Solutions for Problem Sheet 1

(1) $\omega = \sqrt{\frac{1}{0.91\cdot 10^{-30}}} \approx 1.05\cdot 10^{15}$; $\sqrt{\frac{m\omega}{\hbar}} \approx \sqrt{\frac{0.91\cdot 10^{-30}\cdot 1.05\cdot 10^{15}}{1.05\cdot 10^{-34}}} \approx$

$$\approx \sqrt{0.9\cdot 10^{19}} = 3\cdot 10^{9}$$

$\Rightarrow$ 1 length unit $\approx \frac{1}{3\cdot 10^{9}}\, m \approx 3.3\cdot 10^{-10}\, m$

1 time unit $= 0.95\cdot 10^{-15}$ sec.

(2) a) $\partial_{x_i}\psi(x) = -\alpha \frac{x_i}{|x|}\psi(x)$, $\partial_{x_i}^2 \psi(x) = \left(-\alpha\frac{1}{|x|} + \alpha\frac{x_i^2}{|x|^3} + \alpha^2 \frac{x_i^2}{|x|^2}\right)\psi(x)$

$$\Rightarrow \Delta\psi(x) = \left(-\frac{3\alpha}{|x|} + \alpha\frac{|x|^2}{|x|^3} + \alpha^2\frac{|x|^2}{|x|^2}\right)\psi(x) = \left(-\frac{2\alpha}{|x|} + \alpha^2\right)\psi(x)$$

$$\Rightarrow -\frac{1}{2}\Delta\psi(x) - \frac{1}{|x|}\psi(x) = \left(\frac{\alpha}{|x|} - \frac{1}{|x|}\right)\psi(x) + \alpha^2\psi(x) \overset{!}{=} E\psi(x)$$

$$\Rightarrow \alpha = 1,\quad E = -1.$$

b) Position: $\langle x_i\rangle_\psi = 0$ since x_i is odd and ψ is even. Put $x_i = r\cos\theta$.

$$\langle x_i^2\rangle_\psi = \frac{1}{\pi}\int_{\mathbb{R}^3} x_i^2 e^{-2|x|}\,dx = \frac{1}{\pi}\int_0^\infty dr\, r^4 e^{-2r}\int_0^{2\pi} d\phi\int_0^\pi d\theta\, \cos^2\theta\sin\theta, \quad x = \cos\theta$$

$$= \frac{2\pi}{\pi}\int_0^\infty dr\, r^4 e^{-2r}\int_0^\pi d\theta \cos^2\theta\sin\theta \qquad \left[\int_0^\pi \cos^2\theta\sin\theta\, d\theta = 2\int_0^{\pi/2}\cos\theta^2\sin\theta\, d\theta = 2\cdot\int_0^1 x^2 dx = \frac{2}{3}\right]$$

$$= \frac{4}{3}\int_0^\infty dr\, r^4 e^{-2r} \overset{IBP}{=} -\frac{4}{3}\int_0^\infty dr\,(r^4)'\left(-\frac{1}{2}e^{-2r}\right) = \frac{8}{3}\int r^3 e^{-2r}dr \overset{!}{=} \ldots = 1$$

Momentum: $\langle p_i\rangle_\psi = 0$ as before. Put $p_i = r\cos\theta$, as before radial part are $\frac{1}{2}$.

$$i\hbar\,\partial_{x_i}\psi(x) = \frac{x_i}{|x|}e^{-|x|}\, i\hbar$$

$$\Rightarrow \langle p_i^2\rangle_\psi = \langle p_i\psi, p_i\psi\rangle = \frac{\hbar^2}{\pi}\int \frac{x_i^2}{|x|^2}e^{-2|x|}dx = \frac{\hbar^2}{\pi}\int_0^\infty dr\,\frac{r^4}{r^2}e^{-2r}\int_0^{2\pi} d\phi\int_0^\pi \cos^2\theta\sin\theta\, d\theta =$$

$$\frac{\hbar^2}{\pi}\frac{4\pi}{3}\int_0^\infty dr\, r^2 e^{-2r} = \frac{\hbar^2}{3}.$$

So, $(\Delta x_i)_\psi = 1$, $(\Delta p_i)_\psi = \frac{\hbar}{\sqrt{3}} \Rightarrow \Delta x_i \Delta p_i = \frac{\hbar}{\sqrt{3}} \geq \frac{\hbar}{2}$.

c) $\langle \psi, |x|\psi\rangle = \frac{4\pi}{\pi}\int r^3 e^{-2r}\, dr = 4\cdot\frac{3}{8} = \frac{3}{2}$.

$$\frac{1}{(2\pi)^3}\langle\hat\psi, |p|\hat\psi\rangle = \frac{1}{(2\pi)^3}\frac{1}{\pi}4\pi\int_0^\infty \frac{64\pi^2}{(1+r^2)^4}\cdot r^3\, dr$$

$$= \frac{32}{\pi}\int_0^\infty \frac{r^3}{(1+r^2)^4}\, dr = \frac{32}{\pi}\cdot\frac{1}{12} = \frac{8}{3\pi}.$$

d) distance $\sim \frac{3}{4}\cdot 10^{-10}$ m, speed $\sim \frac{8}{10}\cdot 2.2\times 10^6 \frac{m}{s} \approx 1.86\cdot 10^6 \frac{m}{s}$

$\approx$ Speed of light: $\approx 3\cdot 10^8 \frac{m}{s} \approx 100 \times$ speed of el.

Speed of sound $\approx 340 \frac{m}{s} \ll$ speed of electron.

(3) a) Assume $H\psi = E\psi$. Then

$$E\langle\psi,\psi\rangle = \langle\psi, E\psi\rangle = \langle\psi, H\psi\rangle \overset{\text{symm}}{=} \langle H\psi,\psi\rangle = \langle E\psi,\psi\rangle = \bar{E}\langle\psi,\psi\rangle \Rightarrow E = \bar{E}.$$

b) Assume $H\psi_1 = E_1\psi_1$, $H\psi_2 = E_2\psi_2$. Then

$$\bar{E}_1\langle\psi_1,\psi_2\rangle = \langle E_1\psi_1,\psi_2\rangle = \langle H\psi_1,\psi_2\rangle = \langle\psi_1, H\psi_2\rangle = \langle\psi_1, E_2\psi_2\rangle = E_2\langle\psi_1,\psi_2\rangle$$

$$\Rightarrow 0 = (\bar{E}_1 - E_2)\langle\psi_1,\psi_2\rangle \overset{a)}{=} \underbrace{(E_1 - E_2)}_{\neq 0 \text{ assumption}}\langle\psi_1,\psi_2\rangle \Rightarrow \langle\psi_1,\psi_2\rangle = 0.$$

c) Guess: $\langle H\rangle_{1,2} = \frac{1}{2}(\langle H\rangle_1 + \langle H\rangle_2)$; $(\Delta H)_{\psi_{1,2}} = ?$

Compute: $\langle H\rangle_{1,2} = \frac{1}{2}\langle\psi_1+\psi_2, H(\psi_1+\psi_2)\rangle = \frac{1}{2}\langle\psi_1, H\psi_1\rangle + \frac{1}{2}\langle\psi_2, H\psi_2\rangle +$

$$+\frac{1}{2}(\langle\psi_1, H\psi_2\rangle + \langle\psi_2, H\psi_1\rangle) = \frac{1}{2}\langle H\rangle_1 + \frac{1}{2}\langle H\rangle_2 ..$$

$$(\Delta H)_{1,2}^2 = \langle H^2 \rangle_{1,2} - \langle H \rangle_{1,2}^2 =$$

$$= \frac{1}{2}\langle H^2\rangle_1 + \frac{1}{2}\langle H^2\rangle_2 - \left(\frac{1}{2}\left(\langle H\rangle_1 + \langle H\rangle_2\right)\right)^2$$

$$= \frac{1}{2}E_1^2 + \frac{1}{2}E_2^2 - \frac{1}{4}(E_1+E_2)^2 = \frac{1}{4}E_1^2 + \frac{1}{4}E_2^2 - \frac{1}{2}E_1E_2$$

$$= \frac{1}{4}(E_1 - E_2)^2 = \frac{1}{4}\left(\langle H\rangle_1 - \langle H\rangle_2\right)^2$$

$$\Rightarrow (\Delta H)_{1,2} = \frac{1}{2}\left|\langle H\rangle_1 - \langle H\rangle_2\right|.$$

4) a) $[A^2, B] = A^2B - BA^2 = AAB - ABA + ABA - BAA = A[A,B] + [A,B]A.$

b) $[H, X_j] = \frac{1}{2m}[P_j^2, X_j] = \frac{1}{2m}\left(P_j \cdot \frac{\hbar}{i} + \frac{\hbar}{i}P_j\right) = \frac{-i}{m}\hbar P_j$

$[H, P_j] = [V, P_j] = i\hbar\, \partial_j V.$

c) Heisenberg equation: $\frac{d}{dt}A(t) = \frac{i}{\hbar}[H, A(t)].$

$\Rightarrow \frac{d}{dt}P_j(t) = \frac{i}{\hbar}[H, P_j] = -\partial_j V(t)$

i.e. $\frac{d}{dt}\langle \psi(t), P_j \psi(t)\rangle = -\langle \psi(t), \partial_j V \psi(t)\rangle.$

$\frac{d}{dt}X_j(t) = \frac{1}{m}P_j(t).$

5) a) By Proposition 2.7, $(H+i\lambda)^{-1}$ is bounded for $\lambda \in \mathbb{R}, \lambda \neq 0$, the same for $(H-i\lambda)^{-1}$, with $\|(H+i\lambda)^{-1}\| \leq \frac{1}{\lambda}$

b) For $x \in \mathbb{R}$, we have

$$\frac{\lambda^2}{2}\left(\frac{1}{x+i\lambda} + \frac{1}{x-i\lambda}\right) - x = \frac{\lambda^2}{2}\frac{2x}{x^2+\lambda^2} - x = x\left(\frac{\lambda^2}{x^2+\lambda^2} - 1\right) =$$

$$= x \frac{x^2}{x^2+\lambda^2} = x^2\left(\frac{1}{2}\left(\frac{1}{x+i\lambda}+\frac{1}{x-i\lambda}\right)\right).$$

So, $\|(H_\lambda - H)\psi\| = \frac{1}{2}\|(H+i\lambda)^{-1}H^2\psi + (H+i\lambda)^{-1}H^2\psi\| \leq \frac{1}{\lambda}\|H^2\psi\| \xrightarrow{\lambda\to\infty} 0.$

c) $n,m \in \mathbb{N}$:

$$e^{-iH_n} - e^{-iH_m} \overset{\text{FTC}}{=} \int_0^1 \frac{d}{ds}\left(e^{isH_m} e^{i(1-s)H_n}\right) ds$$

$$= \int_0^1 e^{isH_m}(H_m - H_n) e^{i(1-s)H_n}\, ds \quad = (*)$$

now H_λ is self-adjoint, since $\langle \psi, (H+i\lambda)^{-1}\varphi\rangle \overset{\text{e.g. Neumann series}}{=} \langle (H-i\lambda)^{-1}\psi, \varphi\rangle$,

and H_λ has both $(H+i\lambda)^{-1}$ and $(H-i\lambda)^{-1}$ in it.

Thus, for $\psi \in D(H^2)$,

$$\|(e^{-iH_n} - e^{-iH_m})\psi\| \underset{[H_n,H_m]=0}{\leq} \int_0^1 \underbrace{\|e^{isH_m}\|}_{=1}\, \underbrace{\|e^{i(1-s)H_n}\|}_{=1}\, \|(H_n - H_m)\psi\|\, ds \leq \left(\frac{1}{n}+\frac{1}{m}\right) \xrightarrow{n,m\to\infty} 0.$$

6 a) In general, for self-adjoint H, we have

$$\psi \in D(H) \iff \exists (\psi_n) \in D(H) \text{ with } \|\psi_n - \psi\| \xrightarrow{n\to\infty} 0,\ \sup_n \|H\psi_n\| < \infty.$$

Here, put $\phi_n = \sum_{k=1}^{n} \alpha_k \psi_k$.

Now, $\sup_n \|H\phi_n\|^2 = \sup_n \|\sum_{k=1}^{N} \alpha_k \underbrace{H\psi_k}_{=E_k\psi_k}\|^2 = \sup_n \left\langle \sum_{k=1}^{N} \alpha_k E_k \psi_k, \sum_{k=1}^{N} \alpha_k E_k \psi_k\right\rangle$

$$= \sup_n \sum_{\substack{k=1\\ j=1}}^{n} \bar\alpha_j \alpha_k \bar E_j E_k \underbrace{\langle \psi_j, \psi_k\rangle}_{=\delta_{jk}} = \sup_n \sum_{k=1}^{n} |\alpha_k|^2 |E_k|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2 |E_k|^2.$$

b) Define $\psi(t) = \sum_{n=1}^{\infty} \alpha_n e^{-iE_n t}\psi_n$. Then, $\psi(0) = \phi$, and

$$\frac{d}{dt}\psi(t) = \lim_{s\to 0} \frac{1}{s}\cdot\left(\psi(t+s)-\psi(t)\right) =$$

Taylor, $\sigma \in [0,s]$

$$= \lim_{s\to 0}\frac{1}{s}\lim_{n\to\infty}\sum_{k=1}^{n} \alpha_k \left(e^{-iE_n s}-1\right)e^{-iE_n t}\psi_n =$$

$$= \lim_{s\to 0}\frac{1}{s}\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_k\left(-iE_k s - \frac{1}{2}E_k^2 s^2 e^{iE_k\sigma}\right)e^{-iE_k t}\psi_k =$$

$$= \lim_{s\to 0}\frac{1}{s}\left(\sum_{k=1}^{\infty}(-i\alpha_k E_k s)e^{-iE_k t}\psi_k - \frac{1}{2}s^2 \underbrace{\sum_{k=1}^{\infty}E_k^2\alpha_k e^{iE_k(\sigma+t)}\psi_k}_{\in L^2 \text{ by assumption.}}\right)$$

$$= -i\sum_{k=1}^{\infty}E_k\,\alpha_k e^{-iE_k t}\psi_k = -i\,H\psi(t).$$

∎

7 $$E_{quant} - E_{class} = \frac{1}{2}\left(\langle P^2+X^2\rangle_{\psi_0} - \langle P\rangle^2_{\psi_0} - \langle X\rangle^2_{\psi_0}\right) =$$

$$= \frac{1}{2}\left(\left\langle P^2 - \langle P\rangle^2_{\psi_0}\right\rangle_{\psi_0} + \left\langle X^2 - \langle X\rangle^2_{\psi_0}\right\rangle_{\psi_0}\right)$$

$$= \frac{1}{2}\left(\left\langle \left(P-\langle P\rangle_{\psi_0}\right)^2\right\rangle_{\psi_0} + \left\langle\left(X-\langle X\rangle_{\psi_0}\right)^2\right\rangle_{\psi_0}\right)$$

$$= \frac{1}{2}(\Delta P)^2_{\psi_0} + \frac{1}{2}(\Delta X)^2_{\psi_0}.$$

Since $(\Delta P)_{\psi_0}(\Delta X)_{\psi_0} \geq \frac{1}{2}$, we have $(\Delta P)^2_{\psi_0} + (\Delta X)^2_{\psi_0} > 0$.

Minimizing $f(x,y) = x^2+y^2$ under the constraint $xy \geq \frac{1}{2}$ gives $x = y = \frac{1}{\sqrt{2}}$. Thus $E_q - E_{class} \geq \frac{1}{2}$. For the ground state $\psi_0 = e^{-\frac{1}{2}|x|^2}\frac{1}{(\pi)^{1/4}}$, this is minimal.

(8) a) $\psi_1(x)\psi_2(y)$ is an eigenfunction of H:

$$H\,\psi_1(x)\psi_2(y) = -\tfrac{1}{2}(\partial_x^2+\partial_y^2)(\psi_1(x)\psi_2(y)) + (V(x)+V(y))\,\psi_1(x)\psi_2(y)$$

$$= \left(-\tfrac{1}{2}\partial_x^2\psi_1(x) + V(x)\psi_1(x)\right)\psi_2(y) + \left(-\tfrac{1}{2}\partial_y^2\psi_2(y) + V(y)\psi_2(y)\right)\psi_1(x)$$

$$= E_1\psi_1\psi_2 + E_2\psi_1\psi_2 = (E_1+E_2)\psi_1\psi_2.$$

b) Let ψ_n be the n-th oscillator eigenfunction. We have

$$H\psi_n = (n+\tfrac{1}{2})\psi_n.$$

Claim: $\{\psi_n(x)\psi_m(y) : n,m\in\mathbb{N}_0\}$ are all the eigenfunctions.

The eigenvalues are then $\{1+n : n\in\mathbb{N}\}$.

Proof: By a), the above are eigenfunctions, and the eigenvalues of these are as stated. To show that we found all of them, let us prove that $(x,y)\mapsto\psi_n(x)\psi_m(y)$ is a basis of $L^2(\mathbb{R}^2)$. Indeed, consider $f(x,y)\in L^2$. Then for each $\varepsilon>0$ there are functions $g_n(x,y) = 1_{A_n}(x)\cdot 1_{B_n}(y)$, $n\in\mathbb{N}$, such that with $h_N = \sum_{n=1}^{N} g_n(x,y)$, $\|h_N - f\|_{L^2} < \varepsilon$. Now for each g_n, there are ψ_j such that for $\varepsilon'>0$,

$$\|\underbrace{\sum_{j=1}^{m}\psi_j}_{\Psi_m} - g_n\|_{L^2(\mathbb{R})} < \varepsilon'.\ \text{Now}$$

$$\| h_N(x,y) - \sum_{n=1}^{N}\Psi_{m_n}(x)\Psi_{\tilde m_N}(y)\| \le \sum_{n=1}^{N}\|g_n(x,y) - \Psi_m(x)\Psi_{\tilde m}(y)\|.$$

Finally, $\int(1_A(x)1_A(y) - \Psi_m(x)\Psi_{\tilde m}(y))^2\,dx\,dy =$ (since $(a+b)^2 \le 2a^2+2b^2$)

$$= \int\left(1_A(x)1_B(y) - \Psi_m(x)1_B(y) + \Psi_m(x)1_B(y) - \Psi_m(x)\Psi_{\tilde m}(y)\right)^2 \le$$

$$\le 2\int(1_A(x) - \Psi_m(x))^2 1_B(y)\,dx\,dy + 2\int\Psi_m^2(x)\,(1_B(y) - \Psi_{\tilde m}(y))^2\,dx\,dy$$

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$$= 2\| 1_A - \varphi_m \|_{L^2}^2 + 2\| 1_B - \varphi_{\tilde{m}} \|_{L^2}^2 \le 4\varepsilon'.$$

Thus $\| h_N - \varphi_m \varphi_{\tilde{m}} \| \le \varepsilon + 4N\varepsilon' \xrightarrow{\varepsilon' \to 0} \varepsilon$. This is true for all ε, shows the claim.

c) For the eigenvalue $N \in \mathbb{N}$, we need $\varphi_n \varphi_m$ with $n+m = N$.

so the dimension for eigenvalue N is N.

In d dimensions, the dimension for eigenvalue N is

$$\# \{ n_1, n_2, \dots, n_N \in \mathbb{N}_0 : \sum_{j=1}^{N} n_j = N \}.$$

e.g. $N=3$ gives $\frac{N(N+1)}{2}$;