MA4A7 Quantum Mechanics: Basic Principles and Probabilistic Methods

Problem Sheet 2

Volker Betz March, 2012.

---

1. **Dense domains**
   Assume that H is self-adjoint with dense domain. Show that he domain of H^2 is also dense. Do this in three steps:
   (i) Show that $D(H^2) = (H + \mathrm{i})^{-1} D(H)$.
   (ii) Show that $D(H) = \mathrm{Ran(H + i)}^{-1}$, where Ran is the range of an operator, i.e. the set of vectors that can appear as images when the operator is applied to some vector in its domain.
   (iii) Show that for any bounded operator T with dense range, the set $TD(H)$ is dense.
   Now put everything together to prove the claim.

2. **Brownian motion**
   A measure μ on a (possibly infinite-dimensional) metric vector space X is called Gaussian if for any continuous linear functional $\phi : X \to \mathbb{R}$, the image of μ under ϕ is Gaussian. Explicitly, this means that
   $$\int F(\phi(x))\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi v}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2v}(y-m)^2} F(y) \, \mathrm{d}y$$
   for all bounded functions $F : \mathbb{R} \to \mathbb{R}$, with some mean value $m \in \mathbb{R}$ and some variance $v \in \mathbb{R}$ which of course depend on ϕ.
   a) Show that Brownian motion is a Gaussian measure on $C([0, T], \mathbb{R})$, with the topology of uniform convergence. You may use the fact that in this topology, all linear functionals can be approximated by point evaluations $p_t : C([0, T], \mathbb{R}) \to \mathbb{R}, q \mapsto q(t) \equiv q_t$; and the fact that limits of Gaussian measures on $\mathbb{R}$ are also Gaussian.
   b) Compute the mean and the variance of q_t, and compute the covariance $\mathcal{W}^0(q_s q_t)$. Since a Gaussian measure is fully characterized by its mean and covariance, any measure with the same mean and covariance is also Brownian motion.
   c) Using the previous results, show that Brownian motion starting at 0 is invariant under the following transformations on $C([0, T], \mathbb{R})$:
      (i) Scaling: $q_t \mapsto \frac{1}{\sqrt{b}} q_{bt}$,
      (ii) Time reversal: $q_t \mapsto q_{T-t} - q_T$, for $0 \leqslant t \leqslant T$.
      (iii) Time inversion $q_t \mapsto t q_{1/t}$.

3. **Brownian integrals**
   Compute $\mathcal{W}^0(F)$ for the following functions on $C([0, t], \mathbb{R})$:
   a) $F(q) = \int_0^t q_s^2 \, \mathrm{d}s$,
   b) $F(q) = \left(\int_0^t q_s \, \mathrm{d}s \right)^2$,
   c) $F(q) = \left(\int_0^t f'(s) q_s \, \mathrm{d}s \right)^2$.

4. **The simple version of the Feynman-Kac formula**
   Prove the following statement:
   Let $V : \mathbb{R} \to \mathbb{R}$ be a bounded, continuous function, and $H_0 = -\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Prove that for bounded $f, g : \mathbb{R} \to \mathbb{R}$,
   $$\langle f, \mathrm{e}^{-tH} g \rangle_{L^2(\mathbb{R})} = \int f(x) \mathcal{W}^x \left(\mathrm{e}^{-\int_0^t V(q_s) \, \mathrm{d}s} g(q_t) \right) \mathrm{d}x.$$
   Hint: By the Trotter formula,
   $$\mathrm{e}^{-t(H_0+V)} g = \lim_{n \to \infty} (\mathrm{e}^{-\frac{t}{n} H_0} \, \mathrm{e}^{-\frac{t}{n} V})^n g \qquad \text{in } L^2(\mathbb{R}).$$
   By the definition of Brownian motion, the right hand side (before taking the limit) is a Brownian integral. Now use dominated convergence, show that the right hand side converges to the claimed expression as $n \to \infty$.

5. **Kato-class and Kato-decomposable potentials**
   Which of the following potentials are Kato-class, which ones are Kato-decomposable?
   a) $V(x) = -\frac{1}{|x|}$ in dimensions $d = 1, 2, 3$.
   b) $V(x) = \exp(+|x|^2)$ in dimension $d = 3$.
   c) $V(x) = 1$ if each component of x is rational and $V(x) = 0$ otherwise, in dimension $d = 3$.
   d) $V(x) = \sum_{n=1}^{\infty} 1_{\{|x-x_j|<1\}} \frac{j}{|x-x_j|}$, with $x_j = (0, 0, j)$ in $d = 3$.

6. Prove the following statement: If V is Kato-decomposable, then for $\mathcal{W}^x$-almost all Brownian motion paths q_t, the integral $\int_0^t V(q_s)\,\mathrm{d}s$ is finite, and

$$\sup_x \mathbb{W}^x \left(\mathrm{e}^{-\int_0^t V(q_s)\,\mathrm{d}s} \right) < \infty.$$

7. Let V be Kato-decomposable. Use the Markov property of Brownian motion to show that the family of operators P_t with

$$(P_t f)(x) = \mathcal{W}^x \left(\mathrm{e}^{-\int_0^t V(q_s)\,\mathrm{d}s} f(q_t) \right)$$

   is a semigroup on the space of bounded functions. I.e. that $P_s P_t f = P_{t+s} f$ for all bounded f. It is then easy to see (by approximation) that this property is true for all $f \in L^p$, for any $p \in [1, \infty]$.