MA908 Partial Differential Equations in Finance

Exercise Sheet 7: Numerics of ODE and PDE

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1. **Numerical schemes for ODE**

   (a) Prove the discrete Gronwall Lemma: If a sequence $z_0, z_1, z_2, \ldots$ of non-negative numbers satisfies $z_{n+1} \leqslant C z_n + D$ for constants $C, D > 0$, and all $n \in \mathbb{N}$, and it $C \neq 1$, then

   $$z_n \leqslant D \frac{C^n - 1}{C - 1} + z_0 C^{n+1}.$$

   (Hint: use induction, i.e. assume that it holds for some n, and prove that it then also holds for $n+1$. Since it holds for $n = 0$, it holds for all n.) What happens when $C = 1$?

   (b) In the lecture we have only covered first order schemes. Higher order schemes are sometimes better since they converge faster as the discretisation parameter $h \to 0$. Assume that $y(t)$ solves

   $$\partial_t y(t) = F(y(t), t), \qquad y(0) = x.$$

   An obvious choice to improve the order of accuracy is to use Taylor expansion

   $$y(t_{n+1}) = y(t_n + h) = y(t_n) + h y'(t_n) + \frac{h^2}{2} y''(t_n) + \mathcal{O}(h^3).$$

   In the lecture we ignored already the h^2 term and used $y'(t_n) = F(y(t_n), t_n)$. Extend this idea in the following way: in the second order Taylor expansion of $y(t)$, replace all occurrences of $y'(t)$ with $F(y(t), t)$. Be careful to apply the chain and product rule when computing $\frac{\mathrm{d}}{\mathrm{d}t} F(y(t), t)$. Derive from this a second order scheme.

   (c) Evaluating derivatives of F, as in the above scheme, can be computationally expensive in general. To avoid this, expand the function $h \mapsto F(y(t_n) + hF(y(t_n), t_n), t_n + h)$ to first order in h. Compare with the formula from a), and derive a second order numerical scheme involving no derivatives of F.

2. **Naive numerical scheme for the HJB equation:** Consider the non-linear PDE

   $$\partial_t u(x,t) + xr\partial_x u(x,t) + \frac{1}{2}\lambda^2 \frac{(\partial_x u(x,t))^2}{\partial_x^2 u} = 0,$$

   with final condition $u(x, T) = g(x)$. This is the HJB-equation that you have obtained in question 2a) of the previous sheet. Give a simple finite difference scheme for this equation, like we did for the heat equation in the lecture. How do you cope with the fact that now we have a final condition instead of an initial condition? Can you spot any problems with the scheme you obtained?

3. **Programming challenge:** This problem is not relevant for the exam, but it is fun, so do it!

   (a) In the programming language of your choice (e.g. MatLab or C), implement the solution to the heat equation

   $$\partial_t u(x,t) = \frac{1}{2}\sigma^2 \partial_x^2 u(x,t) \qquad (t > 0, 0 < x < 1),$$

   with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition to be fixed later. Use the step size scaling $h_t \equiv h = \mu h_x^2$, with some $\mu > 0$. Try different values of μ on the initial conditions

   $$u_1(x) = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2, \quad u_2(x) = \frac{1}{2} - \left|x - \frac{1}{2}\right|, \qquad u_3 = 1 \text{ if } 1/4 < x < 3/4 \text{ and } 0 \text{ otherwise.}$$

   Try to find, for $\sigma = 1$, the values of μ for which the solution looks sensible when you make h very small, and others where it does not.

(b) Now try to implement the first order finite difference scheme for the HJB equation that you found in problem 2). The first obvious problem is what to do at the spatial boundaries. It is reasonable to keep the $x = 0$ boundary equal to zero, but the other spatial boundary is more tricky (in the original equation, x went up to infinity, but this is obviously not possible here). You can try to extrapolate the last calculated point linearly. Now test your scheme for final condition $g(x) = x^\gamma$ with $0 < \gamma < 1$ and compare with the true solution known from the previous sheet. You will probably find that the scheme is very unstable and at no point the numerical solution resembles the true solution (if you find anything different, let me know, I would be very interested!). So for difficult equations like the HJB, much more sophisticated numerical integration schemes are needed. These are beyond the scope of the courses (and also beyond the knowledge of the lecturer...).