PDE for Finance: Sheet 8 solutions.

1 a) Collect the terms u_j^n and u_j^{n+1} on the two sides:

$$u_j^{n+1} - \mu(1-\theta)\left(u_{j-1}^{n+1} + u_{j+1}^{n+1} - 2u_j^{n+1}\right) = u_j^n + \mu\theta\left(u_{j-1}^n + u_{j+1}^n - 2u_j^n\right)$$

or, with $\underline{u}^n = (u_1^n, \dots u_J^n)$:

$$\left(1 - \mu(1-\theta)A\right)\underline{u}^{n+1} = \left(1 + \mu\theta A\right)\underline{u}^n,$$

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & & & 0 \\ 1 & -2 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & & \end{pmatrix}.$$

b) We know that A has eigenvectors $\underline{v_k}$ with $(v_k)_j = \sin\left(k\pi \frac{j}{J+1}\right)$

and eigenvalues $\lambda_k = 2\left(-1 + \cos\left(\frac{k}{J+1}\pi\right)\right)$.

So, $\left(1 - \mu(1-\theta)A\right)\underline{v_k} = \left(1 - 2\mu(1-\theta)\left(-1 + \cos\left(\frac{k}{J+1}\pi\right)\right)\right)\underline{v_k}$,

$\Rightarrow 1 - \mu(1-\theta)A$ has eigenvalues $1 + \underbrace{2\mu(1-\theta)\left(1 - \cos\left(\frac{k}{J+1}\pi\right)\right)}_{\geq 0} > 1$

and is therefore invertible.

$$\Rightarrow \underline{u}^{n+1} = \left(1 + \mu(1-\theta)A\right)^{-1}\left(1 + \mu\theta A\right)\underline{u}^n,$$

and by the same reasoning as above (i.e.: apply to $\underline{v_k}$),

$\left(1 + \mu(1-\theta)A\right)^{-1}\left(1 + \mu\theta A\right)$ has eigenvalues

$$\tilde{\lambda}_k = \frac{1 - 2\mu\theta\left(1 - \cos\left(\frac{k}{J+1}\pi\right)\right)}{1 + 2\mu(1-\theta)\left(1 - \cos\left(\frac{k}{J+1}\pi\right)\right)}$$

We need $|\tilde{\lambda}_k| < 1$. Since clearly $\tilde{\lambda}_k < +1$, we just need $\tilde{\lambda}_k > -1$

$$\Leftrightarrow \quad 1 - 2\mu\theta\left(1-\cos\left(\tfrac{k}{J+1}\pi\right)\right) > -1 - 2\mu(1-\theta)\left(1-\cos\left(\tfrac{k}{J+1}\pi\right)\right)$$

$$\Leftrightarrow \quad 2 + 2\mu\underbrace{\left(1-\cos\tfrac{k}{J+1}\pi\right)}_{\text{can be just below 2 in the worst case}} - 4\mu\theta\left(1-\cos\tfrac{k}{J+1}\pi\right) > 0$$

$$\Rightarrow \quad 2 + 4\mu - 8\theta\mu > 0 \text{ for } \underline{\text{all}}\ \mu \overset{\text{large } \mu}{\Rightarrow} \theta \text{ needs to be} \le \tfrac{1}{2}.$$

c) We expand $u(x,t)$ around $(x_j, t_{n+\frac12})$:

$$u(x_j, t_{n+1}) = u(x_j, t_{n+\frac12}) + \frac{h}{2}\partial_t u(x_j, t_{n+\frac12}) + \frac{\left(\frac{h}{2}\right)^2}{2}\partial_t^2 u(x_j, t_{n+\frac12}) + \frac{\left(\frac{h}{2}\right)^3}{6}\partial_t^3 u(x_j, t_{n+\frac12}) + \mathcal{O}(h^4)$$

$$u(x_j, t_n) = u(x_j, t_{n+\frac12}) - \frac{h}{2}\partial_t u(x_j, t_{n+\frac12}) + \frac{\left(\frac{h}{2}\right)^2}{2}\partial_t^2 u(x_j, t_{n+\frac12}) - \frac{\left(\frac{h}{2}\right)^3}{6}\partial_t^3 u(x_j, t_{n+\frac12}) + \mathcal{O}(h^4)$$

$$\Rightarrow \frac{1}{h}\left(u(x_j, t_{n+1}) - u(x_j, t_n)\right) = \partial_t u(x_j, t_{n+\frac12}) + \frac{h^2}{24}\partial_t^3 u(x_j, t_{n+\frac12}) + \mathcal{O}(h^3). \quad (1)$$

For the spatial derivatives:

$$u(x_{j\pm1}, t_n) = u(x_j, t_n) \pm h_x \partial_x u(x_j, t_n) + \frac{h_x^2}{2}\partial_x^2 u(x_j, t_n) \pm \frac{h_x^3}{3!}\partial_x^3 u(x_j, t_n) + \frac{h_x^4}{4!}\partial_x^4 u(x_j, t_n) \pm \frac{h_x^5}{5!}\partial_x^5 u(x_j, t_n) + \frac{h_x^6}{6!}\partial_x^6 u(x_j, t_n) + \dots$$

$$\Rightarrow u(x_{j+1}, t_n) + u(x_{j-1}, t_n) - 2u(x_j, t_n) = h_x^2 \partial_x^2 u(x_j, t_n) + \frac{2h_x^4}{4!}\partial_x^4 u(x_j, t_n) + \frac{2h_x^6}{6!}\partial_x^6 u(x_j, t_n) + \mathcal{O}(h_x^8) \quad (*)$$

The same for $u(x_{j\pm1}, t_{n+1})$.

Now expand all terms in addition with respect to t:

$$\overset{\text{at } t_{n+1}}{(*)} = h_x^2 \partial_x^2 u(x_j, t_{n+\frac12}) + \frac{2}{4!} h_x^4 \partial_x^4 u(x_j, t_{n+\frac12}) + \frac{2}{6!} h_x^6 \partial_x^6 u(x_j, t_{n+\frac12})$$

$$+ \frac{h}{2}\left(h_x^2 \partial_t \partial_x^2 u(x_j, t_{n+\frac12}) + \frac{2}{4!} h_x^4 \partial_t \partial_x^4 u(x_j, t_{n+\frac12}) + \dots\right)$$

$$+ \left(\frac{h}{2}\right)^2 \frac{1}{2}\left(\partial_t^2 \partial_x^2 u(x_j, t_{n+\frac12}) + \dots\right)$$

$\overset{\text{at } t_n}{(*)}$ is the same but with $-\frac{h}{2}(\dots)$ in the middle line.

So, $$\theta\, \overset{\text{at } t_n}{(*)} + (1-\theta)\overset{\text{at } t_{n+1}}{(*)} = h_x^2 \partial_x^2 u(x_j, t_{n+\frac12}) + \frac{2}{4!} h_x^4 \partial_x^4 u(x_j, t_{n+\frac12})$$

$$+ \frac{2}{6!} h_x^6 \partial_x^6 u(x_j, t_{n+\frac12}) +$$

$$+ \frac{h}{2}(1-\theta-\theta)\left(h_x^2 \partial_t \partial_x^2 u(x_j, t_{n+\frac12}) + \frac{2}{4!} h_x^4 \partial_t \partial_x^4 u(x_j, t_{n+\frac12}) + \dots\right)$$

$$+ \frac{h^2}{8} h_x^2 \partial_t^2 \partial_x u^2 (x_j, t_{n+\frac12})$$

Multiply this with $-\frac{\sigma^2}{2h_x^2}$ and add to (1) to get

$$T(x_j, h_{n+\frac12}) = \underbrace{\partial_t u(x_j, t_{n+\frac12})}_{\text{cancel (PDE)}} + \frac{h^2}{24} \partial_t^3 u(x_j, t_{n+\frac12}) +$$

$$- \frac{\sigma^2}{2}\left[\partial_x^2 u(x_j, t_{n+\frac12}) + \frac{2}{4!} h_x^2 \partial_x^4 u(x_j, t_{n+\frac12}) + O(h_x^4)\right]$$

$$- \frac{h}{2}\left[(1-2\theta)\, \partial_t \partial_x^2 u(x_j, t_{n+\frac12}) + \frac{2}{4!} h_x^2 \partial_t \partial_x^2 u(x_j, t_{n+\frac12}) + \dots\right]$$

$$- \frac{h^2}{8} \partial_t \partial_x^2 u(x_j, t_{n+\frac12})$$

d) For $\theta = \frac12$, this is <u>second order</u> in h and h_x, otherwise first order.

$\boxed{2}$ We expand everything around $u(x_j, t_n)$:

$$\frac{1}{h}\left(u(x_j, t_{n+1}) - u(x_j, t_n)\right) = \frac{1}{h}\left(u(x_j, t_n) + h\,\partial_t u(x_j, t_n) + \frac{h^2}{2}\partial_t^2 u(x_j, t_n) + \ldots - u(x_j, t_n)\right)$$

$$A := \frac{1}{\delta x_j}\left(u(x_{j+1}, t_n) - u(x_j, t_n)\right) = \frac{1}{\delta x_j}\Big[\underbrace{u(x_j, t_n)}_{\text{cancels}} + \delta x_j\,\partial_x u(x_j, t_n) + (\delta x_j)^2 \frac{1}{2}\partial_x^2 u(x_j, t_n)$$

$$+ \frac{1}{3!}(\delta x_j)^3 \partial_x^3 u(x_j, t_n) + \frac{1}{4!}(\delta x_j)^4 \partial_x^4 u(x_j, t_n) + \ldots - \underbrace{u(x_j, t_n)}_{\text{cancels}}\Big]$$

$$B := \frac{1}{\delta x_{j-1}}\left(u(x_j, t_n) - u(x_{j-1}, t_n)\right) = \frac{1}{\delta x_{j-1}}\Big(\underbrace{u(x_j, t_n) - u(x_j, t_n)}_{=0} + (\delta x_{j-1})\,\partial_x u(x_j, t_n)$$

$$- (\delta x_j)^2 \frac{1}{2}\partial_x^2 u(x_j, t_n) + \frac{1}{3!}(\delta x_j)^3 \partial_x^3 u(x_j, t_n) - \frac{1}{4!}(\delta x_j)^4 \partial_x^4 u(x_j, t_n) + \ldots\Big)$$

the $\partial_x u$-terms cancel!

$$\Rightarrow T_j^n = \frac{1}{h}\left(u(x_j, t_{n+1}) - u(x_j, t_n)\right) - \frac{2}{\delta x_{j+1} + \delta x_j}(A - B) \overset{!}{=}$$

$$= \underbrace{\partial_x u(x_j, t_n)}_{\text{cancels}\ldots} + \frac{h}{2}\partial_x^2 u(x_j, t_n) + \ldots - \underbrace{\frac{2}{\delta x_{j+1} + \delta x_j}\left(\partial_x^2 u(x_j, t_n)\left(\tfrac{1}{2}(\delta x_{j+1} - \delta x_j)\right)\right)}_{\ldots\text{ with this one}}$$

$$+ \frac{2}{\delta x_{j+1} + \delta x_j}\left(\frac{1}{3!}\underbrace{\left[(\delta x_j)^2 - (\delta x_{j-1})^2\right]}_{=(\delta x_j - \delta x_{j-1})(\delta x_j + \delta x_{j-1})} \partial_x^3 u(x_j, t_n)\right)$$

$$+ \frac{2}{\delta x_{j+1} + \delta x_j}\left(\frac{1}{4!}\left((\delta x_j)^3 + (\delta x_{j-1})^3\right)\partial_x^4 u(x_j, t_n)\right) + \ldots$$

$$= \frac{h}{2}\partial_x u(x_j, t_n) - \frac{1}{3}(\delta x_j - \delta x_{j-1})\partial_x^3 u(x_j, t_n) - \frac{1}{12}\left((\delta x_j)^2 + (\delta x_{j-1})^2 - \delta x_j \delta x_{j-1}\right)\partial_x^4 u(x_j, t_n)$$

$+ \ldots$ as claimed.

For equidistant grid points, $\delta x_j = \delta x_{j-1}$ and the truncation error is second order in the spatial grid points.