Solutions to Problem Sheet 2:

(1) (i): Show first $D(H^2) \subset (H+i)^{-1} D(H)$. Assume $\psi \in D(H^2)$.

We need to find $\varphi \in D(H)$ such that

$$\psi = (H+i)^{-1} \varphi .$$

Since $\psi \in D(H^2) \subset D(H)$, we can multiply by $H+i$ on both sides

and find $\varphi = H\psi + i\psi \in D(H)$.

For showing $(H+i)^{-1} D(H) \subset D(H^2)$, consider $\varphi \in D(H)$. Now,

$$H^2 (H+i)^{-1} \varphi = H (H+i-i)(H+i)^{-1} \varphi = H\varphi - i H (H+i)^{-1} \varphi \in L^2,$$

so indeed $(H+i)^{-1} \varphi \in D(H^2)$.

(ii): For $\psi \in D(H)$, we have $\psi = (H+i)^{-1} \underbrace{(H+i)\psi}_{\in L^2} \in \mathrm{Ran}\,(H+i)^{-1}$,

so $D(H) \subset \mathrm{Ran}\,(H+i)^{-1}$.

For $\psi \in \mathrm{Ran}\,(H+i)^{-1}$, there exists (by definition) $\varphi \in L^2$ with

$\psi = (H+i)^{-1}$, and thus $H\psi = (H+i-i)(H+i)^{-1} \varphi = \varphi - i (H+i)^{-1} \varphi \in L^2$,

so $\psi \in D(H)$.

(iii): T has dense range $\Rightarrow$ $\forall \varphi \in L^2 \; \exists \psi \in L^2$ with $\| \varphi - T\psi \|_{L^2} < \varepsilon$

$D(H)$ is dense $\Rightarrow$ $\forall \psi \in L^2 \; \exists f \in D(H)$ with $\| \psi - f \|_{L^2} < \varepsilon$.

Thus $\| \varphi - \underbrace{Tf}_{\in TD(H)} \| \leq \| \varphi - T\psi \| + \| T\psi - Tf \| \leq \varepsilon + \|T\| \underbrace{\|\psi - f\|}_{< \varepsilon}$

$\leq (1 + \|T\|) \varepsilon \Rightarrow$ claim. ∎

(2) a) Let $p_t : C([0,T],\mathbb{R}) \to \mathbb{R}$, $q \mapsto q_t$ be a point evaluation.

Then, by Thm 4.3, we have for $A \subset \mathbb{R}$ measurable:

$$W^x(1_A(p_t(q))) = W^x(q_t \in A) = \frac{1}{(2\pi t)^{1/2}} \int e^{-\frac{1}{2t}|x-\cancel{y}|^2} 1_A(\cancel{x}) d\cancel{x}.$$

Approximating a bounded function $F: \mathbb{R} \to \mathbb{R}$ by step functions, i.e.

$F(x) = \sum_{n=1}^{\infty} 1_{A_n}(x) \beta_n$, we find $W^x(F(p_t(q))) = \frac{1}{(2\pi t)^{1/2}} \int e^{-\frac{1}{2t}|x|^2} F(x) dx$.

So the image of W^x under p_t is Gaussian.

Now write an arbitrary linear functional $\phi: C([0,T],\mathbb{R}) \to \mathbb{R}$ as

$\phi(q) = \sum_{n=0}^{\infty} \beta_n p_{t_n}(q)$ as stated in the examples sheet. Then the image of

W^x under $\phi_N(q) = \sum_{n=0}^{N} \beta_n p_{t_n}(q)$ is Gaussian as the sum of Gaussians,

and its limit ϕ is also Gaussian.

b) $\underbrace{\frac{1}{(2\pi t)^{1/2}} \int e^{-\frac{1}{2t}|x-y|^2} y\, dy}_{W^y(q_t)=} = x$ $\boxed{\text{mean.}}$

$$W^y(q_t^2) - W^y(q_t)^2 = \frac{1}{(2\pi t)^{1/2}} \int e^{-\frac{1}{2t}|x-y|^2} (\cancel{y} - \overset{\text{mean}}{m(x)})^2 dy = \frac{1}{(2\pi t)^{1/2}} \int e^{-\frac{1}{2t}|x-y|^2} (x-y)^2 dy$$

$$= \frac{1}{(2\pi t)^{1/2}} \int e^{-\frac{1}{2t}\cancel{x}^2} y^2 dy = t. \quad \boxed{\text{variance}}$$

covariance: Assume $s<t$: $W^0(q_s q_t) = \int \frac{1}{(2\pi s)^{1/2}} e^{-\frac{1}{2s}|y|^2} y \cdot$

$$\cdots \underbrace{\int \frac{1}{(2\pi(t-s))^{1/2}} e^{-\frac{1}{2(t-s)}(y-z)^2} z \, dz}\, dy$$

$$= \int \frac{1}{(2\pi s)^{1/2}} e^{-\frac{1}{2s}|y|^2} y \cdot y \, dy = s.$$

Assume $t<s$, same calculation gives $W^0(q_s q_t) = t$.

So: $W^0(q_s q_t) = \min\{s,t\}$.

c) (i): Clearly $W^\circ(\frac{1}{\sqrt{b}} q_{bt}) = W^\circ(q_t) = 0$. Now,

$$W^\circ\left(\frac{1}{\sqrt{b}} q_{bs} \frac{1}{\sqrt{b}} q_{bt}\right) = \int \frac{1}{(2\pi s b)^{\frac{1}{2}}} e^{-\frac{1}{bs}x^2} \times \frac{1}{(2\pi(t-s)b)^{\frac{1}{2}}} e^{-\frac{1}{b(t-s)}(y-x)^2} y\, dx\, dy$$

$\frac{x}{\sqrt{b}} = \tilde{x}, \frac{y}{\sqrt{b}} = \tilde{y} \Rightarrow dx = \sqrt{b}\, d\tilde{x}$ etc

$$= \int \frac{1}{(2\pi s)^{\frac{1}{2}}} e^{-\frac{1}{s}x^2} \times \frac{1}{(2\pi(t-s))^{\frac{1}{2}}} e^{-\frac{1}{(t-s)}(y-x)^2} y\, dx\, dy$$

$$= W^\circ(q_s q_t).$$

Since mean and covariance determine a Gaussian process, $\frac{1}{\sqrt{b}} q_{bt} \overset{d}{\sim} q_t$.

(ii): As above, we compute mean and covariance:

$$W^\circ(q_{T-t} - q_T) = W^\circ(q_t) = 0$$

$$W^\circ((q_{T-t} - q_T)(q_{T-s} - q_T)) = W^\circ(q_{T-t} q_{T-s}) + W^\circ(q_T^2) -$$

$$- W^\circ(q_{T-t} q_T) - W^\circ(q_{T-s} q_T) =$$

$$= \min(T-t, T-s) + T - \underbrace{\min(T-t, T)}_{=T-t} - \underbrace{\min(T-s, T)}_{=T-s}$$

$$= \min(T-t, T-s) - T + t + s = T - \max(t,s) - T + t + s$$

$$= \min(t,s).$$

(iii): $W^\circ(t q_{\frac{1}{t}}) = W^\circ(q_t) = 0$;

$$W^\circ(t q_{\frac{1}{t}} s q_{\frac{1}{s}}) = ts \min\left(\frac{1}{t}, \frac{1}{s}\right) = \frac{ts}{\max(t,s)} = \min(t,s). \quad \blacksquare$$

<u>Remark</u>: We also have to check that all processes above are indeed Gaussian, but this is obvious since sums and scalings of Gaussians are Gaussian

(3) a) $W^\circ\left(\int_0^t q_s^2\,ds\right) = \int_0^t ds \int q_s^2 \, dW^\circ(q) =$

$= \int_0^t s\,ds = \frac{1}{2} t^2.$

b) $W^\circ\left[\left(\int_0^t q_s\,ds\right)^2\right] = \int_0^t ds \int_0^t dr \; W^\circ(q_s q_r) =$

$= \int_0^t ds \int_0^t dr \; \min\{s,r\} = \int_0^t ds \left[\int_0^s dr \; r + \int_s^t dr \; s\right] =$

$= \int_0^t ds \left(\frac{1}{2} s^2 + s(t-s)\right) = \int_0^t \left(ts - \frac{1}{2}s^2\right) ds =$

$= \frac{1}{2} t^3 - \frac{1}{6} t^3 = \frac{1}{3} t^3.$

c) $W\left[\left(\int_0^t f'(s) q_s\,ds\right)^2\right] = \int_0^t ds\, f'(s) \int_0^t dr\, f'(r) \overbrace{W(q_s q_r)}^{=\min(s,r)}$

$= \int_0^t ds\, f'(s) \left(\int_0^s f'(r)\, r\, dr + s \int_s^t f'(r)\, dr \right]$

$= \int_0^t ds\, f'(s) \left[\underline{f(s)s} - \cancel{f(0)0} - \int_0^s f(r)\,dr + s\left(f(t) - \underline{f(s)}\right)\right]$

$= \int_0^t ds\, f'(s) \left[s f(t) - \int_0^s f(r)\,dr \right]$

$= f(t)^2 t - f(t) \int_0^t f(r)\,dr - \int_0^t ds\, f(s) f(t) + \int_0^t ds\, f(s)^2$

$= f(t)\left[f(t) t - 2\int_0^t f(s)\,ds\right] + \int ds\, f(s)^2 .$

(4) For f, g bounded, $n \in \mathbb{N}$ we have

$$\left\langle f, \left(e^{-\frac{t}{n}H_0} e^{-\frac{t}{n}V}\right)^n g \right\rangle_{L^2(\mathbb{R}^d)} = \int f(x)\, W^x\left(e^{-\frac{t}{n}\sum_{j=1}^{n} V(q_{jt/n})}\, g(q_t)\right) dx$$

The LHS above converges to $\langle f, e^{-tH} g\rangle$ by the Trotter formula.

The RHS: Since $t \mapsto q_t$ and V are cts, $t \mapsto V(q_t)$ is cts.

So, $\frac{t}{n}\sum_{j=1}^{n} V(q_{jt/n}) \xrightarrow{n\to\infty} \int_0^t V(q_s)\,ds$ as a Riemann sum, for each $q \in C(\mathbb{R}, \mathbb{R}^d)$.

Now, $\left| e^{-\frac{t}{n}\sum_{j=1}^{n} V(q_{tj/n})}\, g(q_t)\right| \leq e^{t \sup_x |V(x)|} \sup_x |g|$,

so $W^x\left(e^{-\frac{t}{n}\sum_{j=1}^{n} V(q_{tj/n})}\, g(q_t)\right) \xrightarrow{n\to\infty} W^x\left(e^{-\int_0^t V(q_s)ds}\, g(q_t)\right)$

by <u>dominated convergence</u>, and

$$\int dx\, f(x)\, W^x\left(e^{-\frac{t}{n}\sum_{j=1}^{n} V(q_{tj/n})}\, g(q_t)\right) \longrightarrow \int dx\, f(x)\, W^x\left(e^{-\int_0^t V(q_s)ds}\, g(q_t)\right)$$

again by dominated convergence.

(5) a) $V(x) = \frac{1}{|x|}$: $\boxed{d=1}$ $\Rightarrow$ $\int_{-1}^{1} V(x)\,dx = \infty$ $\Rightarrow$ $\notin \mathcal{K}(\mathbb{R}),\ \notin \mathcal{K}_\pm(\mathbb{R})$.

$\boxed{d=2}$ $\Rightarrow$ $\int 1_{\{|x-y|\leq r\}} \ln(|x-y|) \frac{1}{|y|}\,dy$

$$= \int_{|y|\leq r} \ln(|y|) \frac{1}{|x+y|}\,dy \leq \int_{|y|\leq r} \ln(|y|)\frac{1}{|y|}\,dy \xrightarrow{r\to 0} 0$$

consider the derivative!

$\Rightarrow \frac{1}{|x|} \in \mathcal{K}(\mathbb{R}^2) \subset \mathcal{K}_\pm(\mathbb{R}^2)$

$\boxed{d=3}$ as above,

$$\int_{|y|\leq r} \frac{1}{|x+y|}\frac{1}{|y|}\,dy \leq \int_{|y|\leq r} \frac{1}{|y|^2}\,dy = \int_{-r}^{r} \rho^2 \frac{1}{\rho^2}\,d\rho\; 4\pi^2 \xrightarrow{r\to 0} 0.$$

$\Rightarrow \frac{1}{|x|} \in \mathcal{K}(\mathbb{R}^3) \subset \mathcal{K}_\pm(\mathbb{R}^3)$

b) $V(x) = e^{|x|^2} \Rightarrow \sup_x \int 1_{\{|x-y|\le r\}} e^{|y|^2} \frac{1}{|x-y|} dy = \infty$

$\Rightarrow V \notin \mathcal{K}(\mathbb{R}^3)$. But for $K \subset \mathbb{R}^3$ compact,

$$\sup_{x \in K} \int 1_{\{|x-y|\le r\}} e^{|y|^2} \frac{1}{|x-y|} dy \le \sup_{y \in K} e^{|y|^2} \int 1_{\{|y|\le r\}} \frac{1}{|y|} dy$$

$\xrightarrow{r\to 0} 0 \Rightarrow V \in \mathcal{K}_{\pm}(\mathbb{R}^3)$ since $V_+ \in \mathcal{K}_{loc}(\mathbb{R}^3)$, $V_- = 0$.

c) $\sup_x \int 1_{\{|x-y|\le r\}} \underbrace{f(y)}_{\le 1} \frac{1}{|x-y|} dy \le \int 1_{\{|y|\le r\}} \frac{1}{|y|} dy \xrightarrow{r\to 0} 0$

$\Rightarrow V \in \mathcal{K}(\mathbb{R}^3)$

d) $V = \sum_{n=1}^{\infty} 1_{\{|x-x_j|\le 1\}} \frac{j}{|x-x_j|} \in \mathcal{K}_{loc}(\mathbb{R}^d)$ as above in a),

But $\sup_{x \in \mathbb{R}^3} \int 1_{\{|x-y|\le r\}} V(y) \frac{1}{|x-y|} dy \overset{\text{for } r \le 1}{\ge}$

$$\ge \int 1_{\{|x-y|\le r\}} \frac{j}{|y-x_j|^2} dy = 4\pi j \int \frac{r^2}{r^2} dr = 4\pi j \quad \forall j$$

$\Rightarrow \sup \ldots = \infty \Rightarrow V \notin \mathcal{K}(\mathbb{R}^d)$.

⑥ By Cor. 4.12, for $V \in \mathcal{K}(\mathbb{R}^d)$ we have $\sup_x W^x(e^{+\int^t V(q_s) ds}) < \infty$,

so $\int_0^t V(q_s) ds < \infty$ almost surely. For $V \in \mathcal{K}_{loc}$, we know $V 1_K \in \mathcal{K}$

for all compact sets K. So, with $K_n = \{x \in \mathbb{R}^d : x \le n\}$,

$$A := \{q : |\int V(q_s) ds| = \infty\} \subset \bigcup_{n \in \mathbb{N}} \{q : |\int V(q_s) 1_{K_n}(q_s) ds| = \infty\} \cup B,$$

with $B = \bigcap_{n \in \mathbb{N}} \{q : \sup_{0 \le s \le t} |q_s| \ge n\}$.

Now $W^x(B) = 0$ by properties of BM, $W^x(\{q : |\int V(q_s) 1_{K_n}(q_s) ds| = \infty\}) = 0$

since $V \in \mathcal{K}_{loc} \Rightarrow W^x(A) = 0$.

The second statement follows by

$$\sup_x W^x\left(e^{-\int_0^t V(q_s)ds}\right) \overset{C.S.}{\leq} \sup_x \Bigg[\underbrace{W^x\Big(\underbrace{e^{-2\int_0^t V_+(q_s)ds}}_{\leq 1}\Big)}_{\substack{\leq 1 \text{ since} \\ W^x \text{ is probab. measure}}} \underbrace{W^x\left(e^{+2\int_0^t V_-(q_s)ds}\right)}_{< \infty \text{ since } V_- \in \mathcal{K}}\Bigg]^{\frac{1}{2}}$$

$< \infty$. ∎

(7) $$P_t[P_s f] = W^x\left(e^{-\int_0^t V(q_r)dr}(P_s f)(q_t)\right) =$$

$$= W^x\left(e^{-\int_0^t V(q_r)dr} W^{q_t}\left(e^{-\int_0^s V(q_r)dr} f(q_s)\right)\right) \overset{\text{Thm 4.4}}{=}$$

$$= W^x\left(e^{-\int_0^t V(q_r)dr} W^x\left(e^{-\int_0^s V(q_{r+t})dr} f(q_{s+t}) \,\Big|\, \mathcal{F}_{\{t\}}\right)\right) \overset{\text{Markov}}{=}$$

$$= W^x\Big(\underbrace{e^{-\int_0^t V(q_r)dr}}_{\mathcal{F}_{[0,t]}\text{-measurable}} W^x\left(e^{-\int_t^{t+s} V(q_r)dr} f(q_{t+s}) \,\Big|\, \mathcal{F}_{[0,t]}\right)\Big) \overset{\substack{\text{Property (ii) of} \\ \text{Cond. exp.}}}{=}$$

$$= W^x\left(e^{-\int_0^t V(q_r)dr} e^{-\int_t^{t+s} V(q_r)dr} f(q_{t+s})\right) =$$

$$= W^x\left(e^{-\int_0^{t+s} V(q_r)dr} f(q_{t+s})\right) = P_{t+s} f(x)$$ ∎