Ergodic Properties of Markov Processes

Exercises for week 6

Exercise 1 Show that a Markov process on a finite state space has a unique invariant measure if and only if there exists one state that can be reached from any other state.

Exercise 2 Let ξ_n be a sequence of i.i.d. random variables that take values $\{0, 1, -1\}$ with equal probabilities. Let $x_n \in \{0, \ldots, 5\}$ be a Markov process given by

$$x_{n+1} = \begin{cases} x_n + \xi_n & \text{if } x \in \{1, \ldots, 4\}, \\ x_n & \text{if } x_n \in \{0, 5\}. \end{cases}$$

Write down the corresponding stochastic matrix and compute $\lim_{n\to\infty} \mathbf{P}(x_n = 0 \,|\, x_0 = j)$ for every j.

Exercise 3 Bill's aunt Martha lives alone and relies very much on her nephew's visits, but Bill doesn't care too much... So every weekend, Bill throws a die. If it shows 1 or 2, he decides to go to pay Martha a visit. Otherwise, he goes out with his mates. However, if Martha hasn't seen Bill for 4 weeks in a row, she complains very badly, so that he feels obliged to go to visit her the following weekend.

How would you model Bill's visit to Martha by a finite-state Markov chain? Using this model, compute how much time elapses on average between two successive visits and how often (on average) Martha has to phone Bill to remind him of his duties.

Exercise 4 Let P be a stochastic matrix on $\mathbf{R}^N$ and fix two states $i \neq j$ in $\{1, \ldots, N\}$. Denote by A_k^{ij} the probability that the process with transition probabilities P starting at k reaches the state i before reaching the state j. Using the Markov property, write down an equation for A_k^{ij}.

Exercise 5 Show that if $\{x_n\}$ is a left-invariant random walk, then $\{x_n^{-1}\}$ is a right-invariant random walk and find its transition probabilities.

Exercise 6 Consider a random walk with transition matrix P on a finite group G and define $\Sigma = \{g \in G \,|\, \bar{P}(g) > 0\}$. Show that P is irreducible if and only if Σ generates G.

Exercise 7 Show that the normalised counting measure $\pi(g) = 1/|G|$ is an invariant measure for every random walk on G.

Exercise 8 Let P be irreducible of period p. Show that, for $n \geq 1$, the period q of P^n is given by $q = p/r$, where r is the greatest common divider between p and n.

* **Exercise 9** Show that the assumption that x is aperiodic is actually not needed in order to prove the law of large numbers.