# Ergodic Properties of Markov Processes

Exercises for week 4

**Exercise 1** Let $x$ be a Markov process with t.p. $P$, let $A \subset \mathcal{X}$ be such that $P(a, A) > 0$ for every $a \in \mathcal{X}$, and define $T = \inf\{n \,|\, x_n \notin A\}$. Define a process $y$ by

$$\mathbf{P}(y_1 \in A_1, \ldots, y_n \in A_n) = \mathbf{P}(x_1 \in A_1, \ldots, x_n \in A_n \,|\, T > n)\;.$$

Show that the process $y$ is again Markov and give an expression for its transition probabilities.

**Exercise 2** Find the Perron-Frobenius vectors of the matrices

$$P_1 = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\;, \quad P_2 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 1 & 0 & 2 \end{pmatrix}\;.$$

(Normalise them so that their entries sum up to $1$.)

**Exercise 3** Draw the incidence graph for the stochastic matrix

$$P = \frac{1}{10}\begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 3 & 0 & 5 \\ 8 & 0 & 6 & 0 \\ 0 & 7 & 0 & 5 \end{pmatrix}\;.$$

Is it irreducible? Aperiodic?

**Exercise 4** Give an example of a stochastic matrix of period $2$ for which there are two possible choices of $A_1, A_2$. Argue that no irreducible matrix with this property can be found.

**Exercise 5** Prove that if there exists $j$ such that $P_{jj} \neq 0$, then the matrix is aperiodic.

**Exercise 6** Let $P$ be an irreducible stochastic matrix. Show that $P$ is aperiodic if and only if $P^n$ is irreducible for every $n \geq 1$.

* **Exercise 7** Show that if $P$ is irreducible and of period $p$, there is only one possible choice of $A_1, \ldots, A_p$. Show also that the period of $P$ is the smallest integer $n$ such that $P^n$ is aperiodic. What is the period of $P^n$ for general $n$?