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The circulation density and its role in 3D turbulence

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Abstract

The scaling argument applied to the vorticity cascade in 2D turbulence results in the well-known k^{-3} energy spectrum. Kelvin theorem for the velocity circulation generalizes the vorticity conservation law on the 3D fluids. Using the Kelvin theorem and the incompressibility of the fluid one can derive Lagrangian conservation of a quantity having the physical meaning of a circulation density. Integrated over the entire space, the powers of the circulation density form a series of the motion integrals which coincides with the vorticity series in the 2D limit. We will discuss the question of applicability of the scaling arguments to the cascades of the geometrical integrals in 3D turbulence. We will see that the cascades of all but one circulation integrals are ruled out by the reconnection kinematics. The only exception is the integral corresponding to the total volume of the vortex tubes, whose cascade corresponds to the k^{-3} energy spectrum. Relation of the circulation density to the Clebsch variables will be considered. We will see that the circulation density can be chosen as one of the Clebsch variables. In the case of the stationary flows, such Clebsch variables becomes an action–angle pair.

1. Lagrangian conservation of the circulation density

Consider an incompressible inviscid fluid. Due to the incompressibility and according to Kelvin theorem the volume σ and circulation c of a closed vortex tube are conserved, so is the ratio σ/c. Considering closed vortex tubes with small cross-sectional areas s we have

$$\sigma/c = \oint \frac{s(l)\,\mathrm{d}l}{c} = \oint \frac{\mathrm{d}l}{|\omega(l)|} = \text{const.}, \qquad (1)$$

where l is the coordinate along the vortex line and $\omega(l)$ is the absolute value of vorticity at the corresponding location on this vortex line. Let us define a function $\alpha(\boldsymbol{r}, t)$ such that

$$\alpha(\boldsymbol{r}, t) = 1 \Big/ \oint \frac{\mathrm{d}l}{|\omega(l, t)|}, \qquad (2)$$

where the integral is taken along the vortex line passing through $\boldsymbol{r}$. Since the integral (1) is conserved for any vortex line moving with the fluid, the function $\alpha(\boldsymbol{r}, t)$ is a Lagrangian invariant, i.e., it is not changing along the fluid–particle trajectories,

$$\partial_t \alpha(\boldsymbol{r}, t) + \boldsymbol{v} \cdot \nabla \alpha(\boldsymbol{r}, t) = 0. \qquad (3)$$

Here $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$ is the velocity. Function $\alpha(\boldsymbol{r}, t)$ has physical meaning of a volume density of the circulation.

For continuous vorticity fields function $\alpha(\boldsymbol{r}, t)$ is continuous between the separating surfaces passing

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through the vorticity nulls and jumps at these surfaces. The vorticity nulls and structure of the vortex lines in their vicinity was considered by Greene [1]. Function $\alpha(\boldsymbol{r}, t)$ can be also defined for open vortex lines if the latter fill ergodically a 2D surface or a 3D volume, see Appendix A. An interesting relation of the circulation density with Eckart's invariants is discussed in Appendix B.

2. Circulation series of integrals

Consider the case in which all the vortex lines are closed. According to (3), any function $F(\alpha)$ integrated over the coordinate space is a motion integral,

$$\int F(\alpha(\boldsymbol{r}, t))\,\mathrm{d}\boldsymbol{r} = \text{const.} \tag{4}$$

In particular, this gives the following series of positive motion integrals:

$$I_n = \int \alpha^n(\boldsymbol{r}, t)\,\mathrm{d}\boldsymbol{r}, \quad n = 0, 1, 2, \ldots \tag{5}$$

We require that the integration in I_0 is carried out over the support of $\alpha(\boldsymbol{r})$ (this is equivalent to the rule $0^0 = 0$ in the integrand). For integral I_0 to have a nontrivial meaning some part of the flow has to be potential. We will call (5) circulation series of integrals. In the 2D limit, the circulation series transform into the well-known vorticity series of integrals,

$$J_n = \int \omega^n(\boldsymbol{r}, t)\,\mathrm{d}\boldsymbol{r}, \quad n = 0, 1, 2, \ldots \tag{6}$$

In this case, the vortex lines are open and the integration in (2) and (5) has to be taken over the unit length along the vorticity.

3. Can the circulation integrals cascade?

Consider now homogeneous turbulence with a source and viscous dissipation well separated in k-space. Let the source term produce only closed vortex lines. In the spirit of Kolmogorov's scaling argument, one can conjecture that there exists an inertial interval for one of the integrals I_n in which the turbulent energy spectrum $E(k)$ is determined only by the flux P_n of this motion integral. Taking into account that $E(k)$ and P_n have dimensions $[L^3/T^2]$ and $[T^{-n-1}L^{-n}]$ correspondingly, we obtain

$$E(k) \propto P_n^{2/(n+1)}\, k^{-(5n+3)/(n+1)}. \tag{7}$$

As we see, the scaling argument applied to different integrals I_n predicts different turbulent spectra. Therefore, the inertial interval in which I_n freely cascade over the scales cannot exist simultaneously for all the circulation integrals.

In contrast, the scaling argument applied to any of the 2D vorticity integrals (6) predicts the same exponent for the energy spectrum. This follows from the fact that the flux Q_n of integral J_n have dimension $[T^{-n-1}]$ which does not contain L. Matching the dimensions, we have

$$E(k) \propto Q_n^{2/(n+1)}\, k^{-3}. \tag{8}$$

The origin of such a difference between the 2D and 3D cases is in the fact that the 3D circulation integrals (5) are nonlocal, whereas the 2D vorticity integrals (6) are local. In the 3D case, any local reconnection event due to small viscosity results in of-order-one global changes in the integrand of (5) along the reconnecting vortex tubes. The only exception is the case $n = 0$ which will be discussed separately. The changes in the integrand of (5) arise via function $\alpha(\boldsymbol{r})$ which experiences a jump due to the changes in the vortex-line length associated with the reconnection. Namely, $\alpha(\boldsymbol{r})$ increases if the reconnecting segments belong to originally different closed vortex lines which form a bigger loop after the reconnection. If the reconnection occurs on the segments of the same vortex loop and thereby splits this loop onto two smaller ones, then $\alpha(\boldsymbol{r})$ will decrease. The change in $\alpha(\boldsymbol{r})$ and, therefore, the integrand of (5) occur at all the scales corresponding to the sizes of the vortex tubes involved in the reconnection process. For small viscosity these scales are much greater than the dissipation scale, and, therefore, the inertial interval for I_n, $n \neq 0$, is absent and these integrals cannot cascade.

A different picture arises for I_0 which is nearly conserved in the scales greater than the dissipation scale if the viscosity is small. Indeed, for small viscosity the total volume of the reconnecting vortex tubes is approximately the same before and after the reconnection. Therefore, I_0 is the only circulation integral for which existence of the inertial interval is not ruled out by the reconnection kinematics. Correspondingly, only the spectrum associated with the cascade of I_0 is selected from the family (7) as possible in 3D turbulence,

$$E(k) \propto P_0^2 k^{-3}. \tag{9}$$

Of course, the cascade of I_0 makes sense only if some part of the flow is potential.

4. Relation of $\alpha(\boldsymbol{r}, t)$ to the Clebsch variables

Function $\alpha(\boldsymbol{r}, t)$ possesses the properties of a Clebsch variable [2,3]: it is a Lagrangian invariant and it is constant along the vortex line. In fact, one can use $\alpha(\boldsymbol{r}, t)$ as the Clebsch variable in situations when it is possible to introduce another Clebsch variable such that

$$\partial_t \beta(\boldsymbol{r}, t) + \boldsymbol{v} \cdot \nabla\beta(\boldsymbol{r}, t) = 0 \tag{10}$$

and the velocity is related to $\alpha(\boldsymbol{r}, t)$ and $\beta(\boldsymbol{r}, t)$ as follows:

$$\boldsymbol{v}(\boldsymbol{r}, t) = \beta\nabla\alpha + \nabla\phi, \tag{11}$$

where ϕ is to be found from the incompressibility condition,

$$\operatorname{div} \boldsymbol{v} = 0. \tag{12}$$

In this case

$$\omega = \nabla\alpha \times \nabla\beta \tag{13}$$

so that the vortex lines lie on the intersection of the surfaces $\alpha(\boldsymbol{r}, t) = \text{const.}$ and $\beta(\boldsymbol{r}, t) = \text{const.}$ Actually, the Clebsch variables can always be introduced if there can be found an arbitrary Lagrangian invariant $\gamma(\boldsymbol{r}, t)$ such that the vortex lines are parallel to $\nabla\alpha \times \nabla\gamma$. In Appendix C we show how to construct in this case Clebsch variable $\beta(\boldsymbol{r}, t)$ satisfying (10) and (11).

Eqs. (3) and (10) can be written in a Hamiltonian form [2],

$$\partial_t \alpha = \delta H / \delta\beta, \tag{14}$$

$$\partial_t \beta = -\delta H / \delta\alpha, \tag{15}$$

where the Hamiltonian H is the kinetic energy of the fluid,

$$H = \frac{1}{2} \int v^2 \, \mathrm{d}\boldsymbol{r}. \tag{16}$$

It is worth mentioning that α^{-1} plays the role of the Jacobian of the transformation from the Lagrangian coordinates to variables α and β. Indeed, for the vortex tube bounded by the surfaces $\alpha(\boldsymbol{r}, t) = \alpha$, $\alpha(\boldsymbol{r}, t) = \alpha + \mathrm{d}\alpha$, $\beta(\boldsymbol{r}, t) = \beta$ and $\beta(\boldsymbol{r}, t) = \beta + \mathrm{d}\beta$ the circulation c is $\mathrm{d}\alpha\, \mathrm{d}\beta$. On the other hand, according to (1) the Lagrangian volume element $\mathrm{d}^3 r = \sigma$ is related to the circulation as $\sigma = c/\alpha$, and therefore $\mathrm{d}^3 r = \alpha^{-1}\, \mathrm{d}\alpha\, \mathrm{d}\beta$.

Eqs. (3) and (10) possess even larger than (4) family of the motion integrals,

$$\int F(\alpha(\boldsymbol{r}, t), \beta(\boldsymbol{r}, t))\, \mathrm{d}\boldsymbol{r}\boldsymbol{u} = \text{const.}, \tag{17}$$

where F is an arbitrary function. Yakhot and Zakharov [3] derived the energy spectrum $E(k) \propto k^{-1}$ assuming that there exists an inertial interval for one of such integrals, the "number of particles",

$$N = \int (\alpha^2 + \beta^2)\, \mathrm{d}\boldsymbol{r}. \tag{18}$$

As it was shown above, the reconnection breaks conservation of the first part of this integral, $\int \alpha^2\, \mathrm{d}\boldsymbol{r}$, in the scales much greater than the dissipation scale. Thus, at least for the considered here choice of the Clebsch variables, the inertial interval for N is absent, which makes realizability of the k^{-1}-spectrum questionable. As pointed out by Newell [4], the nonconservation of N in large scales also follows from the fact that the viscosity term in the equations for the Clebsch variables is nonlocal.

One remarkable property of α is that it is time independent for the stationary flows. This distinguishes it from most of other Clebsch variables (remind that there exist infinitely many Clebsch pairs describing the same flow). The time independence of α for the stationary flows follows directly from definition (2). From (3) one has the following condition for the flow to be stationary:

$$v \cdot \nabla\alpha = 0. \tag{19}$$

On the other hand, the time independence of α means that Hamiltonian (16) is independent of β, see (14). Thus, one can integrate the Eq. (15), which gives

$$\beta = \lambda(\alpha)\, t + \mu(r), \tag{20}$$

where $\lambda(\alpha) = -\delta H/\delta\alpha$ and $\mu(r)$ is a time independent function which satisfies the following equation:

$$v \cdot \nabla\mu = -\lambda. \tag{21}$$

As we see, in the case of stationary flows α and β play the role of an action–angle pair of variables.

5. Summary

The circulation density α defined by (2) is a Lagrangian invariant. This property allows to write the circulation series of the motion integrals (5) which transform into the vorticity series (6) in the 2D limit. In contrast with the vorticity integrals in 2D, the circulation integrals I_n cannot simultaneously cascade over the scales. The nonlocal dependence of the circulation integrals on the vorticity and the vortex reconnection rule out the existence of the inertial interval for all I_n except for the integral I_0. The integral I_0 (the total volume of the vortex tubes) is conserved during the reconnection if the viscosity is small. The k^{-3} energy spectrum, corresponding to the cascade of I_0, is the only one from the family (7) which can possibly develop in 3D turbulence. Of course, passing by the integral I_0 the "reconnection test" does not guarantee the realizability of the k^{-3} spectrum, because it also depends on validity of other assumptions underlying the scaling argument, such as the scale invariance and locality of turbulence.

In addition to being a Lagrangian invariant, the circulation density α is constant along the vortex lines. Combination of these properties allows to choose α as one of the Clebsch variables (for the flows permitting the Clebsch formulation). Advantages of using α as a Clebsch variable are that it is explicitly related with the observables (e.g., vorticity) and that it is time independent for the stationary flows. From the latter property it follows that the Clebsch pair become action–angle variables for the stationary flows, with the circulation density α being the action.

Appendix A

Function $\alpha(r, t)$ can be also defined for open vortex lines if the latter fill ergodically a 2D surface or a 3D volume. The idea for such a definition exploits the fact that an ergodic vortex line starting at any point r will return and approach the starting point as closely as needed thereby forming "loops" with any required accuracy. Consider points r_k, $k = 0, 1, 2\ldots$, such that $r_0 = r$, r_1 is the first point on the vortex line which separated from the starting point r by a given small distance δ, r_2 is the second such point, etc. Since r_k's are not moving exactly with the velocity of the fluid, the integral $\int_{r_k}^{r_{k+1}} \mathrm{d}l/\omega$ is conserved only approximately with the error being proportional to δ. According to Kac's theorem [5], the mean length of the vortex "loops" scales as δ^{-n}, where n is the dimension of the space ergodically filled by the vortex line. Thus, normalizing the averaged $\int_{r_k}^{r_{k+1}} \mathrm{d}l/\omega$ by δ^{-n} and taking the limit $\delta \to 0$ we obtain an exact Lagrangian invariant,

$$\alpha(r, t) = 1/\lim_{\delta\to 0}\lim_{m\to\infty}\left(\delta^n \frac{1}{m}\sum_{k=1}^{m}\int_{r_{k-1}}^{r_k}\frac{\mathrm{d}l}{\omega(l)}\right). \tag{A.1}$$

Note that function $\alpha(r, t)$ is constant on the surface (or volume) filled with a vortex line. In some particular cases $\alpha(r, t)$ can be defined in a more direct way. Consider, for example, the vortex lines filling some nested toroidal surfaces. One can define function $\alpha(r, t)$ as the ratio of the circulation and the volume of a hollow vortex tube lying in between of two adjacent tori. This gives

$$\alpha(\boldsymbol{r},t) = 1 \Big/ \oint\oint \frac{\mathrm{d}\tau \times \mathrm{d}l}{\oint(\omega(\tau,l)\cdot\boldsymbol{n})\,\mathrm{d}\tau}, \tag{A.2}$$

where τ and l are the coordinates on the toroidal surface parametrizing the "small" and the "large" circles correspondingly. Another example is the vortex line filling densely a 3D volume between two nested toroidal surfaces. In this case α is the ratio of the vorticity flux to the volume between the toroidal surfaces.

Appendix B

Let $\boldsymbol{a}(\boldsymbol{r},t)$ be a vector field such that

$$\partial_t \boldsymbol{a} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{a} = 0. \tag{B.1}$$

Eckart's invariants are defined as [6,7]:

$$\omega_a = \nabla_a \times (u\nabla_a x + v\nabla_a y + w\nabla_a z), \tag{B.2}$$

where u, v, w and x, y, z are the velocity and coordinate components correspondingly, which have to be considered as functions of $\boldsymbol{a}$ when differentiated with respect to $\boldsymbol{a}$. The conservation law for ω_a reads

$$\partial_t \omega_a + (\boldsymbol{v}\cdot\nabla)\omega_a = 0. \tag{B.3}$$

One consequence of this conservation law is

$$(\partial_t + (\boldsymbol{v}\cdot\nabla))\oint \frac{\mathrm{d}l_a}{|\omega_a|} = 0, \tag{B.4}$$

where ω_a is expressed in terms of $\boldsymbol{a}$, and the integral is taken in the a-space along such a contour the image of which in the r-space is the vortex line passing the point $\boldsymbol{r}$. One can choose $\boldsymbol{a}(\boldsymbol{r},t)$ to be the initial coordinates. According to (25), in this case $\omega_a = \omega(\boldsymbol{r}(\boldsymbol{a},0),0)$, that is ω_a coincides with the initial value of vorticity at $\boldsymbol{r} = \boldsymbol{a}$. Therefore, the integral in (B.4) is just another representation of the circulation density.

Appendix C

Let us introduce a variable $\gamma(\boldsymbol{r},t)$ such that the vortex lines lie on the intersection of the surfaces $\gamma(\boldsymbol{r},t) = \text{const.}$ and $\alpha(\boldsymbol{r},t) = \text{const.}$, so that the vorticity is directed along $\nabla\gamma\times\nabla\alpha$, and which does not change along the fluid–particle trajectories,

$$\partial_t \gamma(\boldsymbol{r},t) + (\boldsymbol{v}\cdot\nabla)\gamma(\boldsymbol{r},t) = 0. \tag{C.1}$$

According to (1) and (2), one can write the following expression for the absolute value of vorticity:

$$|\omega(\boldsymbol{r})| = \frac{\alpha(\boldsymbol{r})}{s(0)}\oint s(l)\,\mathrm{d}l, \tag{C.2}$$

where $s(l)$ is the cross-sectional area of a thin vortex tube (passing through the point $\boldsymbol{r}$) at the distance l from the point $\boldsymbol{r}$ along the vortex line. Let us consider a vortex-tube bounded by surfaces $\gamma(\boldsymbol{r}) = \gamma_1$, $\gamma(\boldsymbol{r}) = \gamma_2$, $\alpha(\boldsymbol{r}) = \alpha_1$ and $\alpha(\boldsymbol{r}) = \alpha_2$, where $\gamma_1, \gamma_2, \alpha_1$ and α_2 are some constants. The cross-sectional area for such a tube is

$$s = \frac{|\gamma_1-\gamma_2||\alpha_1-\alpha_2|}{|\nabla\gamma\times\nabla\alpha|}. \tag{C.3}$$

Substituting (B.1) into (A.2) and taking into account that the vorticity is directed along $\nabla\gamma\times\nabla\alpha$, we arrive at the expression for the vorticity in terms of variables α and γ,

$$\omega = (\nabla\gamma\times\nabla\alpha)\alpha \oint_{\alpha,\gamma=\text{const.}} \frac{\mathrm{d}l}{|\nabla\gamma\times\nabla\alpha|}. \tag{C.4}$$

Expression of the velocity in terms of α and γ is

$$\boldsymbol{v}(\boldsymbol{r}) = f(\alpha,\gamma)\nabla\alpha + \nabla\phi, \tag{C.5}$$

where

$$f(\alpha,\gamma) = \alpha\partial_\gamma^{-1} \oint_{\alpha,\gamma=\text{const.}} \frac{\mathrm{d}l}{|\nabla\gamma\times\nabla\alpha|} \tag{C.6}$$

and ϕ is to be found from the incompressibility condition,

$$\operatorname{div}\boldsymbol{v} = 0. \tag{C.7}$$

Finally, the Clebsch variables α and β satisfying (3), (10), (11) and (13) are introduced by the change of variable $\gamma \to \beta = f(\alpha,\gamma)$. Note that the possibility to find variable γ with the described above properties exists only for a limited class of motions not including, for example, the nonzero-helicity flows.

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