Surface cyclic quotient singularities and Hirzebruch–Jung resolutions

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Abstract

If V is an affine algebraic variety and $G \subset \operatorname{Aut} V$ a finite group of automorphism of V, the *quotient variety* is an affine algebraic variety V/G with a quotient morphism $V \to X = V/G$. A point of X is an orbit of G on V, and the coordinate ring $k[X]$ is the ring of invariants $k[V]^G$ of the induced action of G on $k[V]$.

This chapter studies the simplest case of this construction, when $V = \mathbb{C}^2$ and $G = \mathbb{Z}/r$ is the cyclic group of order r acting on $\mathbb{C}^2$ by diagonal matrixes; by a slight normalisation, we can assume that

$$G = \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \right\rangle,$$

where $\varepsilon = \exp \frac{2\pi i}{r}$ is a primitive root of 1, and a is coprime to r; these are the *surface cyclic quotient singularities* of type $\frac{1}{r}(1, a)$. This is a famous class of varieties; their theory involves explicit calculations that are a lot of fun, illustrate many ideas of algebraic geometry, and have a number of important applications. Their resolution was a key element in the first completely correct proof of resolution of surface singularities by R.J. Walker in the 1920s, and was reworked in the 1950s by Hirzebruch. Their study leads naturally to the development of toric geometry. My treatment in terms of invariant monomials and continued fractions follows Hirzebruch. A good reference is [R] (in German).

Recently Nakamura and his coworkers have added an interesting new twist to this story (see for example [IN]): the minimal resolution we construct here by explicit computations is a *moduli space*: it naturally parametrises the *G-clusters* or *scheme theoretic G-orbits* $Z \subset \mathbb{C}^2$.

1 Introduction

1.1 Definition of quotient

We use the following standard result, whose proof is an exercise in commutative algebra (compare Mumford [M], Section 7 or Exs 4.3–4.4 for details). The statement is intuitive, and you may take it on trust for now.

Proposition–Definition 1.1 *G is a finite group acting on an affine variety V by algebraic automorphisms. Write $k[V]$ for the coordinate ring of V. Then the quotient $X = V/G$ is an affine variety whose points correspond one-to-one with orbits of the group action, and such that the polynomial functions on X are precisely the invariant polynomial functions on V, that is, $k[X] = k[V]^G$.*

The quotient X as just defined exists: just take $X = \operatorname{Spec} k[V]^G$. In other words, we know that the ring $k[V]^G$ is generated by finitely many polynomials (an application of finiteness of normalisation, etc.). Write $k[V]^G = k[u_1, \dots, u_N]/J$, where u_i are the generators and J the ideal of relations between them. Then V/G is the subvariety of affine space $\mathbb{A}^N_k$ (with coordinates $u_1, \dots, u_N$) defined by the ideal J:

$$X = V(J) = \left\{ \underline{u} \in k^N \mid f(\underline{u}) = 0 \text{ for all } f \in J \right\} \subset k^N.$$

Example 1.2 The baby case $\frac{1}{2}(1,1)$. The simplest example is the group action

$$\mathbb{Z}/2 \quad \text{acting on } \mathbb{C}^2 \text{ by} \quad (x,y) \mapsto (-x,-y).$$

The monomials x^2, xy, y^2 are obviously invariant under the $\mathbb{Z}/2$ action, and it is not hard to see that any $\mathbb{Z}/2$-invariant polynomial in x, y (that is, any *even* polynomial) is a polynomial in these. Consider the polynomial map

$$\varphi\colon \mathbb{C}^2 \to X \subset \mathbb{C}^3 \quad \text{defined by } (x,y) \mapsto (u = x^2, v = xy, w = y^2),$$

where u, v, w are coordinates on $\mathbb{C}^3$; its image X is the ordinary quadratic cone $X : (uw = v^2)$. Then $X = \mathbb{C}^2/(\mathbb{Z}/2)$ is an example of a quotient by a group action.

I explain what resolution of singularities means in this simple context. Write C for the base of the cone $X : (uw = v^2)$, so that C is the standard conic in $\mathbb{P}^2_{\mathbb{C}}$, defined by the same equation viewed as the homogeneous equation $UW = V^2$. It is well known and easy that C is isomorphic to $\mathbb{P}^1$ (see [UAG], §1).

Figure 1: The cylinder resolution of the quadratic cone $X : (uw = v^2)$

Quite generally, a cone X over a nonsingular base variety C has a resolution of singularities given by the corresponding *cylinder*; whereas X is the union of generating lines through the vertex O, the cylinder is the $\mathbb{C}^1$-bundle obtained as the *disjoint* union of these generating lines. For $Q \in C$, write $L_Q \cong \mathbb{C}$ for the generator of the cone through Q, and set

$$Y = \left\{ P \in X, Q \in C \;\middle|\; P \in L_Q \right\} \subset X \times C.$$

In other words, Y is the closed graph of the projection of the cone to its base. The projection $Y \to C$ is the cylinder over C corresponding to the cone X; that is, it is the $\mathbb{C}^1$-bundle obtained as the disjoint union of the generating lines L_Q of X. The second projection $f \colon Y \to X$ is the resolution of singularities of X: this means

(a) Y is nonsingular.

(b) $f \colon Y \to X$ is a projective birational morphism; in particular, f is surjective, and $f^{-1}(0)$ is projective.

(c) $Y \to X$ an isomorphism $Y \setminus f^{-1}(0) \to X \setminus 0$.

To see Y in coordinates, I cover C by two copies of the affine line: the piece $C_0 : (W \neq 0)$ is parametrised by $\xi^2, \xi, 1$, and the piece $C_1 : (U \neq 0)$ by $1, \eta, \eta^2$ (where $\xi = x/y = u/v$ and $\eta = y/x = w/v$). Glueing together these copies of $\mathbb{C}^1$ by $\xi = \eta^{-1}$ (on the open set $UW \neq 0$) makes explicit the identification $C \cong \mathbb{P}^1$.

The cylinder Y is also covered by two affine pieces: since it is a $\mathbb{C}^1$-bundle over C, each piece is $\cong \mathbb{C}^2$, with a coordinate in the base C, and a coordinate in the fibre: this gives $Y = Y_0 \cup Y_1$ where $Y_0 = \mathbb{C}^2$ with coordinates (ξ, w) and $Y_1 = \mathbb{C}^2$ with coordinates (η, u). The glueing between the two pieces where they overlap is given by

$$Y_0 \setminus (\xi = 0) \xrightarrow{\cong} Y_1 \setminus (\eta = 0) \quad \text{by} \quad (\xi, w) \mapsto (\eta = \xi^{-1}, u = w\xi^2).$$

Here ξ and η are parameters on the base $C \cong \mathbb{P}^1$, and u, w linear coordinates in the fibre. The identification $u = w\xi^2$ is the transition function of a $\mathbb{C}^1$-bundle: either u or w can be used as the linear coordinate in the fibre, but the coordinate change between them depends on ξ.

1.2 Notation and aim

The notation $\frac{1}{r}(1,a)$ stands for the action of the cyclic group $\mathbb{Z}/r$ on $\mathbb{C}^2$ given by $(x,y) \mapsto (\varepsilon x, \varepsilon^a y)$, where $\varepsilon = \exp(\frac{2\pi i}{r})$ is a chosen primitive rth root of unity. The aim of this chapter is to treat the quotient singularities $X = \mathbb{C}^2/(\mathbb{Z}/r)$ by analogy with the case $\frac{1}{2}(1,1)$ discussed above. I explain

(1) what it means to give this quotient the structure of an affine algebraic variety,

(2) how to describe its coordinate ring in terms of monomial generators and relations, and

(3) how to construct a resolution of singularities $Y \to X$.

All of this involves very concrete numerical calculations starting from the numbers r, a.

1.3 More examples

I can generalise the baby example in several different directions:

1. $\frac{1}{r}(1,1)$; the quotient is isomorphic to the affine cone over the rational normal curve $C_r \subset \mathbb{P}^r$, and its resolution has exceptional locus $E \cong \mathbb{P}^r$ with $E^2 = -r$.

2. Every finite cyclic subgroup $G \subset \mathrm{SL}(2,\mathbb{C})$ is of the form $\frac{1}{r}(1, r-1)$; the quotient is the hypersurface singularity $uw = v^r$.

3. The general case $\frac{1}{r}(1,a)$ with a coprime to r. There are automatic and very convenient descriptions of the quotient and its resolution in terms of Hirzebruch–Jung continued fractions; this is the main point of this chapter. Moreover, the resolution equals the Hilbert scheme of G-orbits.

4. Non-Abelian finite subgroups $G \subset \mathrm{SL}(2,\mathbb{C})$; these are classified as the binary dihedral, binary tetrahedral, binary octahedral and binary dihedral groups, and correspond to the Du Val singularities D_n, E_6, E_7, E_8. Work of Ito and Nakamura [IN] shows that in these cases also, the minimal resolution equals the Hilbert scheme of G-orbits.

5. Non-Abelian finite subgroups $G \subset \mathrm{GL}(2,\mathbb{C})$; these are classified in A, D, E terms, and the resolutions are known. At the present time, whether the minimal resolution equals the Hilbert scheme of G-orbits is an interesting open question.

2 Hirzebruch–Jung continued fractions and the Newton polygon of the lattice $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(1, b)$

To treat more general cases, we need some more notation and some fairly easy combinatorics.

Definition 2.1 Let r, b be coprime integers with $r > b > 0$. Then the Hirzebruch–Jung *continued fraction* of r/b is the expression

$$\frac{r}{b} = a_1 - \cfrac{1}{a_2 - \frac{1}{a_3 - \cdots}} = [a_1, a_2, \ldots, a_k]. \tag{2.1}$$

For example,

$$\frac{7}{3} = [3, 2, 2] = 3 - \cfrac{1}{2 - \frac{1}{2}}.$$

Here $3 = \left\lceil \frac{7}{3} \right\rceil$ is the *roundup* (the least integer $\geq$ a fraction). The overshoot is $3 - \left\lceil \frac{7}{3} \right\rceil = \frac{2}{3}$, and we take its reciprocal to get $\frac{3}{2} = 2 - \frac{1}{2}$. See the homework sheet for more examples.

In general, (2.1) means the following: if r and b are given, write $a_1 = \left\lceil \frac{r}{b} \right\rceil$ for the roundup; $a_1 \geq 2$ because $r > b$. If $b = 1$ then $r = r/b = [a_1]$ is an integer, so we stop. Failing that, there is a fractional overshoot of $a_1 - \frac{r}{b}$, and we take its reciprocal $\frac{b}{ba_1 - r} = \frac{r_1}{b_1}$ as a new fraction (still with $r_1 > b_1 > 0$ and coprime), and calculate $\frac{r_1}{b_1} = [a_2, \ldots, a_k]$ recursively. In other words, replace

$$\frac{r}{b} \mapsto \frac{r_1}{b_1} \quad \text{where} \quad \begin{pmatrix} r_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_1 \end{pmatrix} \begin{pmatrix} r \\ b \end{pmatrix} = \begin{pmatrix} b \\ ba_1 - r \end{pmatrix} \tag{2.2}$$

and continue likewise.

Conversely, if $[a_1, \ldots, a_k]$ are given with each $a_i \geq 2$, then we calculate from the other end: set $[a_k] = a_k$, and for $i = k, k-1, \ldots, 2$

$$[a_{i-1}, \ldots, a_k] = a_{i-1} - \frac{1}{[a_i, a_{i+1}, \ldots, a_k]}$$

See the homework sheet for an interpretation of $[a_1, \ldots, a_k]$ in terms of

$$\begin{pmatrix} 0 & 1 \\ -1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & a_k \end{pmatrix}$$

the product of elementary matrixes in $\mathrm{SL}(2, \mathbb{Z})$. You can view this as the composite of k changes of bases from e_0, e_1 to e_1, e_2, etc. to e_k, e_{k+1}.

Proposition 2.2 *Let* $r > b > 1$ *be coprime integers and consider the lattice* $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(1, b) \subset \mathbb{R}^2$. *Then* L *contains the lattice* $\mathbb{Z}^2$ *as a sublattice of index* r*, and its other cosets are represented by the* $r - 1$ *lattice points* $\frac{1}{r}(i, \overline{ib})$ *contained in the unit square of* $\mathbb{R}^2$ *(see Figure 2). Define the* Newton

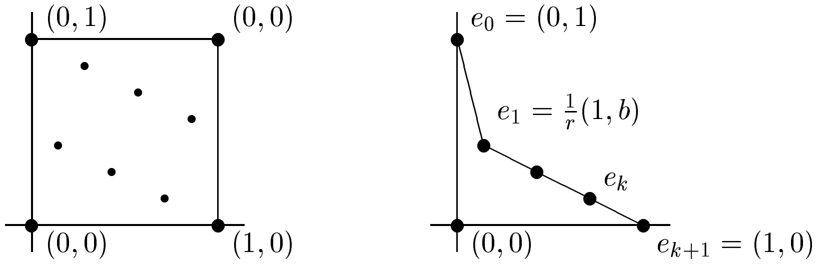

Figure 2: The lattice L and its Newton polygon

polygon *as the convex hull* Newton L *in* $\mathbb{R}^2$ *of all the nonzero lattice points in the positive quadrant.*

Write

$$e_0 = (0,1), \quad e_1 = \frac{1}{r}(1,b), \quad e_2, \dots, e_k, \quad e_{k+1} = (1,0)$$

for the lattice points on the boundary of Newton L.

Then

(I) Any two consecutive lattice points e_i, e_{i+1} *for* $i = 0, \dots, k$ *form an oriented basis of* L.

(II) Any three consecutive lattice points e_{i-1}, e_i, e_{i+1} *for* $i = 1, \dots, k$ *satisfy a relation*

$$e_{i+1} + e_{i-1} = a_i e_i \quad \textit{for some integer } a_i \geq 2.$$

(III) The integers $a_1, \dots, a_k$ *in (II) are the entries of the continued fraction:*

$$\frac{r}{b} = [a_1, \dots, a_k].$$

The relation in (II) can be viewed as a coordinate change *from the basis* e_{i-1}, e_i *to the next basis* e_i, e_{i+1} *expressed by the matrix* $\begin{pmatrix} 0 & 1 \\ -1 & a_i \end{pmatrix}$, *that is,*

$$\begin{array}{rcccl} e_i & = & (0,1) & = & e_i, \\ e_{i+1} & = & (-1, a_i) & = & -e_{i-1} + a_i e_i. \end{array}$$

Proof (I) Consider the parallelogram $\Pi = \langle 0, e_i, e_{i+1}, e_i + e_{i+1} \rangle$; by construction of Newton L, the lower triangle $\Delta = \langle 0, e_i, e_{i+1} \rangle$ has no lattice point other than its vertices. For any point v in the upper triangle $\Delta^+ = \langle e_i, e_{i+1}, e_i + e_{i+1} \rangle$, the point $e_i + e_{i+1} - v$ is in the lower triangle $\Delta = \langle 0, e_i, e_{i+1} \rangle$; thus the upper triangle also has no lattice points other than its vertices. The conclusion is that Π has no lattice points other than its vertices. It follows that Π is a fundamental domain for L acting by translation on $\mathbb{R}$, and therefore e_i, e_{i+1} is a basis of L.

Remark 2.3 Thus for plane lattices, the convexity condition is very strong, and implies that we get a $\mathbb{Z}$ basis of a lattice. This part of the argument fails in dimension ≥ 3.

(II) Since e_{i-1}, e_i is a basis, I can certainly write $e_{i+1} = \alpha e_{i-1} + \beta e_i$ with $\alpha, \beta \in \mathbb{Z}$. On the other hand, for e_i, e_{i+1} to be a basis, e_{i-1} must be expressed as an integral linear combination of them, and therefore $\alpha = \pm 1$. But from the figure, e_i is a positive combination of e_{i-1} and e_{i+1} so that $\alpha = -1$ and the relation is $e_{i-1} + e_{i+1} = \beta e_i$ with $\beta > 0$; if $\beta = 1$ then e_i is in the interior of Newton L, so $\beta \geq 2$. This proves (II).

(III) We have $e_0 + e_2 = a_1 e_1$ where a_1 is as in (II). Thus $e_2 = \frac{1}{r}(a_1, a_1 b - r)$ is in the unit square. Therefore $a_1 b - r \geq 0$; moreover, $a_1 b - r \geq b$ is impossible, since then e_2 would be above e_1, contradicting the construction of e_2 as a point on the boundary of Newton L. Therefore $a_1 = \left\lceil \frac{b}{r} \right\rceil$.

The argument for $a_2, \dots, a_k$ works recursively: write

$$L = (\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_{k+1}) + \mathbb{Z} \cdot e_2.$$

If I take a new copy of $\mathbb{Z}^2$ with basis e_1, e_{k+1} (*not* e_0, e_{k+1}) then since $e_2 = \frac{1}{b}(e_1 + (a_1 b - r) e_{k+1})$, I have $L = \mathbb{Z}^2 + \mathbb{Z}\frac{1}{r_1}(1, b_1)$, where r_1, b_1 are as in (2.2). The Newton polygon of L in the smaller cone spanned by e_1, e_{k+1} is the same as before, so that by the preceding argument, I have $e_1 + e_3 = a_2 e_2$ where $a_2 = \left\lceil \frac{r_1}{b_1} \right\rceil$. The result follows by recursion. Q.E.D.

Example 2.4 Figure 2 illustrates $\mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{7}(1,3)$. The Newton boundary points are

$$e_0 = (0,1), \quad e_1 = \frac{1}{7}(1,3), \quad e_2 = \frac{1}{7}(3,2), \quad e_3 = \frac{1}{7}(5,1), \quad e_4 = (1,0)$$

We have $\frac{7}{3} = [3,2,2]$; then $e_0 + e_2 = 3e_1$, which in the figure means that the Newton boundary is strictly convex at e_1. The coefficients 2 at e_2, e_3 reflect the fact that the boundary is a straight line.

Corollary 2.5 *As in 1.2, write $\frac{1}{r}(1,a)$ for the action of $\mathbb{Z}/r$ on $\mathbb{C}^2$ by $(x,y) \mapsto (\varepsilon x, \varepsilon^a y)$. Then the invariant monomials of the action are generated by*

$$u_0 = x^r, \quad u_1 = x^{r-a}y, \quad u_2, \dots, u_k, \quad u_{k+1} = y^r;$$

these satisfy

$$u_{i-1}u_{i+1} = u_i^{a_i} \quad \text{for } i = 1, \dots, k, \tag{2.3}$$

where the exponents a_i are the entries of $\frac{r}{r-a} = [a_1, \dots, a_k]$.

Note that this calculates the ring of invariant monomials $\mathbb{C}[x,y]^G$ for us: it is generated by the monomials u_i, and the relations (2.3) (while only a subset of all the relations) are already enough to specify all the u_i as rational expressions in u_0, u_1, or in any two consecutive monomials u_i, u_{i+1}.

Thus the morphism $\mathbb{C}^2 \to X \subset \mathbb{C}^{k+2}$ is the quotient of Proposition-Definition 1.1. The relations (2.3) determine the image X set-theoretically; for a full set of generators of the ideal $I(X)$ we also need relations for u_iu_j with $|i-j| \geq 2$.

Since the group acts on $x^\alpha y^\beta$ by $\varepsilon^{\alpha+b\beta}$, the invariant monomials are $x^\alpha y^\beta$ where $\alpha + b\beta \equiv 0 \bmod r$. The exponents form the lattice

$$M = \mathbb{Z}^2 \langle (r,0), (0,r) \rangle + \mathbb{Z} \cdot (r-a, 1),$$

to which Proposition 2.2 applies.

Example 2.6 The calculation comes out in a nice uniform way in two important cases. The first is $\frac{1}{r}(1,1)$, in which the invariant monomials are $u_0 = x^r, u_1 = x^{r-1}y, \dots, u_i = x^{r-i}y^i, \dots, u_{r+1} = y^r$. The relations holding between these are

$$\operatorname{rank} \begin{pmatrix} u_0 & u_1 & \dots & u_{r-1} \\ u_1 & u_2 & \dots & u_r \end{pmatrix} \leq 1.$$

In particular, all the $a_i = 2$, giving $\frac{r}{r-1} = [2, 2, \dots, 2]$ with $r-1$ repetitions of 2.

The other case is $\frac{1}{r}(1, r-1)$. Then the invariant monomials are generated by $u = x^r, v = xy, w = y^r$, with the single relation $uw = v^r$. The continued fraction is $\frac{r}{1} = [r]$.

3 Resolution of singularities

Example 3.1 I now discuss the case $\frac{1}{r}(1,1)$ and its resolution, which is very similar to the quadratic cone of Example 1.2. Here the cyclic group $\mathbb{Z}/r$ acts on $\mathbb{C}^2$ by $(x,y) \mapsto (\varepsilon x, \varepsilon y)$. All the monomials of degree r in x, y are invariant: write

$$u_i = x^{r-i}y^i \quad \text{for } i = 0, \dots, r. \tag{3.1}$$

It is easy to see that the u_i generate $k[x,y]^{\mathbb{Z}/r}$, with relations

$$\operatorname{rank} \begin{pmatrix} u_0 & u_1 & \dots & u_{r-1} \\ u_1 & u_2 & \dots & u_r \end{pmatrix} \le 1 \tag{3.2}$$

(that is, $u_i u_{j+1} = u_{i+1} u_j$ for $0 \le i < j \le r-1$. This is true because

$$\frac{u_{i-1}}{u_i} = \frac{x}{y} \quad \text{for } i = 1, \dots r.$$

Write $X \subset \mathbb{C}^{r+1}$ for the variety defined by these equations. This is a cone over the rational normal curve $\mathbb{P}^1 \cong C_r \subset \mathbb{P}^r$. Here I'm more interested in its resolution. Compare Example 1.2.

One sees from (3.2) that outside the origin, one of u_0 or $u_r \neq 0$, so that the ratio between the two rows of the matrix in (3.2) is well defined as a point of $\mathbb{P}^1$; this ratio is just $x : y$. Introducing this ratio as a new function tells us exactly how to resolve the singularity of X. In fact $\eta = u_1/u_0 = y/x$ is a regular function at any point where $u_0 \neq 0$ and (3.2) gives

$$u_i = u_0 \eta^i \quad \text{for } i = 0, \dots, r.$$

so that X is parametrised by $u_0 = x^r$ and $\eta = u_1/u_0 = y/x$.

More precisely, write Y_0 for a copy of $\mathbb{C}^2$ with coordinates ξ, w, where $\xi = x/y$. Then

$$\sigma_0 \colon (\xi, w) \mapsto (u_0 = \xi^r w, u_1 = \xi^{r-1} w, \dots, u_r = w), \tag{3.3}$$

defines a morphism $\sigma_0 \colon Y_0 \to X$ that restricts to an isomorphism

$$Y_0 \setminus \{\xi\text{-axis}\} \xrightarrow{\cong} X \setminus \{u_0\text{-axis}\},$$

and contracts the ξ-axis $w = 0$ to the origin $(0, \dots, 0) \in X$. The inverse rational map is given by

$$w = u_r, \quad \xi = \frac{u_{r-1}}{u_r} = \dots = \frac{u_0}{u_1}.$$

The "cylinder" resolution of X is $Y = Y_0 \cup Y_1$, where $Y_0 = \mathbb{C}^2$ with coordinates (ξ, w) and $Y_1 = \mathbb{C}^2$ with coordinates (η, u). The glueing between the two pieces where they overlap is given by

$$Y_0 \setminus (\xi = 0) \xrightarrow{\cong} Y_1 \setminus (\eta = 0) \quad \text{by} \quad (\xi, w) \mapsto (\eta = \xi^{-1}, u = w\xi^r). \tag{3.4}$$

The morphism $\sigma\colon Y \to X$ is given by (3.3) on Y_0 and something similar on Y_1.

It is clear from (3.4) that Y has a morphism $p\colon Y \to \mathbb{P}^1$, defined by $(\xi : 1)$ on Y_0 and $(1 : \eta)$ on Y_1, and that it is a $\mathbb{C}^1$ bundle over $\mathbb{P}^1$. The exceptional locus of the morphism $p\colon Y \to X$ is the zero section of the bundle p, defined by $u = w = 0$. The twisting of the $\mathbb{C}^1$ bundle is given by the transition function ξ^r in (3.4).

Theorem 3.2 *As usual, consider the action of $\mathbb{Z}/r$ on $\mathbb{C}^2$ of type $\frac{1}{r}(1,a)$ and the cyclic quotient singularity $X = \mathbb{C}^2/(\mathbb{Z}/r)$ of Corollary 2.5. Write L for the overlattice $L = \mathbb{Z}^2 + \mathbb{Z}\cdot\frac{1}{r}(1,a)$ of $\mathbb{Z}^2$, and*

$$M = \big\{(\alpha,\beta) \bigm| \alpha + a\beta \equiv 0 \ mod\ r\big\} \subset \mathbb{Z}^2$$

for the dual lattice of invariant monomials.

Write $\frac{r}{a} = [b_1, \dots, b_l]$, and let $e_0, e_1, \dots, e_{k+1}$ be as in Figure 2 of Proposition 2.2. For each $i = 0, \dots, k$, let ξ_i, η_i be monomials forming the dual basis of M to e_i, e_{i+1}; that is, such that

$$e_i(\xi_i) = 1, e_i(\eta_i) = 0, \quad e_{i+1}(\xi_i) = 0, e_{i+1}(\eta_i) = 1.$$

Then X has a resolution of singularities $Y \to X$ constructed as follows:

$$Y = Y_0 \cup Y_1 \cup \cdots \cup Y_l, \tag{3.5}$$

where each $Y_i \cong \mathbb{C}^2$ with coordinates ξ_i, η_i.

The glueing $Y_i \cup Y_{i+1}$ in (3.5) and the morphism $Y \to X$ are both determined automatically by the definition of ξ_i, η_i consists of

$$Y_i \setminus (\eta_i = 0) \xrightarrow{\cong} Y_i \setminus (\xi_{i+1} = 0) \quad \textit{defined by} \quad \xi_{i+1} = \eta_i^{-1}, \eta_{i+1} = \xi_i \eta_i^{b_i}.$$

Remark 3.3 Don't confuse the continued fraction of $\frac{r}{r-a}$ used for the calculation of invariant monomials in Corollary 2.5 with that of $\frac{r}{a}$ in Theorem 3.2. The first refers to the Newton polygon of the sublattice M of invariant monomials, whereas the second refers to the dual overlattice L.

Example 3.4 Consider $\frac{1}{7}(1,4)$. By Corollary 2.5, $\frac{7}{7-4} = [3,2,2]$ and the invariant monomials are generated by

$$u_0 = x^7, u_1 = x^3y, u_2 = x^2y^3, u_3 = xy^5, u_4y^7 \tag{3.6}$$

with $u_1^3 = u_0u_2$, etc. This provides the ring of invariants $k[x,y]^{\mathbb{Z}/r}$ and the definition of the quotient variety X.

The resolution of singularities is given by $\frac{7}{4} = [2,4]$ and the consecutive lattice points on the Newton boundary of $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{7}{4}$; these are

$$e_0 = (0,1), e_1 = \frac{1}{7}(1,4), e_2 = \frac{1}{7}(2,1), e_3 = (1,0).$$

Taking dual bases gives

$$\xi_0 = x^7, \eta_0 = y/x^4, \quad \xi_1 = x^4/y, \eta_1 = y^2/x, \quad \xi_2 = x/y^2, \eta_2 = y^7.$$

Thus $Y = Y_0 \cup Y_1 \cup Y_2$, with 3 copies of $\mathbb{C}^2$ glued by

$$\xi_1 = \eta_0^{-1}, \eta_1 = \xi_0\eta_0^2, \quad \xi_2 = \eta_1^{-1}, \eta_2 = \xi_1\eta_1^4.$$

4 Exercises

Exercise 4.1 Check all the assertions in 1.2 concerning the quotient $\frac{1}{2}(1,1)$. (Compare [Ch], Chapter 4, 4.1.)

Exercise 4.2 In the exercise $\frac{1}{7}(1,4)$, verify that the fibres of the quotient morphism $\mathbb{C}^2 \to \mathbb{C}^5$ (defined by $u_0, \dots, u_4$ in (3.6)) are exactly the group orbits.

Exercise 4.3 Prove $k[V]^G$ is f.g.

Exercise 4.4 Verify that the morphism $\pi\colon V \to X = V/G$ constructed in Proposition–Definition 1.1 is

1. well defined;
2. injective;
3. surjective.

Thus π is a set theoretic 1-to-1 correspondence from the orbits of G on V to the points of X (as stated implicity in (1.1)). [Hint: π well defined is clear. For injective, write $O_x = G \cdot x$ for the orbit; if $O_x \cap O_y = \emptyset$ then use the NSS to deduce the existence of a polynomial that is zero on O_x and nonzero at a point of O_y; a suitable combination of these, f say, is nonzero at every point of O_y. Now take a product g^*f over $g \in G$ to obtain a G-invariant function on X separating O_x and O_y. For surjective, use the fact that the extension $k[V]^G \subset k[V]$ is finite, plus generalities on finite maps (see e.g., [UCA], Chapter 4.]

Exercise 4.5 In Example 3.1, prove that the u_i generate $k[x,y]^{\mathbb{Z}/r}$; also, every relation between the monomials is in the ideal generated by (3.1) [Hint: This is combinatorics with monomials. Generators: the only invariant monomial of degree $< n$ is 1; any monomial of degree $\geq n$ is divisible by one or more of the u_i. Relations: the monomials of degree $d \geq 1$ in $u_0, \dots, u_r$ correspond (many-to-one) to monomials of degree nd in x, y; choose a subset that correspond one-to-one (for example, the first monomial in the u_i in lex ordering corresponding to $x^{nd-i}y^i$). Then the relations (3.1) can be used to translate any monomial of degree $d \geq 1$ in $u_0, \dots, u_r$ into one of the chosen set.]

Exercise 4.6 For the quotient singularity $\frac{1}{r}(1, r-1)$ of Example 2.6, show how to write down explicitly r affine pieces $Y_i \cong \mathbb{C}^2$ and glueing maps $Y_i^{(0)} \cong Y_j^{(0)}$ between their dense open pieces so that $Y = \bigcup Y_i$ is a resolution of $X : uw = v^r \subset \mathbb{C}^3$. [Hint: $Y_i := \mathbb{C}^2$ with coordinates λ_i, μ_i where $\lambda_i = x^{n-i}/y^i$ and $\mu_i = y^{i+1}/x^{n-i-1}$.]

Exercise 4.7 For the group action $\frac{1}{r}(1, r-1)$ of Example 2.6, show that every G-cluster $Z \subset \mathbb{C}^2$ is defined by

$$x^{n-i} = \lambda_i y^i, \quad y^{i+1} = \mu_i x^{n-i-1}, \quad xy = \lambda_i \mu_i$$

for some i and some λ_i, μ_i.

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5 Orbifold Hilbert schemes and resolution of toric singularities

Ito and Nakamura introduced the notion of orbifold Hilbert scheme of a quotient by a group orbit, and used it to provide a construction of the minimal resolution $Y \to X$ of the Gorenstein surface quotient singularities $X = \mathbb{C}^2/G$ for any finite subgroup $G \subset \mathrm{SL}(2, \mathbb{Z})$, related to the Gonzalez-Sprinberg–Verdier form of the McKay correspondence. This paper extends the definition of orbifold Hilbert scheme to any[1] toric variety X in terms of the algebra of Weil divisor classes, and shows that it provides a distinguished toric resolution of singularities $Y \to X$ (not necessarily minimal).

I learn how to calculate the Ito–Nakamura Hilbert scheme for certain classes of Abelian quotient singularities. I prove in particular that it provides the minimal resolution of the Hirzebruch–Jung cyclic quotient singularities $\frac{1}{r}(1, a)$ and a distinguished choice of crepant resolutions for Gorenstein 3-fold Abelian quotient singularities (despite the apparently widespread belief among 3-folders that this is impossible because of flops).

5.1 First definitions

I start by reproducing the definition of Ito and Nakamura in a convenient form.

Definition 5.1 G acts linearly on $\mathbb{C}^n$. An *IN ideal* is an ideal $J \subset \mathcal{O}_{\mathbb{C}^n}$ with the two properties:

1. J is invariant under G;
2. the quotient $\mathcal{O}_{\mathbb{C}^n}/J$ is the regular representation of G (that is, isomorphic to $k[G]$ as G-module); in particular, $J \subset \mathcal{O}_{\mathbb{C}^n}$ has colength $N = \operatorname{card} G$.

Equivalently, I can take $J \subset k[\mathbb{C}^n] = k[x_1, ..x_n]$ (more commonly $k[x, y]$, $k[x, y, z]$, etc.).

I assume from now on that G is an Abelian group acting diagonally (it is of course an exceedingly interesting problem to generalise this assumption). Write $\pi\colon \mathbb{C}^n \to X = \mathbb{C}^n/G$ for the quotient map; then $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathbb{C}^n}^G$ is the sheaf of invariants of G in $\mathcal{O}_{\mathbb{C}^n}$.

[1]This is based on an old version, and "any" is certainly exaggerated

Write $G^\vee = \mathrm{Hom}(G, \mathbb{C}^\times)$ for the character group; a character $a \in G^\vee$ is a function $a(g)$ on G with values in the roots of unity, and such that $a(g_1 g_2) = a(g_1)a(g_2)$. It corresponds to a divisorial sheaf $\mathcal{O}_X(a)$ on X, consisting of functions on $\mathbb{C}^n$ that are eigenfunctions of the action of each $g \in G$ with eigenvalue $a(g)$:

$$\mathcal{O}_X(a) = \left\{ f \in \pi_* \mathcal{O}_{\mathbb{C}^n} \;\middle|\; g^*(f) = a(g) f \right\};$$

this is the *eigensheaf* corresponding to a. Although sections of $\mathcal{O}_X(a)$ are not themselves functions on X, the *ratio* of any two nonzero sections of $\mathcal{O}_X(a)$ can be viewed as a rational function on X (just as ratios f_d/g_d of forms of the same degree are rational functions on projective space). It is well known that the sheaves $\mathcal{O}_X(a)$ for $a \in G^\vee$ correspond one-to-one to divisorial sheaves on X up to isomorphism, so that

$$G^\vee = \mathrm{Cl}\, X,$$

where $\mathrm{Cl}\, X$ is the group of Weil divisor classes. An element $g \in G$ is a *quasireflection* (sometimes *unitary reflection*) if it fixes pointwise a hyperplane $\mathbb{C}^{n-1} \subset \mathbb{C}^n$, or in other words, g is conjugate to $\mathrm{diag}(\varepsilon, 1, \dots, 1)$. I assume here, without too much loss of generality, that G has no quasireflections.

Now attend carefully to Condition 2 in Definition 5.1. Because an IN-ideal J is G-invariant, the exact sequence

$$0 \to J \subset \pi_* \mathcal{O}_{\mathbb{C}^n} \to \mathcal{O}/J \to 0$$

splits up as a direct sum of sequences

$$0 \to J(a) \subset \mathcal{O}_X(a) \to (\mathcal{O}/J)(a) \to 0 \quad \text{for } a \in G^\vee;$$

Condition 2 says that each quotient $\mathcal{O}_X(a)/J(a)$ is isomorphic to the eigenspace $k[G]^a \subset k[G]$, so is one dimensional. This proves the following proposition:

Proposition 5.2 *An IN-ideal $J \subset \mathcal{O}_{\mathbb{C}^n}$ is exactly the same thing as a choice of a maximal subsheaf of $\mathcal{O}_X$ submodules*

$$J(a) \subset \mathcal{O}_X(a) \quad \textit{for all } a \in G^\vee$$

such that the product

$$\mathcal{O}_X(a) \cdot J(b) \subset J(a+b) \quad \textit{for all } a, b \in G^\vee.$$

Here a *maximal subsheaf* of $\mathcal{O}_X$ submodules or *maximal submodule* means simply a sheaf of submodules of colength 1. Maximal *proper* subsheaf would be more precise, but I omit the verbose epithet by analogy with maximal ideals: note that $J(0) \subset \mathcal{O}_X = \mathcal{O}_X(0)$ is a maximal ideal. The condition $\mathcal{O}_X(a) \cdot J(b) \subset J(a+b)$ just ensures that the sum $\bigoplus \mathcal{O}_X(a)$ is an ideal.

Now for a single sheaf $\mathcal{F}$, the set of all possible choices of a maximal subsheaf of $\mathcal{F}$ is parametrised by the blowup of $\mathcal{F}$ in X. Assume for simplicity that $\mathcal{F}$ is torsion free and rank 1, so locally isomorphic to an ideal sheaf $\mathcal{F} \subset \mathcal{O}_X$.

Proposition 5.3 *Let $\mathcal{F}$ be a rank 1 torsion free sheaf. Then there are canonical isomorphisms*

$$\mathrm{Quot}(\mathcal{F})_1 = \mathrm{Hilb}(\max, \mathcal{F}) = \mathrm{Bl}_{\mathcal{F}} X,$$

where the first item is the Quot *scheme of quotients of $\mathcal{F}$ isomorphic to $\mathbb{C}$ ($= \mathcal{O}_X/m_x$ for some $x \in X$), the second is the Hilbert scheme of maximal subsheaves of $\mathcal{F}$, and the third is the blowup* $\mathrm{Proj} \bigoplus_{n \geq 0} \mathcal{F}^n$, *where* $\mathcal{F}^n = \mathcal{F}^{\otimes k}/\mathrm{torsion}$.

Proof The first equality is more or less the definition. For the second, suppose that X is affine, and that $s_1, \dots, s_k$ generate $\mathcal{F}$ as $\mathcal{O}_X$ module; let $\mathcal{F}' \subset \mathcal{F}$ be a maximal submodule. Then $\mathcal{F}'$ does not contain all the s_i, so that $s_1 \notin \mathcal{F}'$, say. Then by the codimension 1 condition, $\mathcal{F} = \mathcal{F}' \oplus \mathbb{C}s_1$, and hence

$$\text{for each } j = 2, \dots, k, \quad \text{there exists } a_j \in \mathbb{C} \text{ such that} \quad s_j - a_j s_1 \in \mathcal{F}'.$$

Then obviously $\mathcal{F}' = \langle s_j - a_j s_1 \rangle$. In other words, to specify a point of $\mathrm{Hilb}(\max, \mathcal{F})$ is the same thing as to specify the ratio $s_1 : s_2 : \cdots : s_k$. The blowup has exactly the same interpretation. Q.E.D.

Example 5.4 The Hilbert scheme $\mathrm{IN}(X)$ of Ito–Nakamura ideals can be calculated for the surface cyclic quotient singularities $\frac{1}{r}(1, a)$, using the Hirzebruch–Jung continued fractions, and turns out to be exactly the minimal resolution of singularities. I just do one example $\frac{1}{5}(1, 2)$. (The general case is more-or-less the same, up to spending a lot of time introducing notation.)

The invariant monomials are x^5, x^3y, xy^2, y^5, and these generate the coordinate ring $k[X]$ of the quotient $\mathbb{C}/(\mathbb{Z}/5)$. The nontrivial eigensheaves are generated over $\mathcal{O}_X$ as follows:

$$\mathcal{O}_X(1) = (x, y^3)$$
$$\mathcal{O}_X(2) = (x^2, y)$$

In principle I should also write down the remaining eigensheaves $\mathcal{O}_X(3) = (x^3, xy, y^4)$ and $\mathcal{O}_X(4) = (x^4, x^2y, y^2)$, but these are *inactive* in a sense to be discussed in more detail below: they introduce no new ratios, because

$$\mathcal{O}_X(1) \otimes \mathcal{O}_X(2) \twoheadrightarrow \mathcal{O}_X(3) \quad \text{and} \quad \mathcal{O}_X(2) \otimes \mathcal{O}_X(2) \twoheadrightarrow \mathcal{O}_X(4)$$

are surjective. Now every IN ideal J contains two nontrivial relations

$$p_1x - q_1y^3 \quad \text{and} \quad p_2x^2 - q_2y$$

between the generators of $\mathcal{O}_X(1)$ and $\mathcal{O}_X(2)$. If x, y are not both in $\operatorname{rad} J$, then it is easy to see that

$$J = \left\langle x^5 = a^5, x^2y = a^2b, xy^3 = ab^3, b^3x = ay^3, bx^2 = a^2y \right\rangle$$

for some $a, b \in \mathbb{C}^2 \setminus (0,0)$. This corresponds to a free orbit of $\mathbb{Z}/5$ on $\mathbb{C}^2$. Otherwise, x and y are both vanishingly small at J, and it is then not possible for both of the ratios x/y^3 and x^2/y to be nontrivial. Thus either

$$p_1 \neq 0, q_2 = 0 \quad \text{and} \quad J = \left\langle x - (q_1/p_1)y^3, y^5 \right\rangle$$

or

$$p_1 = 0, q_2 \neq 0 \quad \text{and} \quad J = \left\langle x^5, y - (p_2/q_2)x^2 \right\rangle.$$

This shows that the Hilbert scheme $\mathrm{IN}(X)$ coincides with the minimal resolution. This calculation is more or less the same as Ito–Nakamura's but the assumption that $G \subset \mathrm{SL}(2, \mathbb{C})$ is not relevant.

The set of IN ideals is naturally parametrised by a subvariety $\mathrm{IN}(G)$ of Hilb^G_N, the Hilbert scheme[2] of G-invariant clusters $Z \subset \mathbb{C}^n$ of length $N = |G|$.

From now on, assume for simplicity that $G = \mathbb{Z}/r$ is cyclic with $r = N$, and that the x_i are eigencoordinates with $a_i = \operatorname{wt} x_i$ coprime to r for some

[2]For some reason, the argument has shifted to $\mathbb{C}^n$ for any n. It is known that Hilb^N is then pathological, and can (say) have components of the wrong dimension. We seem to be trying to show that there is a good component birational to $\mathbb{C}^n/G$.

i. Then $\mathrm{IN}(G)$ is birational to the orbit space A/G. More precisely, for $i = 1, \dots, n$, write $x = x_i$ and

$$\mathrm{Main}(x) = \left\{ J \in IN(G) \;\middle|\; \begin{array}{l} \text{the set of pure powers } 1, x, x^2, \dots, x^{r-1} \\ \text{is a vector space basis of } \mathcal{O}_{\mathbb{C}^n}/J \end{array} \right\}$$

Then any $J \in \mathrm{Main}(x)$ contains relations $y = bx^c$ (for each $y = x_j$), so that a minimum set of generators is given by

$$J = \langle x^r = a^r, y = bx^c \rangle$$

(in more detail, $x_i^r = a_i^r$ for the chosen value of i, and $x_j = b_j x_i^{c_{ij}}$ for all $j \neq i$). Thus J is a complete intersection ideal. Every J not contained in the exceptional locus of $\mathrm{IN}(G) \to \mathbb{C}^n/G$ is in $\mathrm{Main}(x_i)$ for some i.

Example 5.5 For the surface cyclic quotient singularities, everything can be calculated using the Hirzebruch–Jung continued fractions. I just do one example $\frac{1}{27}(1,8)$. (The others are more-or-less the same, up to introducing notation).

(a) Using $27/(27-8) = [2,2,4,3]$, we get the invariant monomials:

$$y^{27}, y^{19}z, y^{11}z^2, y^3z^3, yz^{10}, z^{27}$$

(b) Using $27/8 = [4,2,3,2]$, we get the Newton polynomial of weights to be

$(0,27)$

$(1,8)$

$(4,5)$

$(7,2)$

$(17,1)$

$(27,0)$

(c) Each of the junior weights corresponds to a character of G for which the corresponding eigenspace contains two monomials:

$$L(1) : (y, z^{17}), \quad L(2) : (y^2, z^7), \quad L(5) : (y^5, z^4), \quad L(8) : (y^8, z).$$

Therefore every IN ideal must contain relations

$$\begin{aligned}
p_1 y &= q_1 z^{17}\\
p_2 y^2 &= q_2 z^7\\
p_3 y^5 &= q_3 z^4\\
p_4 y^8 &= q_4 z
\end{aligned}$$

with $(p_i, q_i) \neq (0,0)$. The study of ideals $J \in \mathrm{IN}(G)$ thus breaks up into cases according to whether $p_i = 0$ or not.

(d) Case division

Case $p_1 \neq 0$ Then obviously $y = (q_1/p_1)z^{17}$ and $J \in \mathrm{Main}(z)$. Then

$$J = \left\langle z^{27} = b^{27}, y = (q_1/p_1)z \right\rangle .$$

Either case $b = 0$ or $b \neq 0$ is allowed. If $b \neq 0$ then we can rescale to get

$$p_1 = b^{17}, \quad q_1 = a \tag{5.1}$$
$$p_2 = b^7, \quad q_2 = a^2 \tag{5.2}$$
$$p_3 = b^4, \quad q_3 = a^5 \tag{5.3}$$
$$p_4 = b, \quad q^4 = a^8 \tag{5.4}$$

This is the orbit of a point $y = a, z = b \in \mathbb{C}^2$ with $b \neq 0$. If $p_1 \neq 0$ then either $q_1 \neq 0$ or $q_2 = q_3 = q_4 = 0$.

Similarly, $q_4 \neq 0$ if and only if $J \in \mathrm{Main}(y)$. These two cases obviously include all cases with $(y,z) \neq (0,0)$ (and a few more).

Case $p_1 = 0$**,** $p_2 \neq 0$ Then

$$J = \left\langle z^{17} = 0, y^2 = (q_2/p_2)z^7, yz^{10} = 0 \right\rangle$$

Here z^{17} comes from the basic relation (5.1) with $p_1 = 0, q_1 \neq 0$. The first two relations imply that also $y^a \in J$ for some a, so that the ideal is exceptional, and must contain the invariant monomial yz^{10}. In this case, a basis of $\mathcal{O}_{\mathbb{C}^n}/J$ is given by

$$\begin{array}{llllllll}
1 & z & z^2 & \dots & z^9 & & \dots & z^{16} \quad * \\
y & yz & yz^2 & \dots & yz^9 & * & & \\
* & & & & & & &
\end{array}$$

where the $*$s mark the ends corresponding to the 3 generators of J. Note that $y^5 = 0$, so that in this case necessarily $q_3 = q_4 = 0$.

Case $p_1 = p_2 = 0$**,** $p_3 \neq 0$ Then

$$J = \left\langle z^7 = 0, y^5 = (q_3/p_3)z^4, y^3z^3 \right\rangle$$

The generators arise from (5.2), (5.3) and the invariant monomial y^3z^3. A basis of $\mathcal{O}_{\mathbb{C}^n}/J$ is given by

$$\begin{array}{cccccccc}
1 & z & z^2 & z^3 & z^4 & z^5 & z^6 & * \\
y & yz & yz^2 & yz^3 & yz^4 & yz^5 & yz^6 & \\
y^2 & y^2z & y^2z^2 & y^2z^3 & y^2z^4 & y^2z^5 & y^2z^6 & \\
y^3 & y^3z & y^3z^2 & * & & & & \\
y^4 & y^4z & y^4z^2 & & & & & \\
* & & & & & & &
\end{array}$$

In this case necessarily $q_4 = 0$.

Case $p_1 = p_2 = p_3 = 0$**,** $p_4 \neq 0$ Then

$$J = \left\langle z^4 = 0, y^8 = (q_4/p_4)z, y^3z^3 \right\rangle$$

A basis of $\mathcal{O}_{\mathbb{C}^n}/J$ is given by

$$\begin{array}{ccccc}
1 & z & z^2 & z^3 & * \\
y & yz & yz^2 & yz^3 & \\
y^2 & y^2z & y^2z^2 & y^2z^3 & \\
y^3 & y^3z & y^3z^2 & * & \\
y^4 & y^4z & y^4z^2 & & \\
y^5 & y^5z & y^5z^2 & & \\
y^6 & y^6z & y^6z^2 & & \\
y^7 & y^7z & y^7z^2 & & \\
* & & & &
\end{array}$$

It is almost obvious that $\mathrm{IN}(G)$ equals the minimal resolution of $\mathbb{C}^2/G$ in this case.

6 Appendix: open problems on Hilbert schemes and McKay correspondence

This section dates back a couple of years and is not up-to-date.

I have just realised one very simple thing: it is perfectly possible for the Hilbert scheme to determine a unique crepant resolution of $\mathbb{C}^3/G$, even if there are lots of flips. I only explain it in the cyclic case $\frac{1}{r}(a,b,c)$. Draw the usual picture of the junior face of the cube with its junior weights; each junior weight $\frac{1}{r}(d,e,f)$ corresponds to a surface in the resolution which is generically the weighted projective space $\mathbb{P}(d,e,f)$. The general point of $\mathbb{P}(d,e,f)$ is defined by "active" ratios of eigenmonomials (i.e., not a product of simpler ratios). These ratios define an Ito–Nakamura ideal $J \subset \mathcal{O}_{\mathbb{C}^3}$. (i.e. J is G-invariant and $\mathcal{O}/J = k[G]$ is the regular representation).

My observation is the following: two junior weights are joined in the Hilbert scheme resolution if and only if they have a common active ratio. e.g., for $\frac{1}{11}(1,2,8)$ write $A = 731$, $B = 614$, $C = 128$, $D = 245$, $E = 362$ for the junior weights. Then AD is joined by a line with parameter $y^3 : xz^2$ (both monomials have A weight 9 and D weight 12), and BD are joined by a line with parameter $z^2 : xy^2$ (both monomials have B weight 8 and D weight 10).

6.1 Some conjectures on the McKay correspondence for a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$

Notation: $f\colon \mathbb{C}^3 \to \mathbb{C}^3/G = X$ is the quotient map, $Y \to X$ a resolution of singularities, ρ an irreducible representation of G. Set $\mathcal{A} = f_*\mathcal{O}_{\mathbb{C}^3}$, and break it up as an $\mathcal{O}_X[G]$ module as a direct sum $\bigoplus_\rho F_\rho \otimes \rho$, where F_ρ is a sheaf on X, which is a vector bundle of rank $\deg \rho$ outside the singular locus.

(At least in the Abelian case)

1. Define the character sheaves $\mathcal{O}_X(\rho)$ and their birational transform $|D(\rho)|$ on Y (any crepant resolution).

Conjecture 6.1 *The $D(\rho)$ span the movable cone* $\mathrm{Mov}(Y)$.

2. If ρ is of sum type (that is, $\mathcal{O}_X(\rho)$ is the surjective image of $\mathcal{O}_X(\rho_1) \otimes \mathcal{O}_X(\rho_2)$ for some ρ_1, ρ_2 then $D(\rho) = D(\rho_1) + D(\rho_2)$, therefore it is not needed as a generator of $\mathrm{Mov}(Y)$.

Conjecture 6.2 *The edges of* $\mathrm{Mov}(Y)$ *are $D(\rho)$ for ρ of nonsum type.*

3. If Y is a crepant resolution, there is an ample cone $\mathrm{Amp}(Y) \subset \mathrm{Mov}(Y)$.

Conjecture 6.3 *The edges of* $\mathrm{Amp}(Y)$ *are $D(\rho)$ for some subset of nonsum characters. I call these* junior. $\mathrm{Amp}(Y)$ *should be a simplicial cone.*

4.

Conjecture 6.4 *If ρ is nonsum, but not junior for Y then $D(\rho)$ is not nef.*

5.

Conjecture 6.5 *The junior $D(\rho)$ form a $\mathbb{Z}$-basis of $H^2(Y,\mathbb{Z})$. (I guess the $\mathbb{Q}$ statement is obvious.)*

6. How do we get a relation between representations and H^4? I hope to find a way of mapping the sum type and the nonsum but not nef characters to H^2. One possibility is: if $D(\rho) \twoheadleftarrow D(\rho_1) + D(\rho_2)$, make $D(\rho)$ correspond to the intersection class $D(\rho_1) \cap D(\rho_2)$. In order for this to be well defined, we need to know either that ρ_1 and ρ_2 are unique (which seems to be true in small example), or at least that $D(\rho_1) \cap D(\rho_2)$ does not depend on the choice of ρ_1 and ρ_2.

I now have some additional ideas, still based on toric valuations, that I believe should allow us to go directly from the group conjugacy classes to algebraic cohomology classes on any resolution, and to some representations. When there is a crepant resolution, the age corresponds to the codimension, and if we are lucky we should be able to prove that these classes base the cohomology and correspond bijectively to the irreducible representations. This is a much stronger version of the old conjectures, because it is essentially independent of which crepant resolution we choose. Part of the reason for my optimism is the talk of Batyrev and Kontsevich, but I'm very confident that we can beat anything they can do.

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