MA135—VECTORS AND MATRICES
Example Sheet 3

All questions in Sections A and B must be handed in to your supervisor via the supervisor's pigeon loft by **3pm Monday, Week 4.** Section C questions should be **attempted** by students who hope to get a 1st or 2:1 degree, but are not to be handed in.

Section A

A1 Let $\mathbf{u} = (-1, 2, 1, 0)$, $\mathbf{v} = (0, 1, 3, 1)$, $\mathbf{w} = (-2, 3, 0, 5)$. Calculate:

(i) $2\mathbf{u} - \mathbf{v} + \mathbf{w}$ (ii) $\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}$ (iii) $(2\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$.

A2 Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^7$ and λ be a scalar. Which of the following operations are **not** defined: $2\lambda + \mathbf{v}$, $\mathbf{w} + \lambda\mathbf{v}$, $\mathbf{u}/\mathbf{v}$, $\|\mathbf{v}\| - \lambda\mathbf{u}$, $\|\mathbf{u}\|\,\mathbf{v} - \lambda\mathbf{u}$, $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$, $\|\mathbf{u} \cdot \mathbf{v}\|$.

A3 Find the cosine of the angle determined by the given pair of vectors, and state whether the angle is acute, obtuse or a right-angle:

(i) $(1, 0, 2)$, $(-2, 1, 1)$ (ii) $(1, -1)$, $(1, -2)$

(iii) $(4, 1, -1, 2)$, $(1, -4, 2, -1)$

A4 Let $\mathbf{v}$ be a non-zero vector and $\mathbf{w}$ the unit vector of the opposite direction to $\mathbf{v}$. Write $\mathbf{w}$ in terms of $\mathbf{v}$.

A5 Suppose $\mathbf{v} \in \mathbb{R}^3$. Show that $\mathbf{v} = (\mathbf{v} \cdot \mathbf{i})\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\mathbf{j} + (\mathbf{v} \cdot \mathbf{k})\mathbf{k}$.

A6 Let L_1 and L_2 be the straight lines given by the vector equations

$$L_1 : \mathbf{x} = (0, 1, 1) + t(2, 2, 2), \quad L_2 : \mathbf{x} = (1, 2, 2) + t(1, 1, 1).$$

Show that both lines pass through both of the points $(0, 1, 1)$ and $(1, 2, 2)$. Does that mean that L_1 and L_2 are the same line?

Section B

B1 Let $\mathbf{r}_0 = (-1, 1)$. Find all vectors $\mathbf{r} = (x, y)$ satisfying

$$\|\mathbf{r}\| = \|\mathbf{r} - \mathbf{r}_0\| = \sqrt{5}.$$

Make a rough sketch to interpret this equation.

B2 Let L be the line in the plane given by the equation $y = mx + c$. What is the vector equation of L?

B3 Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are orthogonal non-zero vectors in Euclidean n-space, and that a vector $\mathbf{v}$ is expressed as

$$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n.$$

Show that the scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ are given by

$$\lambda_i = \frac{\mathbf{v} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2}, \qquad i = 1, 2, \ldots, n.$$

What are λ_i if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are orthonormal?

Section C

C1 Let $\mathbf{u}$, $\mathbf{v}$ be vectors in $\mathbb{R}^n$, and let A be the area of the parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides. Show that

$$A^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.$$

C2 **(The Cauchy-Schwartz Inquality)** Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are real numbers. Show that

$$|u_1 v_1 + u_2 v_2 + \cdots + u_n v_n| \\ \leq \left(u_1^2 + u_2^2 + \cdots + u_n^2\right)^{1/2} \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)^{1/2}.$$

Hint: Think about what the inquality is saying in terms of vectors.

C3 **(The Triangle Inequality)** Let $\mathbf{u}$, $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Hint: Start with $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$ and after expanding the brackets use the Cauchy-Schwartz inequality.

C4 Let $L_1 : \mathbf{x} = \mathbf{u}_1 + t\mathbf{v}_1$ and $L_2 : \mathbf{x} = \mathbf{u}_2 + t\mathbf{v}_2$ be straight lines in $\mathbb{R}^n$, where $\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_2 \neq \mathbf{0}$ and $\mathbf{u}_1 \neq \mathbf{u}_2$. Show that L_1 and L_2 are the same line if and only if the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{u}_1 - \mathbf{u}_2$ are parallel.