MA136 Introduction to Abstract Algebra

Homework Assignment 1

Attempt all the questions on this sheet, and hand in solutions to A1, A2, B1, B2, B3. Solutions to your supervisor's pigeon-loft by **2pm Tuesday, week 7.** Your supervisor will mark some, but not all, of these questions. They don't know which ones yet, so there's no point asking them.

(A1) In the following, is $\circ$ a binary operation on A? If so, is it commutative? Is it associative? In each case justify your answer.
  (a) $A = \mathbb{R}$ is the set of real numbers and $a \circ b = a/b$.
  (b) $A = \{1, 2, 3, 4, \dots\}$ is the set of positive integers and $a \circ b = a^b$.
  (c) $A = \{\dots, 1/8, 1/4, 1/2, 1, 2, 4, 8, \dots\}$ is the set of powers of 2 and $a \circ b = ab$.
  (d) $A = \mathbb{C}$ is the set of complex numbers and $a \circ b = |a - b|$.

(A2) Give counterexamples to show that the following do not necessarily hold for elements a, b of D_4:
  (a) $(ab)^2 = a^2b^2$;
  (b) $(ab)^{-1} = a^{-1}b^{-1}$;
  (c) $a^{-1}ba = b$;
  (d) $b^{-1}a = ab^{-1}$.

(B1) In this exercise you will write out the composition table for the group D_3 which is the group of symmetries of an equilateral triangle. Make sure you have read and understood pages 32–36 of the lecture notes about D_4, the group of symmetries of a square. Sketch an equilateral triangle and label the vertices 1, 2, 3 in anticlockwise order. Label the centre of the triangle with O. Let ρ_0, ρ_1, ρ_2 denote anticlockwise rotations about O through angles 0, $2\pi/3$ and $4\pi/3$. Let σ_1, σ_2, σ_3 denote reflections about the lines respectively joining vertices 1, 2, 3 to O. Let

$$D_3 = \{\rho_0, \rho_1, \rho_2, \sigma_1, \sigma_2, \sigma_3\}.$$

Write down a composition table for D_3 and explain why it is a group. Is it abelian? It has six subgroups; write them down. Draw a figure to show how these subgroups fit inside each other (such as the one on page 36 of the notes).

(B2) Write down the symmetries of a triangle that is isoceles but not equilateral and a composition table for them. Do they form a group?

(B3) Let G be a group satisfying $a^2 = 1$ for all a in G. Show that G is abelian.

(C) Let $M_2(\mathbb{R})$ be the set of 2×2 matrices with entries in $\mathbb{R}$. Let $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let

$$U = \{A \in M_2(\mathbb{R}) : A^2 = I_2\}.$$

Is $(U, \cdot)$ a group?