MA908L Partial Differential Equations in Finance

Exercise Sheet 3: Heat equation

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1. **Heat equation with discontinuous initial condition**
Consider the heat equation $\partial_t u = \frac{1}{2}\partial_x^2 u$ in one dimension, with initial data

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

   (a) Use the solution formula to show that

   $$u(x,t) = N\Big(\frac{x}{\sqrt{t}}\Big),$$

   where

   $$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-s^2/2}\,\mathrm{d}s.$$

   (b) Explore the solution by answering the following questions: what is $\max_x \partial_x u(x,t)$ as a function of time t? Where is it achieved? What is $\max_x \partial_x u(x,t)$? For which x is $\partial_x u(x,t)$ smaller than $1/10 \max_x \partial_x u(x)$? Sketch the graph of $\partial_x u$ as a function of x at a time $t > 0$ of your choice.

   (c) Let u be the solution for the problem above. Show that $v(x,t) = \int_{-\infty}^{x} u(z,t)\,\mathrm{d}z$ solves $\partial_t v = \partial_x^2 v$. What is the initial condition? Deduce the qualitative behaviour of v as a function of x and t: how rapidly does v tend to 0 as $x \to -\infty$? What is the behaviour of v as $x \to \infty$? What is the value of $v(0,t)$? Sketch the graph of $v(x,t)$ as a function of x for some $t > 0$ of your choice.

2. **Pricing of barrier options** Let $\mathrm{d}y_t = \mu y_t\,\mathrm{d}t + \sigma y_t\,\mathrm{d}B_t$ be a stock price modelled by geometric Brownian motion. Recall that the price P_t of an option with maturity T and payout Φ can be obtained by solving the Black-Scholes PDE

$$\partial_t P + \frac{1}{2}\sigma^2 x^2 \partial_x^2 P + b(x\partial_x P - P) = 0,$$

with final condition $P(x,T) = \Phi(x)$, and where b is the discount rate.
In the lectures we showed that after the coordinate transform $y = \ln x$ and $\tau = \frac{1}{2}\sigma^2(T-t)$, the function

$$u(y,\tau) = P(\mathrm{e}^y, T - \frac{2}{\sigma^2}\tau)\,\mathrm{e}^{-\alpha y - \beta\tau},$$

with $\alpha = \frac{1-k}{2}, \beta = \frac{(k-1)^2}{4}, k = \frac{2b}{\sigma}$, solves the heat equation

$$\partial_\tau u - \partial_y^2 u = 0,$$

with initial condition $u(y,0) = \mathrm{e}^{\frac{1}{2}(k-1)y}\,\Phi(\mathrm{e}^y)$.
Consider now the knockout European call option with strike price K and knock-out value K. This means final condition $\Phi(x) = (x-K)^+$, and boundary condition $P(K,t) = 0$ for all t in the Balck-Scholes PDE.

   (a) Determine the initial and boundary conditions for the corresponding heat equation.

   (b) Use the symmetry trick to obtain the solution u to the heat equation with the initial and boundary values you found.

   (c) Transform the solution back to the original variables; for this, you may assume that you know the pricing function $C(x,t)$ for the vanilla European call option (it is explicit and given in most texts about financial mathematics). Then, express the solution $P(x,t)$ in terms of $C(x,t)$.

3. **Initial and boundary value problems** Give the solution (in terms of an integral, as usual), for the heat equation

$$\partial_t w_t - \frac{1}{2}\partial_x^2 w = 0, \qquad \text{for } x > 0, t > 0,$$

and with the following initial and boundary conditions:

(a) $w(0,t) = 0$ and $w(x,0) = 1$. How does the solution relate to the solution u of question 1a)?

(b) $w(0,t) = 0$ and $w(x,0) = (x-K)^+$, with $K > 0$. How does this relate to the solution v of question 1c)?

(c) $w(0,t) = 0$ and $w(x,0) = (x-K)^+$ with $K < 0$. This does no longer relate directly to question 1, but you can use the symmetry trick. How does the solution behave near $(0,0)$?

(d) $w(0,t) = 1$ and $w(x,0) = 0$. Hint: it is closely related to the solution of a).

Interpret all the above solution as the expected payouts of some options where the underlying performs Brownian motion and starts at x. (See sheet 1 for how expected payouts connect to PDE.) E.g.: the solution of a) is is an option that pays 1 at maturity if the underlying not cross below zero, and nothing otherwise.