ON THE NONLOCAL TURBULENCE OF DRIFT TYPE WAVES

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Two new effects the drift turbulence can display are disclosed: (1) the turbulence spectrum in k-space separates into unconnected components of large and small scales, (2) the very presence of weak small-scale turbulence imposes rigid restrictions on powerful large-scale components.

1. A wide variety of problems in the physics of the atmosphere and the ocean [1,2], in physics of magnetized plasmas [3–6] and in astrophysics leads to a study of drift-type waves of Rossby waves having the dispersion law

$$\omega=\frac{\beta k_x}{1+\rho^2k^2} \tag{1}$$

($\boldsymbol{k}=(k_x, k_y)$ is a wave vector; β, ρ are constants). The nonlinear interaction between waves may be different (see refs. [1–9]). For definiteness we consider the nonlinear dynamics of the waves to be described by the Charney–Hasegawa–Mima (CHM) equation [1,3,4]:

$$\frac{\partial}{\partial t}(\rho^2\Delta\psi-\psi)-\beta\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x}=0\,. \tag{2}$$

When the nonlinearity is small enough, then for a statistical description of the system of drift-type waves one can use the kinetic equation for waves [10]:

$$\begin{aligned}\frac{\partial n_k}{\partial t}=4\pi\int &|V_{k,k_1,k_2}|^2\\ &\times\delta(k+k_1+k_2)\delta(\Omega_k+\Omega_{k_1}+\Omega_{k_2})\\ &\times[n_{k_1}n_{k_2}+n_kn_{k_1}\operatorname{sign}(\omega_k\omega_{k_2})\\ &+n_kn_{k_2}\operatorname{sign}(\omega_k\omega_{k_1})]\,\mathrm{d}k_1\,\mathrm{d}k_2+\gamma_kn_k\,,\end{aligned} \tag{3}$$

where $n_k=\epsilon_k/\omega_k$ is the wave action spectrum (or the number of quasiparticles with momentum k), ϵ_k is the spectrum of the energy

$$E=\tfrac{1}{2}\int[\psi^2-(\nabla\psi)^2]\,\mathrm{d}x\,\mathrm{d}y\,,$$

$n_k=n_{-k}$, γ_k is the linear growth rate, V_{k,k_1,k_2} are the matrix elements of the wave interaction; in the case of the CHM equation we have [11]:

$$\begin{aligned}V_{k,k_1,k_2}=&-\frac{\mathrm{i}}{4\pi}|\beta k_xk_{1x}k_{2x}|^{1/2}\\ &\times\left(\frac{k_y}{1+\rho^2k^2}+\frac{k_{1y}}{1+\rho^2k_1^2}+\frac{k_{2y}}{1+\rho^2k_2^2}\right).\end{aligned} \tag{4}$$

In eq. (3) we have used the function

$$\Omega_k=\beta k_x-\omega_k=\frac{\beta\rho^2k_xk^2}{1+\rho^2k^2}\,,$$

which will be significant later on; the quantity $-\Omega_k$ is the wave frequency in the reference frame moving with the drift velocity β. Under the condition $\gamma_k \equiv 0$ eq. (3) conserves the energy E, the enstrophy (x-momentum) P_x and y-momentum P_y:

$$E = \tfrac{1}{2} \int |\omega_k| n_k \, dk ,$$

$$\boldsymbol{P} = (P_x, P_y) = \tfrac{1}{2} \int k \operatorname{sign} k_x \, n_k \, dk .$$

2. In the investigation of turbulence one often considers only near-scale interaction to be essential. By means of this interaction the energy (or some other conserved quantity) cascades step by step from large to small scales (or conversely). This assumption, called the hypothesis of locality of turbulence, is a ground for the Kolmogorov–Obukhov [12,13] spectrum of hydrodynamic turbulence; this spectrum is completely determined by the energy flux from the large to small scales. The spectra, determined by the flux of energy or enstrophy through the scales, have also been found for weak (i.e. with small nonlinearity) turbulence of the drift-type waves (see refs. [14–16]). The analysis of these spectra shows that the turbulence of the drift type waves appears to be nonlocal (these results will be published separately [17–19]). On the basis of this analysis one can assume that the evolution of the turbulence is mainly determined by the interaction with only large-scale turbulence. This is also indicated by some experimental facts. For example the observation of plasma turbulence in the F-layer of the equatorial ionosphere in the range of scales appropriate to the drift waves (5 m) $\lesssim |k|^{-1} \lesssim$ 100 m, in this connection $\rho \sim 5$ m) show that the spectrum n_k behaves as $|k|^{-6.5}$. Since the collision integral diverges on such spectra at $k \to 0$ [18], according to these observations the nonlocal interaction with the large scales seems to be dominant. Computer experiments also show a strong concentration of the spectrum in the region of small $|k|$ [21,6].

3. Thus we presuppose that the evolution of the spectrum n_k (when $|k|$ is not too small) is mainly determined by the interaction with the large-scale turbulence characterized by small wave vector p: $|p| \ll |k|$, $\rho |p| \ll 1$. Then the kinetic equation (3) may be written as follows:

$$\frac{\partial n_k}{\partial t} = 2 \int 4\pi |V_{k,p,-k-p}|^2 \times \delta(\Omega_{k+p} - \Omega_k) n_p (n_{k+p} - n_k) \, dp + \gamma_k n_k ,$$
$$|p| \ll |k|, \rho^{-1} , \tag{5}$$

the term $n_k n_{k+p}$ is neglected since the quantity n_p is considered to be sufficiently large; the quantity Ω_p is also neglected because it is of third order in $|p|$ while the difference $\Omega_{k+p} - \Omega_k$ is of first order. In the reference frame moving with the drift velocity β the large-scale turbulence may be considered to be frozen ($\Omega_p \approx 0$). On the background of this turbulence the high-frequency quanta are moving with their energy Ω_k conserved.

In accordance with eq. (5) the spectrum evolution occurs in k-space independently of each of the curves

$$\Omega_k = v = \text{const} \tag{6}$$

shown in fig. 1. Under the condition $\gamma_k = 0$ eq. (5) conserves the number of quasiparticles on each curve (6):

$$N(v) = \int n_k \delta(\Omega_k - v) \, dk ,$$
$$\dot{N}(v) = \int n_k \gamma_k \delta(\Omega_k - v) \, dk . \tag{7}$$

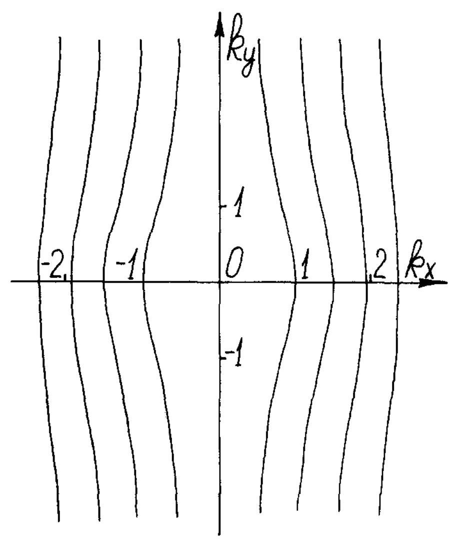

Fig. 1.

As the quantity $|p|$ is considered to be small eq. (5) reduces to the differential equation

$$\frac{\partial n_k}{\partial t}=\frac{\partial \Omega_k}{\partial k_x}\frac{\mathrm{D}}{\mathrm{D}k_y}S\frac{\mathrm{D}}{\mathrm{D}k_y}n_k+\gamma_k n_k\,, \tag{8}$$

which describes one-dimensional diffusion of the high-frequency quanta along the curves (6); $\mathrm{D}/\mathrm{D}k_y$ denotes differentiation with respect to k_y under constant $v=\Omega_k$:

$$\frac{\mathrm{D}}{\mathrm{D}k_y}=-\varphi\frac{\partial}{\partial k_x}+\frac{\partial}{\partial k_y}\,,$$

where

$$\varphi=\frac{\partial\Omega_k/\partial k_y}{\partial\Omega_k/\partial k_x}=\frac{2k_xk_y}{k^2(1+\rho^2k^2)+2k_x^2}\,.$$

The diffusion coefficient S is of the form

$$S=\left(\frac{\partial\Omega_k}{\partial k_x}\right)^{-2}\times\int_{-\infty}^{\infty}4\pi[\,|V_{k,p,-k-p}|^2 n_p]_{p_x=-\varphi p_y}p_y^2\,\mathrm{d}p_y\,;$$

in the case of the matrix element (4)

$$S=f(k)\int_{-\infty}^{\infty}p_y^5 n(-\varphi p_y,p_y)\,\mathrm{d}p_y\,, \tag{9'}$$

where

$$f(k)=\frac{8\pi}{\beta\rho^4}\frac{k^8(3+\rho^2k^2)^2(1+\rho^2k^2)\,|k_x^3k_y|}{[k^2(1+\rho^2k^2)+2k_x^2]^5}\,.$$

Redistribution of turbulence between the curves (6) is a slower process than diffusion along the curves, and it is determined by some corrections to eq. (8) which may turn out to be not reducible to the form of a differential operator. The possibility to disregard these corrections is in principle based on the fact that the curves (6) are non-closed and go to infinity, where strong enough dissipation always takes place. Indeed, if the quantity γ_k was positive everywhere on some curve (6), the number of particles on this curve would grow until infinity (see (7)). In this case it is necessary to consider the departure of the particles from this curve, and eq. (8) without corrections proves to be insufficient. It can be shown [19] that for nonlocal turbulence in the case of isotropic media the wave interaction in the lowest order occurs also along curves in k-space, but in this case these curves have the form of circles $|k|=$const. As a rule the quantity γ_k is everywhere positive on some of these circles; and therefore in this case taking account of the nonlocal interaction in the lowest order only is not sufficient.

4. Let us look for the behaviour of the energy and the momentum on a curve (6):

$$E(v)=\tfrac{1}{2}\int|\omega_k|\,n_k\delta(\Omega_k-v)\,\mathrm{d}k\,,$$

$$P(v)=\tfrac{1}{2}\int k\,\mathrm{sign}\,k_x\,n_k\delta(\Omega_k-v)\,\mathrm{d}k\,.$$

According to eq. (8)

$$2\dot{E}(v)\,\mathrm{sign}\,v=\beta\int_{-\infty}^{\infty}\varphi Q\,\mathrm{d}k_y+\int_{-\infty}^{\infty}\omega_k\gamma_k n_k\left(\frac{\partial\Omega_k}{\partial k_x}\right)^{-1}\mathrm{d}k_y\,,$$

$$2\dot{P}_x(v)\,\mathrm{sign}\,v=-\int_{-\infty}^{\infty}\varphi Q\,\mathrm{d}k_y+\int_{-\infty}^{\infty}k_x\gamma_k n_k\left(\frac{\partial\Omega_k}{\partial k_x}\right)^{-1}\mathrm{d}k_y\,,$$

$$2\dot{P}_y(v)\,\mathrm{sign}\,v=\int_{-\infty}^{\infty}Q\,\mathrm{d}k_y+\int_{-\infty}^{\infty}k_y\gamma_k n_k\left(\frac{\partial\Omega_k}{\partial k_x}\right)^{-1}\mathrm{d}k_y\,, \tag{10}$$

where $Q=-S\,\mathrm{D}n_k/\mathrm{D}k_y$ is the flux of particles along the curve (6). In these integrals the integration is carried out along the curve (6), i.e. the variable k_x is considered to be a function of the variable k_y implicitly defined by the equation $\Omega(k_x, k_y)=v$. As far as the total energy and impulse are conserved when $\gamma_k=0$ the first terms in expression (10) taken with minus sign give the fluxed of energy and momentum into the large-scale turbulence. Formulae (10) have an intuitive physical meaning. When the particles are moving along the curve (6) from the source in the range of small $|k_y|$ to the region of dissipation in the range of large $|k_y|$, they are going over to the states with smaller values of ω_k and k_x but with large values of $|k_y|$. In this way the particles lose their energy and enstrophy but gain y-momentum.

5. If the spectrum n_p of the large-scale turbulence is given the quantity S is a fixed function of the wave vector k. Then the character of the evolution of the spectrum of the small-scale turbulence is determined by the solution of the eigenvalue problem:

$$\lambda n_k = \frac{\partial \Omega_k}{\partial k_x} \frac{\mathrm{D}}{\mathrm{D}k_y} S \frac{\mathrm{D}}{\mathrm{D}k_y} n_k + \gamma_k n_k ,$$

$$Q = \mathrm{o}(1/k_y), \quad k_y \to \pm\infty , \tag{11}$$

where the quantity γ_k is analogous to the potential in the Schrödinger equation. For the drift-type waves in different physical situations the form of the function γ_k proves to be roughly the same (see refs. [20–22,6]); as a rule γ_k is positive in some domain adjoining the axis k_x and reaches its maximum value at some point of the axis k_x. As time tends to infinity the spectrum on the curve (6) approaches the eigenfunction $\bar{n}(k_y, v)$ corresponding to the maximum eigenvalue $\bar{\lambda}(v)$. This eigenfunction is everywhere positive. If for some v the quantity $\bar{\lambda}(v) < 0$ then the spectrum on the curve (6) exponentially tends to zero. The rate of departure of the particles into the dissipation region ($\gamma_k < 0$) by means of diffusion is greater than the supply of particles by the source ($\gamma_k > 0$). If $\bar{\lambda}(v) > 0$ then the spectrum on the curve (6) exponentially grows. The supply of particles by the source is greater than the rate of their departure into the dissipation region.

Thus, if there exist values of v such that $\bar{\lambda}(v) > 0$ (these ones correspond to the curves (6) with large enough growth rates $\gamma_k > 0$) then according to (10) the energy flux to the large-scale turbulence grows and causes an increase of its spectrum n_p. It is natural to expect that this leads to an increase of the diffusion coefficient (9) (i.e. to an increase of the departure of particles into the dissipation region), and consequently results in a decrease of the eigenvalues $\bar{\lambda}(v)$. As far as each curve (6) passes over the region of strong enough dissipation the eigenvalue $\bar{\lambda}(v)$ always becomes negative under sufficiently large increase of the diffusion coefficient S. Therefore the interval of those v for which $\bar{\lambda}(v) > 0$ will narrow with time until it degenerates to a point v_0, with $\bar{\lambda}(v_0) = 0$ (on the curve $\Omega(k) = v_0$ the growth rate is greatest in some sense). Thus the turbulence spectrum in k-space will be separated in k-space into two unconnected components: the large-scale turbulence and the jet spectrum of the small-scale turbulence concentrated on the curve $\Omega(k) = v_0$ (intermediate scales will "die out").

6. It follows from the above reasoning that the presence of small-scale turbulence imposes rigid restrictions on the large-scale turbulence (although the latter possesses much more energy than the former one). For the wave vectors k from some curve (6) the quantity $S(k)$ depends on the magnitude of the large-scale turbulence spectrum n_p only in the sector

$$|p_x/p_y| < \varphi_{\max}(v) = \max|\varphi(k_y, v)| ,$$

$$-\infty < k_y < \infty \tag{12}$$

(see (9)), which proves to be comparatively narrow, $\varphi_{\max}(v) \to 0$ when $v \to \infty$ (see fig. 2). Let the sector (12) at $v = v_0$ be so narrow that the large-scale turbulence spectrum n_p in this sector (12) is defined by its asymptotics for small p_x:

$$n(p_x, p_y) \simeq |p_x|^{\tau} g(p_y) ,$$

were τ is a number, and $g(p_y)$ is some function (e.g. if the wave action spectrum has no singularity and does not become zero on the axis k_y, then $\tau = 0$, $g(p_y) = n(0, p_y)$; if the energy spectrum E_k possesses such a property, then $\tau = -1$, $g(p_y) = E(0, p_y)$). In this situation expression (9′) for the quantity S factorizes:

$$S = \mu f(k) \varphi^{\tau} ,$$

where μ does not depend on the vector k:

$$\mu = \int_{-\infty}^{\infty} p_y^{5+\tau} g(p_y) \, \mathrm{d}p_y . \tag{13}$$

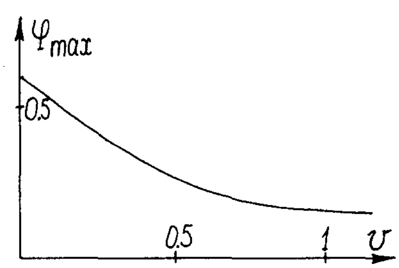

Fig. 2.

To find the "survival curve" (i.e. the curve $\Omega(k)=v_0$, on which the jet spectrum is eventually concentrated) we need to solve for all curves (6) the eigenvalue problem for the equation

$$\mu \frac{\mathrm{D}}{\mathrm{D}k_y} f\varphi^\tau \frac{\mathrm{D}}{\mathrm{D}k_y} n_k + \left(\frac{\partial \Omega_k}{\partial k_x}\right)^{-1} \gamma_k n_k = 0 ,$$

with the boundary conditions (11) and under the condition $n_k > 0$. In this way we find for each curve (6) the eigenvalue and the eigenfunction $\bar{n}(k_y, v)$. The "survival curve" $\Omega(k)=v_0$ corresponds to that value v_0 which is the maximum point of the function $\mu(v)$. Hence the large-scale turbulence must evolve to the state with the integral characteristics equal to the eigenvalue $\mu_0 = \mu(v_0)$:

$$\int_{-\infty}^{\infty} p_y^{5+\tau} g(p_y)\,\mathrm{d}p_y = \mu_0 = \max_{0<v<\infty} \mu(v) .$$

Thus the presence of weak small-scale turbulence fixed the level of large-scale turbulence.

The spectrum of the small-scale turbulence evolves to the jet spectrum, whose shape is defined by the eigenfunction $\bar{n}(k_y, v)$ and whose amplitude is found from the condition that the energy flux from the "survival curve" ought to be equal to the rate of dissipation in the large-scale turbulence. If the dissipation in the large scales is practically absent, then the amplitude of the jet spectrum becomes zero. Nevertheless the integral characteristic (13) must be equal to the eigenvalue μ_0.

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