Ergodic Properties of Markov Processes

Exercises for week 8

Exercise 1 Consider the movement of a king on a chessboard who, at every timestep, chooses one of the possible 8 (or less if he is on the border) moves independently and with equal probabilities. Show that the position of the king forms a Markov process which is reversible with respect to its invariant measure and give and expression for the invariant measure π.

Exercise 2 Let $\mathcal{M}$ be a subset of the set of probability measures on $\mathbf{R}^n$. Show that a sufficient condition for $\mathcal{M}$ to be tight is that there exists a function $F\colon \mathbf{R}^n \to \mathbf{R}_+$ with $\lim_{R\to\infty} \inf\{F(x) \,|\, \|x\| \geq R\} = \infty$ and a constant C such that

$$\int_{\mathbf{R}^n} F(x)\,\mu(dx) < C\;,$$

for every $\mu \in \mathcal{M}$.

Exercise 3 Show that a sequence of delta-measures $\{\delta_{x_n}\}_{n\geq 0}$ on a complete separable metric space $\mathcal{X}$ converges weakly to a delta measure δ_x if and only if the sequence $\{x_n\}$ converges to x. Similarly, show that the sequence is tight if and only if the closure of the set $\{x_n \,|\, n \geq 0\}$ is compact. **Note:** consider first the case $\mathcal{X} = \mathbf{R}$.

Exercise 4 Use the central limit theorem to show that the sequence $\{\mu_n\}$ of measures on $\mathbf{Z}$ given by the laws of the simple random walk at time n is *not* tight.

Exercise 5 Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and define a sequence of measures μ_n by

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{e_k}\;.$$

Show that this sequence of measures is not tight in $\mathcal{H}$. Let $\mathcal{H}'$ be the Hilbert space obtained by completing $\mathcal{H}$ under the norm

$$\|x\|^2 = \sum_{n=1}^{\infty} \frac{\langle e_n, x\rangle}{n}\;.$$

Show that the sequence of measures μ_n viewed as measures on $\mathcal{H}'$ is tight and actually converges weakly to a limit. Give this limit.

Exercise 6 Let $\{\xi_n\}$ be a sequence of i.i.d. random variables with values in the space of continuous functions $\mathcal{C}([0,1],\mathbf{R})$ and such that $\mathbf{E}\sup_{t\in[0,1} |\xi_n(t)|^2 < \infty$. Let x_n be the real-valued Markov process defined so that given x_n, x_{n+1} is the solution at time 1 to the differential equation

$$\frac{dx(t)}{dt} = x(t) - x^3(t) + \xi_n(t)\;,\quad x(0) = x_n\;.$$

Show that this Markov process admits an invariant probability measure.

Hint: Show that there exists a constant C such that $\mathbf{E}(x_{n+1}^2 \,|\, x_n) \leq C + \frac{1}{2}x_n^2$. And conclude that the law of x_n generates a tight family of probability measures on $\mathbf{R}$.

**** Exercise 7 (Wiener measure)** Let $\mathcal{X}$ be the space of continuous functions from $[0,1]$ into **R** and let $\{\xi_n\}_{n\geq 0}$ be a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables. Let $\varphi\colon \mathbf{R} \to [0,1]$ be the function defined by $\varphi(t) = \max\{\min\{t,1\},0\}$ and define a sequence $\{x_n\}$ of $\mathcal{X}$-valued random variables by

$$x_n(t) = \sum_{n=0}^{N-1} \frac{\xi_n}{\sqrt{N}} \varphi(t - \tfrac{n}{N}) \;.$$

Show that the sequence μ_n of measures on $\mathcal{X}$ given by the laws of x_n is tight so that there exists a probability measure $\mathcal{W}$ on $\mathcal{X}$ and a subsequence n_k such that $\mu_{n_k} \to \mathcal{W}$ weakly. Show that one has actually $\mu_n \to \mathcal{W}$ weakly by showing that the law of $(x_{t_1}, \ldots, x_{t_k})$ under μ_n converges to a limiting distribution as $n \to \infty$.

Hint: Remember that the Arzela-Ascoli theorem states that a set $A \subset \mathcal{X}$ is relatively compact if and only if it is bounded and the functions in A are equicontinuous.