PDE for Finance, Solutions Sheet 6

$\boxed{1}$ a) The assumption that is silently made is that buying and selling stock/bond is much faster than their price fluctuations. Under this assumption (which may not always be justified!), the value of the portfolio is constant on the time scale it takes to sell the bond and buy stock, or vice versa. Thus,

$$A_s p_s + B_s q_s = \text{const.} \qquad (*)$$

By the same assumption, p_s and q_s are individually constant on the relevant time scale. Differentiating $(*)$ then gives

$$A_s' p_s + B_s' q_s = 0,$$ which is what was claimed.

b)
$$\alpha(s) = \frac{\text{wealth in stock}}{\text{total wealth}} = \frac{A_s p_s}{y_s}$$

$$1-\alpha(s) = \frac{\text{wealth in bond}}{\text{total wealth}} = \frac{B_s q_s}{y_s}$$

So,

$$dy_s = A_s\, dp_s + B_s\, dq_s = \alpha(s)\frac{y_s}{p_s}\, dp_s + (1-\alpha(s))\frac{y_s}{q_s}\, dq_s$$

$$= \alpha(s)\frac{y_s}{p_s}\left(\mu p_s\, ds + \sigma p_s\, dW_s\right) + (1-\alpha(s))\frac{y_s}{q_s}\, r q_s\, ds$$

$$= \alpha(s)\, y_s\left(\mu\, ds + \sigma\, dW_s\right) + (1-\alpha(s))\, y_s\, r\, ds.$$

This only works so nicely since <u>both</u> dp_s and dq_s are proportional to p_s and q_s, respectively! In particular, it is important that the stock price is modelled by geometric BM.

$\boxed{2}$ a) The HJB equation is the one given in the lecture (general case), with final condition h (instead of g) and no running payoff.

In the notation there, we have

$$f(y_s, \alpha_s) = \alpha(s)\, y_s\, \mu + (1-\alpha(s))\, y_s\, r$$

$$\sigma(y_s, \alpha_s) = \alpha(s)\, y_s\, \sigma .$$

So the HJB-equation is

$$\partial_t u + \max_\alpha \left((xr + \alpha(\mu - r)x)\, \partial_x u + \tfrac{1}{2} x^2 \sigma^2 \alpha^2 \partial_x^2 u \right) = 0$$

with final condition $u(x,T) = h(x)$.

The maximal α can be computed by differentiation:

$$(\mu - r)x\, \partial_x u + \sigma^2 x^2 \alpha\, \partial_x^2 u = 0$$

$$\Rightarrow \quad \alpha^* = -\frac{(\mu - r)\, \partial_x u}{\sigma^2 x\, \partial_x^2 u} .$$

This gives the final HJB-equation

$$\partial_t u + xr\, \partial_x u - \frac{(\mu - r)^2}{\sigma^2} \frac{(\partial_x u)^2}{\partial_x^2 u} + \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} \frac{(\partial_x u)^2}{\partial_x^2 u} = 0$$

or, simplified and with $\lambda^2 = \frac{(\mu - r)^2}{\sigma^2}$:

$$\partial_t u + xr\, \partial_x u + \frac{1}{2} \lambda^2 \frac{(\partial_x u)^2}{\partial_x^2 u} = 0. \qquad (*)$$

This is a very non-linear PDE!

b) Since $u(x,T) = x^\gamma$, we are tempted to try $u(x,t) = g(t)\, x^\gamma$.

Indeed, this works! Putting it into $(*)$, we find

$$\partial_t u = g'(t)\, x^\gamma ,$$

$$r x \partial_x u = r x \gamma x^{\gamma-1} g(t) = g(t)\, r \gamma x^\gamma$$

$$\left.\begin{aligned} (\partial_x u)^2 &= \gamma^2 x^{2\gamma-2} g(t)^2 \\ \partial_x^2 u &= \gamma(\gamma-1) x^{\gamma-2} g(t) \end{aligned}\right\} \Rightarrow \frac{(\partial_x u)^2}{\partial_x^2 u} = \frac{\gamma}{\gamma-1} x^\gamma g(t)$$

So $(*)$ becomes (after dividing by x^γ):

$$g'(t) + r\gamma g(t) + \frac{1}{2}\lambda^2 \frac{\gamma}{\gamma-1} g(t) = 0$$

$$\Leftrightarrow \quad g'(t) = -\left(r\gamma + \frac{\lambda^2 \gamma}{2(\gamma-1)}\right) g(t) \quad ; \quad g(T) = 1 .$$

$$\text{So,} \quad g(t) = \exp\left(-\left(r\gamma + \frac{\lambda^2\gamma}{\gamma-1}\right)(t-T)\right)$$

$$\Rightarrow \quad u(x,t) = x \exp\left(-\left(r\gamma + \frac{\lambda^2\gamma}{\gamma-1}\right)(t-T)\right)$$

$$\alpha^*(x,t) = -\frac{(\mu-r)}{\sigma^2 x} \frac{\partial_x u}{\partial_x^2 u} = -\frac{\mu-r}{\sigma^2 x} \frac{\gamma x^{\gamma-1} g(t)}{\gamma(\gamma-1) x^{\gamma-2} g(t)} =$$

$$= -\frac{\mu-r}{\sigma^2} \frac{1}{\gamma-1} \quad \text{indep. of } x,t \,!$$

This makes sense as long as $0 \le \alpha^* \le 1$, thus we need $\mu > r$ (as $\gamma < 1$),

and $\frac{\mu-r}{\sigma^2} \le 1-\gamma$.

If $\mu < r$, there is no point taking the risk associated with the stock, and the optimal control is $\alpha \equiv 0$. If $\frac{\mu-r}{\sigma^2} \ge 1-\gamma$, the return rate of the stock is so much better that $\alpha \equiv 1$.