Ergodic Properties of Markov Processes

Exercises for week 7

Exercise 1 Given a transition probability P with invariant measure π on a state space $\mathcal{X}$, check that the family of measures $\mathbf{P}_\pi^n$ on $\mathcal{X}^{2n+1}$ given by

$$\int f(x_{-n},\ldots,x_n)\,\mathbf{P}_\pi^n(dx) = \int_{\mathcal{X}}\int_{\mathcal{X}}\cdots\int_{\mathcal{X}} f(x_{-n},\ldots,x_n)\,P(x_{n-1},dx_n)\cdots P(x_{-n},dx_{1-n})\pi(dx_{-n})\;.$$

is consistent. In other words, show that if the function f does not depend on x_{-n} and x_n, then one has $\int f(x_{1-n},\ldots,x_{n-1})\,\mathbf{P}_\pi^n(dx) = \int f(x_{1-n},\ldots,x_{n-1})\,\mathbf{P}_\pi^{n-1}(dx)$.

Exercise 2 Show that a random walk on a group G is reversible if and only if $\bar{P}(g) = \bar{P}(g^{-1})$, where $\bar{P}$ is the probability measure on G that gives the distribution of the 'steps'.

Exercise 3 Let (V,E) be a non-oriented connected graph and let x be a simple random walk on V. This is defined in such a way that if $x_n = v \in V$, then x_{n+1} is equal to one of the adjacent edges to v and all of them have equal probabilities. Guess the invariant measure for this process and show that your guess was correct (and therefore that the process is reversible with respect to this measure).

Exercise 4 Consider a random walk on the positive integers. At each step, the probability of increasing the number by one is $1/3$ and the probability of decreasing it by one is $2/3$. When the process is at 0, it moves to 1 with probability $1/3$ and stays at 0 with probability $2/3$.

1. Compute the invariant measure for this Markov process and show that it is reversible.
2. Let T be the first time when the process first hits 0. Find an expression for $\mathbf{E}(T \mid x_0 = n)$.

* **Exercise 5 (Parrondo's paradox)** Fix a small value ε and consider the following two games between Greg and a professional gambler. In the first game, Greg tosses a biased coin which shows heads with probability $\frac{1}{2} - \varepsilon$. If the coin shows heads, he earns 1 pound from the gambler. If it shows tails, he pays 1 pound to the gambler. In the second game, Greg first counts his money. If it is a multiple of 3, he wins 1 pound from the gambler with probability $\frac{1}{10} - \varepsilon$ and looses 1 pound with probability $\frac{9}{10} + \varepsilon$. If it is not a multiple of three, Greg wins 1 pound with probability $\frac{3}{4} - \varepsilon$ and looses 1 pound with probability $\frac{1}{4} + \varepsilon$.

1. Show that if $\varepsilon = 0$, both games are fair (in the long run).
2. Show that if Greg always plays the first game, he looses money at an average rate of 2ε pence per round.
3. Model the second game by a Markov chain on $\mathbf{Z}_3$ and show that if Greg always plays the second game, he looses money at an average rate of roughly $\frac{294}{169}\varepsilon$ pence per round.
4. Greg proposes to toss a fair coin at the beginning of each round and to play one round of the first game if the coin shows heads and one round of the second game if it shows tails. Of course, the professional gambler accepts. If $\varepsilon = 0.05$, who is smarter: Greg or the gambler? What if $\varepsilon = 0.005$?

Note: The calculations in points 3 and 4 are almsot impossible to do by hand. It is a good exercise to solve them using Maple or Mathematica.

* **Exercise 6 (Metropolis algorithm)** Let (V, E) be a non-oriented connected regular graph (*i.e.* a graph where every vertex has the same number of edges attached to it), let $H \colon V \to \mathbf{R}_+$ be an arbitrary function on the set of vertices, and let $\beta > 0$. Define a Markov process on V as follows. Let's say the process starts at vertex v. Choose uniformly a vertex v' among the vertices such that $vv' \in E$ (in other words such that there exists an edge joining v and v'). If $H(v') \leq H(v)$, go to v'. If $H(v') > H(v)$, go to v' with probability $\exp(-\beta(H(v') - H(v)))$ and stay in v otherwise.

1. Write down the transition probabilities for this process.
2. Guess the invariant measure for this process and show that your guess is correct (and therefore that the process is reversible with respect to this measure).
3. What happens in the limit $\beta \to \infty$? (This is called 'simulated annealing'.)

Note: If you find the exercise as it stands too difficult, choose one particular graph (*e.g.* a square) and solve it there. You should then be able to gess the general answer.