Ergodic Properties of Markov Processes

Exercises for week 9

Exercise 1 Let x be the autoregressive process on **R** defined by

$$x_{n+1} = \alpha x_n + \xi_n \;,$$

for a sequence of i.i.d. normal random variables $\{\xi_n\}$.

a. Show that $V(x) = |x|^p$ is a Lyapunov function for this system for every value of $p \geq 1$ if and only if $|\alpha| < 1$.
b. Show that this system has a unique invariant measure if $|\alpha| < 1$.
c. Show that this system has no invariant probability measure if $|\alpha| \geq 1$.

Exercise 2 Let x be a biased random walk on the natural numbers **N**. More precisely, for some $p \in (0,1)$, we suppose that its transition probabilities are given by

$$P_{i+1,i} = p \;, \qquad P_{i-1,i} = 1 - p \;,$$

for $i > 0$ and by $P_{1,0} = p$, $P_{0,0} = 1 - p$.

a. For which values of λ and p is $k \mapsto \lambda^k$ a Lyapunov function for x?
b. Compute the invariant measure for this random walk when $p \leq 1/2$.

Exercise 3 Let $F\colon \mathcal{S}^1 \to \mathcal{S}^1$ be an arbitrary continuous map from the circle $\mathcal{S}^1$ to itself and let μ be a probability measure on $\mathcal{S}^1$ which has a continuous density ϱ with respect to the Lebesgue measure. Assume furthermore that $\varrho(x) > 0$ for every $x \in \mathcal{S}^1$. Define a Markov process on $\mathcal{S}^1$ by

$$x_{n+1} = F(x_n) \cdot \xi_n \;,$$

where ξ_n is a sequence of i.i.d. random variables with law μ and the multiplication operation on $\mathcal{S}^1$ is the one obtained by identifying $\mathcal{S}^1$ with the unit circle in **C**.

* **Exercise 4** Let x be the Markov process on $\mathbf{R}_+$ defined by

$$x_{n+1} = \sqrt{x_n \xi_n} \;,$$

for a sequence of i.i.d. random variables $\{\xi_n\}$ that are uniformly distributed in the interval $[1, 2]$. Show that

a. The corresponding transition probabilities are Feller.
b. The function $V(x) = x + \frac{1}{x}$ is a Lyapunov function for this system.
c. Any invariant measure must have as its support the interval $[1, 2]$.

Exercise 5 Let $\{\gamma_n\}$ be any sequence of numbers such that $\sum_n \gamma_n^2 < \infty$. Show that there always exists a positive increasing sequence $\{r_n\}$ with $r_n \to +\infty$ such that $\sum_n r_n^2 \gamma_n^2 < \infty$.

* **Exercise 6** Let $\{r_n\}$ be any increasing sequence of positive numbers such that $\lim_{n\to\infty} r_n = \infty$. Show that the 'hyperball' $\Gamma = \{x \in \ell^2 \mid \sum_{n=1}^{\infty} r_n^2 x_n^2 \leq C\}$ is compact in ℓ^2 for any value of C.

Exercise 7 (Construction of infinite-dimensional Gaussian measures) Let $\{\gamma_n\}$ be as above and let ν_n denote the Gaussian measure on $\mathbf{R}^n$ given by

$$\nu_n = \mathcal{N}(0, \gamma_1^2) \times \mathcal{N}(0, \gamma_2^2) \times \ldots \times \mathcal{N}(0, \gamma_n^2) .$$

Let $\iota_n \colon \mathbf{R}^n \to \ell^2$ be the canonical injection given by

$$\iota_n(x_1, \ldots x_n) = (x_1, \ldots, x_n, 0, \ldots)$$

and define $\mu_n = \iota_n^* \nu_n$. Using the previous two exercises and the Tchebycheff inequality, show that the sequence $\{\mu_n\}$ of probability measures on ℓ^2 is tight.