact, this result was already obtained before in [15] for a narrower class of 3-wave systems (see footnote 2). Note that we still have not used any assumption about the statistics of A's and, therefore, (15) could be used in future for studying systems with random phases but correlated amplitudes.

Taking the inverse Laplace transform of (15) we have the following equation for the PDF,

$$\dot{\mathcal{P}} = -\int \frac{\delta F_j}{\delta s_j}\, dk_j, \tag{16}$$

where F_j is a flux of probability in the space of the amplitude s_j,

$$\begin{aligned} -\frac{F_j}{4\pi\epsilon^2 s_j} = \int \bigg\{ & \big(\big|V^j_{mn}\big|^2 \delta\big(\omega^j_{mn}\big)\delta^j_{m+n} + 2\big|V^n_{jm}\big|^2 \delta\big(\omega^n_{jm}\big)\delta^n_{j+m}\big) s_n s_m \frac{\delta\mathcal{P}}{\delta s_j} \\ & + 2\mathcal{P}\big(\big|V^n_{jm}\big|^2 \delta\big(\omega^n_{jm}\big)\delta^n_{j+m} - \big|V^j_{mn}\big|^2 \delta\big(\omega^j_{mn}\big)\delta^j_{m+n}\big) s_m \\ & + 2\big(\big|V^n_{jm}\big|^2 \delta\big(\omega^n_{jm}\big)\delta^n_{j+m} - 2\big|V^j_{mn}\big|^2 \delta\big(\omega^j_{mn}\big)\delta^j_{m+n}\big) s_n s_m \frac{\delta\mathcal{P}}{\delta s_m} \bigg\} dk_m\, dk_n. \end{aligned} \tag{17}$$

This equation is identical to the Zaslavskii–Sagdeev (ZS) [13] equation (Brout–Prigogine in the physics of crystals context [15,16]). Note that ZS equation was originally derived in [13] for a much narrower class of systems, see footnote 2, whereas the result above indicates that it is also valid in the most general case of 3-wave systems. Here we should again emphasize the importance of the order of limits, $N \to \infty$ first and $\epsilon \to 0$ second. Physically this means that the frequency resonance is broad enough to cover a great many modes. Some authors, e.g., ZS and BP leave the sum notation in the PDF equation even after the $\epsilon \to 0$ limit taken giving $\delta(\omega^n_{jm})$. One has to be careful interpreting such a formula because formally the RHS is null in most of the cases because there may be no exact resonances between the discrete k modes (as it is the case, e.g., for the capillary waves). Thus, our functional integral notation is a more accurate way to write the result.

5. In what sense are the amplitudes independent?

Obviously, the variables s_j do not separate in the above equation for the PDF. Substituting

$$\mathcal{P}^{(N,a)} = P^{(a)}_{j_1} P^{(a)}_{j_2} \cdots P^{(a)}_{j_N} \tag{18}$$

(compare with (10)) into the discrete version of (17) we see that it turns into zero on the thermodynamic solution with $P^{(a)}_j = \omega_j \exp(-\omega_j s_j)$. However, it is not zero for the one-mode PDF $P^{(a)}_j$ corresponding to the cascade-type Kolmogorov–Zakharov (KZ) spectrum n^{kz}_j, i.e., $P^{(a)}_j = (1/n^{kz}_j)\exp(-s_j/n^{kz}_j)$ (see next section), nor it is likely to be zero for any other PDF of form (18). This means that, even if initially independent, the amplitudes will correlate with each other at the nonlinear time. Does this mean that the existing WT theory, and in particular the kinetic equation, is invalid?

To answer to this question let us differentiate the discrete version of Eq. (15) with respect to λ's to get equations for the amplitude moments. We can easily see that

$$\partial_t\big(\big\langle A^2_{j_1} A^2_{j_2}\big\rangle - \big\langle A^2_{j_1}\big\rangle\big\langle A^2_{j_2}\big\rangle\big) = O\big(\epsilon^4\big) \quad (j_1, j_2 \in \mathcal{B}_N) \tag{19}$$

if $\langle A_{j_1}^2 A_{j_2}^2 A_{j_3}^2\rangle = \langle A_{j_1}^2\rangle\langle A_{j_2}^2\rangle\langle A_{j_3}^2\rangle$ (with the same accuracy) at $t=0$. Similarly, in terms of PDF's

$$\partial_t\big(P^{(2,a)}_{j_1,j_2}(s_{j_1},s_{j_2}) - P^{(a)}_{j_1}(s_{j_1})P^{(a)}_{j_2}(s_{j_2})\big) = O\big(\epsilon^4\big) \quad (j_1, j_2 \in \mathcal{B}_N) \tag{20}$$

if $P^{(4,a)}_{j_1,j_2,j_3,j_4}(s_{j_1},s_{j_2},s_{j_3},s_{j_4}) = P^{(a)}_{j_1}(s_{j_1})P^{(a)}_{j_2}(s_{j_2})P^{(a)}_{j_3}(s_{j_3})P^{(a)}_{j_4}(s_{j_4})$ at $t=0$. Here $P^{(4,a)}_{j_1,j_2,j_3,j_4}(s_{j_1},s_{j_2},s_{j_3},s_{j_4})$, $P^{(2,a)}_{j_1,j_2}(s_{j_1},s_{j_2})$ and $P^{(a)}_{j}(s_j)$ are the four-particle, two-particle and one-particle PDF's obtained from $\mathcal{P}$ by integrating out all but 4, 2 or 1 arguments, respectively. One can see that, with accuracy ϵ^2, the Fourier modes will remain independent of each other in any pair over the nonlinear time if they were independent in every triplet at $t=0$.

Similarly, one can show that the modes will remain independent over the nonlinear time in any subset of $M<N$ modes with accuracy M/N (and ϵ^2) if they were initially independent in every subset of size $M+1$. Namely

$$P^{(M,a)}_{j_1,j_2,\dots,j_M}(s_{j_1},s_{j_2},\dots,s_{j_M}) - P^{(a)}_{j_1}(s_{j_1})P^{(a)}_{j_2}(s_{j_2})\cdots P^{(a)}_{j_M}(s_{j_M}) = O(M/N) + O\big(\epsilon^2\big) \quad (j_1, j_2, \dots, j_M \in \mathcal{B}_N) \tag{21}$$

if $P^{(M+2,a)}_{j_1,j_2,\dots,j_{M+2}} = P^{(a)}_{j_1}P^{(a)}_{j_2}\cdots P^{(a)}_{j_{M+2}}$ at $t=0$.

The mismatch $O(M/N)$ arises from some terms in the ZS equation with coinciding indices j. For $M=2$ there is only one such term in the N-sum and, therefore, the corresponding error is $O(1/N)$ which is much less than $O(\epsilon^2)$ (due to the order of the limits in N and ϵ). However, the number of such terms grows as M and the error accumulates to $O(M/N)$ which can greatly exceed $O(\epsilon^2)$ for sufficiently large M.

We see that the accuracy with which the modes remain independent in a subset is worse for larger subsets and that the independence property is completely lost for subsets approaching in size the entire set, $M\sim N$. One should not worry too much about this loss because N is the biggest parameter in the problem (size of the box) and the modes will be independent in all M-subsets no matter how large. Thus, the statistical objects involving any *finite* number of particles are factorisable as products of the one-particle objects and, therefore, the WT theory reduces to considering the one-particle objects. This results explains why we re-defined RPA in its relaxed "essential RPA" form. Indeed, in this form RPA is sufficient for the WT closure and, on the other hand, it remains valid over the nonlinear time. In particular, only property (19) is needed, as far as the amplitude statistics is concerned, for deriving the three-wave kinetic equation, and this fact validates this equation and all of its solutions, including the KZ spectrum which plays an important role in WT.

The situation where modes can be considered as independent when taken in relatively small sets but should be treated as dependent in the context of much larger sets is not so unusual in physics. Consider for example a distribution of electrons and ions in plasma. The full N-particle distribution function in this case satisfies the Louville equation which is, in general, not a separable equation. In other words, the N-particle distribution function cannot be written as a product of N one-particle distribution functions. However, an M-particle distribution can indeed be represented as a product of M one-particle distributions if $M\ll N_D$ where N_D is the number of particles in the Debye sphere. We see an interesting transition from an individual to collective behavior when the number of particles approaches N_D. In the special case of the one-particle function we have here the famous mean-field Vlasov equation which is valid up to $O(1/N_D)$ corrections (representing particle collisions).

6. One-particle statistics

We have established above that the one-point statistics are at the heart of WT theory. All one-point statistical objects can be derived from the one-point amplitude generating function,

$$Z_a(\lambda_j) = \big\langle e^{\lambda_j A_j^2}\big\rangle,$$

which can be obtained from the N-point Z by taking all μ's and all λ's, except for λ_j, equal to zero. Substituting such values into (15) we get the following equation for Z_a,

$$\frac{\partial Z_a}{\partial t} = \lambda_j \eta_j Z_a + \left(\lambda_j^2 \eta_j - \lambda_j \gamma_j\right) \frac{\partial Z_a}{\partial \lambda_j}, \tag{22}$$

where,

$$\eta_j = 4\pi\epsilon^2 \int \left(\left|V^j_{lm}\right|^2 \delta^j_{lm} \delta\left(\omega^j_{lm}\right) + 2\left|V^m_{jl}\right|^2 \delta^m_{jl} \delta\left(\omega^m_{jl}\right)\right) n_l n_m \, dk_l \, dk_m, \tag{23}$$

$$\gamma_j = 8\pi\epsilon^2 \int \left(\left|V^j_{lm}\right|^2 \delta^j_{lm} \delta\left(\omega^j_{lm}\right) n_m + \left|V^m_{jl}\right|^2 \delta^m_{jl} \delta\left(\omega^m_{jl}\right)(n_l - n_m)\right) dk_l \, dk_m. \tag{24}$$

Correspondingly, for the one particle PDF $P_a(s_j)$ we have

$$\frac{\partial P_a}{\partial t} + \frac{\partial F}{\partial s_j} = 0, \tag{25}$$

with F is a probability flux in the s-space,

$$F = -s_j \left(\gamma P_a + \eta_j \frac{\delta P_a}{\delta s_j}\right). \tag{26}$$

Eqs. (22) and (25) where previously obtained and studied in [21] in for four-wave systems. The only difference for the four-wave case was different expressions for η and γ. For the three-wave case, the equation for the PDF was not considered before, but equations for its moments were derived and solved in [19]. In particular, the equation for the first moment is nothing but the familiar kinetic equation $\dot{n} = -\gamma n + \eta$ which gives $\eta = \gamma n$ for any steady state. This, in turn means that in the steady state with $F = 0$ we have $P_j^{(a)} = (1/n_j)\exp(-s_j/n_j)$ where n_j can be any steady state solution of th kinetic equation including the KZ spectrum which plays the central role in WT [2,17]. However, it was shown in [21] that there also exist solutions with $F \neq 0$ which describe WT intermittency.

7. Discussion

In the present Letter, we considered the evolution of the full N-particle objects such as the generating functional and the probability density function for all the wave amplitudes and their phase factors. We proved that the phase factors, being statistically independent and uniform on S^1 initially, remain so over the nonlinear evolution time. This result does not rely on any assumptions about the amplitude statistics and, therefore, can be used in future for studying systems with correlated amplitudes (but random phases). If in addition the initial amplitudes are independent too, then they remain so over the nonlinear time in a coarse-grained sense. Namely, all joint PDF's for the number of modes $M \ll N$ split into products of the one-particle densities with $O(M/N)$ accuracy. Thus, the full N-particle PDF does not get factorized as a product of N one-particle densities and the Fourier modes in the set considered as a whole are not independent. However, the wave turbulence closure only deals with the joint objects of the finite size M of variables while taking the $N \to \infty$ limit. These objects do get factorized into products and, for the WT purposes, the Fourier modes can be interpreted as statistically independent. These results reduce the WT problem to the study of the one-particle amplitude PDF's and they validate the generalized RPA technique introduced in [19,21]. Such a study of the one-particle PDF and the high-order momenta of the wave amplitudes was done in [19,21] and the reader is referred to these papers for the discussion of WT intermittency.

Finally, we would like to mention the role of quasi-resonant interactions which, as we saw, do not produce any long-term effect at the ϵ^2 order considered in this Letter. However, these interactions do modify statistics at ϵ^4 order as was shown in [22]. The ϵ^4 correction can be important for the real space correlators which have Gaussian values at the ϵ^2 order for any (not necessarily Rayleigh) amplitude distributions.

Acknowledgements

We thank Alan Newell for the feedback he gave us about our results during his RPI visit in March 2004, particularly for pointing out that a very similar PDF equation was derived in Prigogine's book. Later we also discovered a brilliant paper by Zaslavskii and Sagdeev who applied Brout–Prigogine's ideas to wave turbulence—a work which has been largely forgotten by the WT community. Should we have known about this paper in the beginning of our work, ZS would have been a great inspiration and would have probably influenced our method (which is now different and more general than ZS). We also thank Colm Connaughton for his comments and for the proofreading of the manuscript. Yeontaek Choi's work is supported by KOSEF M07-2003-000-10003-0. Yuri Lvov acknowledges support provided by NSF CAREER grant DMS 0134955 and by ONR YIP grant N000140210528.

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