Ergodic Properties of Markov Processes

Exercises for week 5

* **Exercise 1** Show that the three following conditions are equivalent:

(a) P is irreducible and aperiodic.

(b) P^n is irreducible for every $n \geq 1$.

(c) There exists $n \geq 1$ such that $(P^n)_{ij} > 0$ for every $i, j = 1, \ldots, N$.

Exercise 2 Indicate all the communication classes together with their partial ordering for the stochastic matrix

$$P_1 = \frac{1}{4}\begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 \end{pmatrix}$$

Exercise 3 Let P be the stochastic matrix given by

$$P = \frac{1}{10}\begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 5 & 0 \\ 0 & 3 & 0 & 5 & 0 & 4 \\ 1 & 0 & 6 & 0 & 5 & 0 \\ 0 & 3 & 0 & 5 & 0 & 6 \end{pmatrix},$$

and denote by x a Markov process with transition probabilities given by P.

1. Draw the incidence graph associated to P and classify the states $\{1, \ldots, 6\}$ into communication classes of recurrent states and transient states.
2. Compute $\mathbf{P}(x_3 = 5 \,|\, x_1 = 3)$ and $\mathbf{P}(x_4 = 6 \,|\, x_0 = 3)$.
3. What are all the invariant probability measures for P?

Exercise 4 Let P be the stochastic matrix given by

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \end{pmatrix}$$

What is the period of P? Compute its Perron-Frobenius vector π. How does the value $\mathbf{P}(x_n = 4 \,|\, x_0 = 1)$ behave for large values of n?

* **Exercise 5** Let P be an arbitrary stochastic matrix. Show that the set of all normalised positive vectors π such that $P\pi = \pi$ consists of all convex linear combinations of the Perron-Frobenius vectors of the restrictions of P to its recurrent communication classes.