Gravity Wave Turbulence in a Laboratory Flume

Petr Denissenko,[1] Sergei Lukaschuk,[1] and Sergey Nazarenko[2]
[1]Fluid Dynamics Laboratory, University of Hull, Hull, HU6 7RX, United Kingdom
[2]Mathematics Institute, University of Warwick, Coventry, CV4-7AL, United Kingdom
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We present an experimental study of the statistics of surface gravity wave turbulence in a flume of a horizontal size 12×6 m. For a wide range of amplitudes the wave energy spectrum was found to scale as $E_\omega \sim \omega^{-\nu}$ in a frequency range of up to one decade. However, ν appears to be nonuniversal: it depends on the wave intensity and ranges from about 6 to 4. We discuss our results in the context of existing theories and argue that at low wave amplitudes the wave statistics is affected by the flume finite size, and at high amplitudes the wave breaking effect dominates.

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Energy spectra of gravity surface waves and their probability density functions (PDF) contain important information about nonlinear mechanisms of the wave interaction. There are two most celebrated theories about the mechanisms that determine the shape of the energy spectra in the inertial (universal) interval of scales. The first theory was suggested by Phillips (PH) [1] who argued that sharp wave crests of breaking waves play the dominant role in the short-wave asymptotic of the spectrum, and this approach has been followed by many researchers (see, e.g., [2]). The second theory, by Zakharov and Filonenko (ZF) [3], considers the wave energy scaling in the inertial range as the result of four-wave resonant interactions of random weakly nonlinear waves. Importance of short-wave asymptotic for oceanographic and climate applications stimulated a long-term discussion and motivated further research. Although significant progress has been made recently both in numerical and field experiments [4–9], an unresolved issue still remains which of the theories, and under what conditions, could be applicable to the random waves generated in laboratory wave tanks. For example the experiments [8,10] used wind-wave forcing, which is spread over the entire frequency range. Thus, there was no well-defined inertial range and no statistical isotropy due to high tank aspect ratio (typical for wind-wave tunnels); both conditions are used in deriving ZF spectrum. In order to achieve statistical isotropy and to obtain a wide inertial range of scales, one should use flumes with the horizontal aspect ratios closer to one and with a forcing localized at low frequencies. Here considerable progress was made for smaller tanks, which are relevant to the studies of capillary [11,12] and gravity-capillary waves [13]. As far as we know, no attempts were made to study nonlinear evolution of random gravity waves in large flumes with close to 1 aspect ratio. Here we report the results of this kind of experiment. To introduce reference points for our interpretations, we start with a summary of existing theoretical predictions.

Consider the wave energy spectrum

$$E_\omega = g \int e^{i\omega t'} \langle \eta(\mathbf{x}, t)\eta(\mathbf{x}, t + t')\rangle \, dt',$$

where $\eta(\mathbf{x}, t)$ is the surface elevation field in the horizontal plane and the angle brackets mean the ensemble averaging. Most of the theories predict a power-law scaling

$$E_\omega \propto \epsilon^\alpha \omega^{-\nu} \tag{1}$$

where ϵ is the energy dissipation rate and the exponents ν and α depend on a particular theory.

Weak turbulence theory (WTT) considers weakly nonlinear random-phase waves in an infinite box limit. For the wave spectrum, these assumptions lead to the Hasselmann equation [14]. Zakharov-Filonenko (ZF) spectrum $E_\omega \propto g^2 \epsilon^{1/3} \omega^{-4}$ is an exact solution of the Hasselmann equation which describes a steady state with energy cascading through an inertial range from large scales where it is produced to the small scales where it is dissipated. It is important that in deriving WTT the limit of an infinite box is taken before the limit of small nonlinearity. This means that in any large but finite box, the wave intensity should be strong enough so that the nonlinear resonance broadening is much greater than the spacing of the k grid. As estimated in [15], this implies a condition on the minimal angle of the surface elevation

$$\gamma > 1/(kL)^{1/4}, \tag{2}$$

where L is the size of the basin. This is a very severe restriction meaning, e.g., that for a ten-kilometer wide gulf and meter-long waves one should have $\gamma > 0.1$. On the other hand, even for very small amplitudes some resonances survive [6,7,16], and it is possible that they can support the energy cascade through scales even when condition (2) is not satisfied.

PH spectrum is derived by assuming that g and ω (and not ϵ) are the only relevant dimensional physical quantities, which gives [1]

$$E_\omega \propto g^3 \omega^{-5} \qquad (3)$$

[i.e., $\alpha = 0$ in (1)]. PH spectrum is associated with the sharp crested waves, so that the short-wave Fourier asymptotics are dominated by discontinuous slopes. Assuming first that discontinuity is happening at an isolated point, we get for the one-dimensional energy spectrum $E_k \propto k^{-3}$. Second, assuming that the transition from the k space to the ω space should be done according to the linear wave relation $\omega = \sqrt{gk}$, we arrive at PH spectrum (3). Recently Kuznetsov [17] questioned both of these assumptions and argued that (i) the slope breaks on one-dimensional lines (ridges) rather than zero-dimensional points (peaks), and (ii) the wave crest is propagating with a preserved shape; i.e., $\omega \propto k$ should be used instead of $\omega = \sqrt{gk}$. This gives $E_\omega \propto \omega^{-4}$, i.e., formally the same scaling as ZF, even though the physics behind it is different. Finally, it was proposed in [18] that wave crest ridges may have a noninteger fractal dimension in the range $0 < D < 2$. For $\omega \propto k$, we have $E_\omega \sim g^{1+D} \epsilon^{(2-D)/3} \omega^{-3-D}$.

Discrete WTT [15] appeals to an observation that condition (2) is often violated for weak waves in finite basins, so that the number of exact and approximate four-wave resonances will be drastically depleted [15,19,20]. This arrests the energy cascade and, therefore, leads to energy accumulation near the forcing interval of scales. The accumulation will proceed until the intensity is strong enough for the nonlinear broadening to become comparable to the k-grid spacing, i.e., when the condition (2) will become marginally satisfied. At this point, the four-wave resonances will get engaged and the spectrum "sandpile" will tip over towards the higher wave numbers. This process will proceed until the whole k space will be filled by the spectrum having a critical slope determined by the condition that the resonance broadening is of the order of the k-grid spacing for all modes in the inertial range. This gives $E_\omega \sim g^{7/2} L^{-1/2} \omega^{-6}$ (hence $\alpha = 0$ in this case).

PDF of wave crest heights in homogeneous isotropic wave fields with random independent phases would have a Gaussian shape. For stronger nonlinearities, PDF for the wave crests was predicted by Tayfun [21] based on a model where the wave field is made of independent weakly nonlinear Stokes waves whose first harmonics are Gaussian. Tayfun distribution was found to be in a good agreement with numerical simulations with wide-angle quasi-isotropic wave fields [22] and to a much lesser extent in narrow-angle distributions [23]. Additional important information is contained in PDF of Fourier amplitudes. Based on a generalized WTT, Choi *et al.* [16] obtained solutions for such a PDF which has an enhanced probability of strong waves with respect to Gaussian fields.

The experiments were conducted in a rectangular tank with dimensions $12 \times 6 \times 1.5$ m filled with water up to the depth of 0.9 m. The wave maker consists of 8 piston-type paddles covering the full span of a short side of the tank. The oscillation amplitude, frequency, and phase can be set for each paddle independently allowing control of the directional distribution of the generated waves. In the experiments described here, we used different excitations, the geometry of the flume, and the water depth. In most cases the excitation was a superposition of two waves with frequencies 0.973 and 1.14 Hz with one wave directed along the normal to the wave maker and the other at the angle of 7° to it. In some other cases we used monochromatic excitation at 2 Hz, or broadband excitations in the range from 0.8 to 1.2 Hz (with wave vectors distributed over a 90° sector). Some measurements were done in a perfectly rectangular flume and some with the back wall tilted at 20°. Comparing the data related to different excitation conditions we did not find any quantitative difference in the inertial interval of spectra. No changes in the inertial interval of spectra were found even when the depth of water was decreased to 0.45 m.

The surface elevation $\eta(t)$ was measured by capacitance gauges simultaneously at two points in the middle part of the tank. The minimum signal acquisition time was 2000 s. The degree of nonlinearity can be characterized by the mean slope at the energy containing scale, $\gamma = k_m A$, where k_m is the wave number corresponding to the maximum in the energy spectrum and A is the rms of $\eta(t)$. In most experiments $k_m \geq 4.0\ \text{m}^{-1}$, which corresponds to the wavelength $\lambda \leq 1.6$ m. Our experiments covered the range from weak waves with mostly smooth surface and occasional seldom wave breaking at $\gamma \approx 0.052$ to stronger waves characterized by a choppy surface with the numerous wave breaking events at $\gamma \approx 0.21$.

To calculate wave power spectra we used the Welch algorithm with averaging over 1000 spectral estimates. The typical spectra are shown in Fig. 1. It shows much steeper slopes ν for the weak waves than for the strong ones. On the low amplitude end, we present only data where the excitation amplitudes are strong enough for the

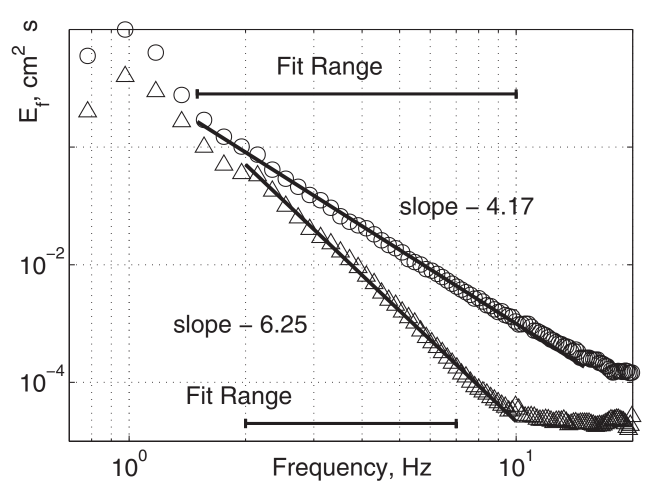

FIG. 1. Typical results for the energy spectra. △: an experiment with low wave intensity, $\gamma \approx 0.09$; ○: an experiment with high wave intensity, $\gamma \approx 0.25$.

scaling to be well defined and not obstructed much by the harmonics of the forcing frequency. On the high amplitude side, we are restricted by intense splashing. For the experiments with the minimal intensity, we get $\nu \approx 6$, which agrees with the discrete WTT [15]. Thus, we confirm that the k-space discreteness is important at low wave amplitudes. In this case the condition (2) is not satisfied, $\gamma \approx 0.1 < 1/(k_m L)^{1/4} \approx 0.4$. A factor of 4 deficiency here indicates that many of the four-wave resonances are "switched off" and most of energy is carried from low to high wave numbers by a small number of active wave number quartets. This causes an extra anisotropy and inhomogeneity of wave turbulence seen in our experiment for weak waves.

Figure 2 shows the spectra slopes obtained in the experiments with different wave intensities and different excitation conditions. The x axis represents a combination $J = E_f/f^{-\nu}$, which is a good measure of the wave field intensity because, according to (1), it is expected to be frequency independent in the scaling range $E_f/f^{-\nu} \propto \varepsilon^{\alpha}$ (here $f = 2\pi\omega$ and $E_f = E_\omega/2\pi$). At large wave field intensities, the scaling range reaches one decade in ω in width, which is significantly greater than the scaling ranges previously observed in the numerics and in the field observations. There is a range of intensities where PH slope $\nu = 5$ is observed, and we note that wave breaking events were common for such intensities. At higher intensities, the slope close to $\nu = 4$ is observed as it was predicted by both ZF and Kuznetsov theories [3,17]. However, the water surface was visibly very choppy with numerous wave-breaking and high values of the surface slope $\gamma > 0.15$,

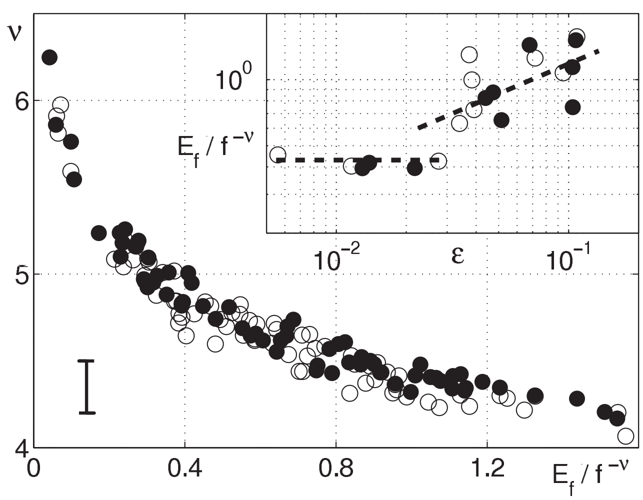

FIG. 2. Plot of the spectral slopes vs the wave intensity $J = E_f/f^{-\nu}$ measured within the inertial interval. Here, γ varies in the range from 0.08 to 0.25. Inset: Dependence of the spectral intensity J on the energy dissipation rate. E_f is measured in cm^2 s, f is in Hz. Solid and open circles correspond to the signals measured simultaneously in two different points separated by a distance of 0.4 m.

thus invalidating the weak nonlinearity assumption of ZF theory [3]. Kuznetsov theory [17] is more likely to be relevant to these conditions. We have not reached a plateau in the vicinity of $\nu = 4$: ν gradually decreases reaching $\nu = 4.1$ for the maximum intensity $\gamma \approx 0.25$. The gradual change of ν could be explained by changing the fractal dimension of the wave crest ridges, from $D = 2$ "surface filling" lines giving $\nu = 5$ at lower amplitudes to a set of 1D lines giving $\nu = 4$ at larger amplitudes (Phillips-Kuznetsov scenario).

The inset in Fig. 2 shows the dependence of the wave intensity $J = E_f/f^{-\nu}$ on the energy dissipation rate ϵ. The value of ϵ was found independently from the decay rate of the wave energy decay measured immediately after switching off the wave maker via fitting the dependence of energy on time by an exponential function. For different wave intensities the decay rates were constant within a time interval of at least 500 excitation wave periods. Despite a significant scattering of data points (this is mainly because of errors in ϵ due to the limited length of the fit interval and the chaotic nature of signals), one can see two different regimes. At higher dissipation rates, J is an increasing function of ϵ. For comparison we show the slope $\alpha = \frac{1}{3}$ corresponding to the WTT prediction. At low excitations (up to the values corresponding to $\nu \sim 5$), intensity J is nearly constant independent of ϵ, which agrees with predictions of $\alpha = 0$ by both the discrete WTT (for weaker waves) and by PH spectrum (for stronger waves). In this regime, the wave intensity J saturates at an ϵ-independent level and the excesses of energy are released downscale via sandpile collapses, the later being stronger or/and more frequent for larger values of ϵ.

Figure 3 shows the PDF of the wave crests in a case with strong wave intensity. There is a good qualitative agreement of this PDF with Tayfun distribution except for rather irregular deviations near the PDF maximum. Figure 4

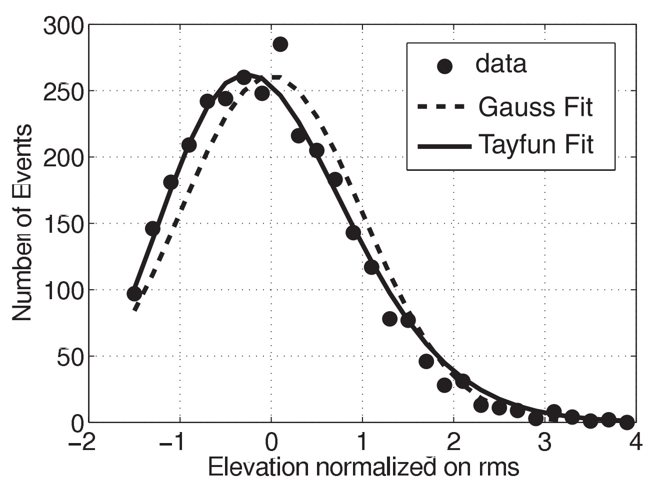

FIG. 3. PDF of the wave crests for high wave intensity, $\gamma \approx 0.25$.

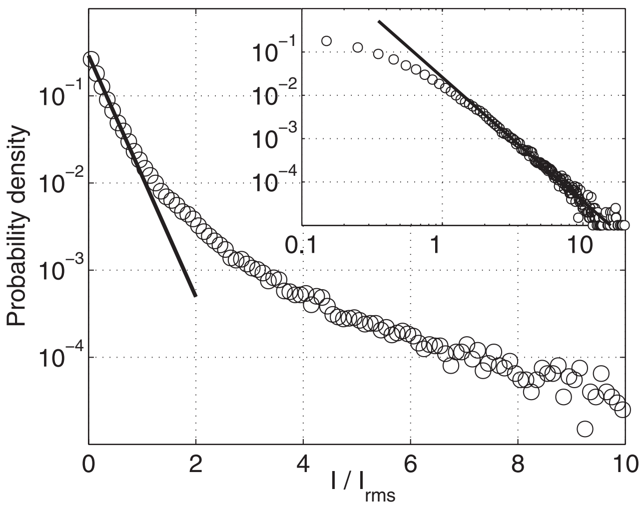

FIG. 4. PDF of the spectral intensity bandpass filtered with frequency window $\Delta f = \pm 1$ Hz centered at $f = 6$ Hz, $\gamma \approx 0.16$. The inset shows the same plot in log-log coordinates.

shows a log-lin plot of the PDF of the spectral intensity I obtained as the square of the bandpass filtered signal with a pass window Δf around a particular frequency f ($\Delta f \ll f$). The straight line on the plot shows Rayleigh distribution. The inset shows the same PDF in log-log representation. In the case of weak waves (not shown here) the PDF follows closely Rayleigh shape up to $I \sim 6\ I_{\rm rms}$. This is natural because Rayleigh distribution of intensities corresponds to Gaussian wave fields, which are expected to occur when the waves are weakly nonlinear. However, in the case of stronger waves the PDF deviates from Rayleigh distribution in the tail. A power law I^{-3} provides a good fit for the range $2I_{\rm rms} < I < 10I_{\rm rms}$ (a straight line in the inset), which indicates a very strong intermittency. This picture is similar to the one observed in numerical simulations [6,16].

Our main conclusion is that the slope of the wave spectrum is not universal: it takes high values, about 6, for weak wave fields and gradually decreases to about 4 when the wave field intensity increases and the water surface becomes very choppy with a lot of wave breaking. At low wave amplitudes our results agree with the prediction of [15] for the critical slope 6, which indicates that the finite size effects of the flume are important at these amplitudes and the four-wave resonances are greatly depleted. The gradual decrease of the spectral slope at larger amplitudes is possibly due to the sharp wave crests whose fractal dimension decreases with increasing wave intensity—from $D \sim 2$ for lower intensities to $D = 1$ for Kuznetsov spectrum. At this point dependence of D on the wave intensity is speculative, and a further study of the wave breaking morphology is needed. The important feature of our experiments was that the weak turbulence regime was unlikely to have ever been achieved: with increasing wave intensity the nonlinearity becomes strong before the system looses sensitivity to the k-space discreteness. PDFs of the wave crests were found to agree well with Tayfun distribution except for the region of small amplitudes. For PDFs of Fourier modes, we observed an enhanced (with respect to Gaussian) probability of strong wave amplitudes.

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