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# Gibbs measures relative to Brownian motion and Nelson's model

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# Introduction

Some of the basic ideas of this work can be most easily seen in a simple example. Consider a quantum mechanical particle described by the Schrödinger operator (Hamiltonian) $H = -\frac{1}{2}\Delta + V$ in $L^2(\mathbb{R}^d)$. $V : \mathbb{R}^d \to \mathbb{R}$ acts as a multiplication operator, and we choose $V$ in such a way that $H$ is self-adjoint and bounded below. Thus the spectrum of $H$ is a subset of $\mathbb{R}$, and its infimum $E_0$ is finite.

Assume that we have found a ground state of $H$, i.e. an eigenfunction $\psi \in L^2(\mathbb{R}^d)$ with $H\psi = E_0\psi$. Now we want to find out something about the properties of $\psi$; of course $\psi$ is square integrable by definition, but we might e.g. be interested in detailed information about the decay of $\psi$ at infinity.

Two very different methods exist for getting answers to such questions. One of them, which one could call the analytic approach, uses the eigenvalue equation $H\psi = E_0\psi$ along with techniques from partial differential equations. This is a very natural thing to do and is extremely successful in many cases [42].

The other approach uses methods from probability theory and is known as functional integration. In the example given above, one would study the measure

$$d\mu_T(x) = \frac{1}{Z_T} f(x_{-T}) \exp\left(-\int_{-T}^{T} V(x_s)\, ds\right) f(x_T)\, d\mathcal{W}(x), \tag{0.1}$$

defined on the space $C(\mathbb{R}, \mathbb{R}^d)$ of continuous functions $x = (x_t)_{t\in\mathbb{R}}$. In (0.1), $\mathcal{W}$ is the (infinite mass) Wiener measure, $T > 0$, $f \in L^2(\mathbb{R}^d)$, and $Z_T$ normalizes $\mu_T$ to a probability measure. To obtain a first understanding of the measure $\mu_T$, note that it gives high probability to paths that spend a lot of time in regions where $V$ is small, while suppressing paths that spend too much time in regions where $V$ is large.

In order to understand what (0.1) has to do with the ground state of $H = -\frac{1}{2}\Delta + V$, we need the Feynman-Kac formula: it tells us that

$$\left(e^{-T(H-E_0)} f\right)(q) = \int e^{-\int_0^T (V(x_s)-E_0)\, ds} f(x_t)\, d\mathcal{W}^q(x) \tag{0.2}$$

where $\mathcal{W}^q$ is the measure of Brownian motion starting in $q \in \mathbb{R}^d$. The strongly continuous contraction semigroup $e^{-T(H-E_0)}$ on the left hand side of (0.2) is derived from $H$ via functional calculus. We now take two copies of the right hand side of (0.2), reverse time in one of them (resulting in a measure on $C(]-\infty, 0], \mathbb{R}^d)$) and glue the two parts together at time 0. Since Brownian motion is time reversible and has the Markov property, integrating over $q \in \mathbb{R}^d$ with respect to Lebesgue measure then yields (0.1). This explains the connection between (0.1) and the Hamiltonian $H$.

The relevance of (0.1) for the study of the ground state $\psi$ now comes from the fact that by spectral theory,

$$\lim_{T\to\infty} e^{-T(H-E_0)} f = \frac{\langle \psi, f\rangle_{L^2(\mathbb{R}^d)}}{\|\psi\|^2_{L^2(\mathbb{R}^d)}} \psi \tag{0.3}$$

in $L^2(\mathbb{R}^d)$ for all $f \in L^2(\mathbb{R}^d)$. Thus for large $T$, the Lebesgue-density of the time zero distribution of $\mu_T$, i.e. of the image measure of $\mu_T$ under the point evaluation map $x \mapsto x_0$, will be very close to $\psi^2$. This gives us a strategy to study $\psi$: first ensure that (0.1) has a $T \to \infty$ limit, and then study the $t = 0$ distribution of the limiting measure.

At first sight, a method involving objects like Wiener measure looks much more complicated and less practical than the alternative of just studying the Schrödinger operator directly via the eigenvalue equation. In fact, in cases of quantum systems with few degrees of freedom (like the example above), the more powerful results have been obtained by the analytic approach, although even in this setup functional integration has been applied successfully [45, 46]. The situation where functional integration really becomes superior are systems of quantum field theory which have infinitely many degrees of freedom. Here often functional integrals are the only method available, and in some cases they are even used to define the Hamiltonian itself [19, 44].

In my thesis I consider the Nelson model, which describes a quantum particle coupled to a bosonic scalar field. Nelson's model is a borderline case. While it does feature infinitely many degrees of freedom, it can still be treated by the analytic method, and in fact most of the results in the literature have been obtained that way [6, 20]. One purpose of the present work is to see what functional integration gives us when applied to Nelson's model. When we write down the functional integral for Nelson's model, an infinite dimensional Ornstein-Uhlenbeck process appears in addition to the Wiener measure from the example above, and in the exponent we have an additional integral due to the interaction of the particle with the field. The question now is how good a control we can obtain over these objects.

One strategy is to immediately integrate over the field degrees of freedom, which can be done explicitly since the coupling of the field to the particle is linear in the field variables, and the corresponding path measure is Gaussian. The result is an expression of the form

$$d\mu_T(x) = \frac{1}{Z_T} \exp\left( - \int_{-T}^{T} V(x_s)\, ds - \int_{-T}^{T} ds \int_{-T}^{T} dt W(x_t - x_s, t - s) \right) d\mathcal{W}(x), \tag{0.4}$$

where $W : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ comes from the interaction with the field. This technique, which goes back to R. Feynman [14], has been used by E. Nelson [32] to investigate the ultraviolet divergence in the model, and by H. Spohn [47] to estimate the effective mass of the polaron.

An advantage of the above strategy is that we do not have to deal with stochastic processes on infinite dimensional state spaces. A disadvantage is that only information about the particle, not about the field, can be obtained. In particular, the method can not answer questions like

a) What is the expected number of bosons of given momentum in the ground state?

b) What is the expected number of bosons with position in $M \subset \mathbb{R}^d$?

c) What is the average strength of the bosonic field and how large is its variance?

In order to answer such questions, we will have to deal directly with the infinite dimensional Ornstein-Uhlenbeck process at first, using the linearity in the field variables only at a later stage of our considerations.

It is obvious that in functional integration, measures of the form (0.1) and (0.4) play a crucial role. They are, at least optically, very similar to a class of probability measures appearing in statistical mechanics, the Gibbs measures. As a crucial difference however, Gibbs measures are usually defined on a countable product of measurable spaces. More concretely, if (0.4) was to fit into the classical theory of Gibbs measures, $\mathcal{W}$ would have to be a countable product of measures on $\mathbb{R}^d$, which it is obviously not. Measures of the type (0.4) will be called Gibbs measures relative to Brownian motion. A systematic study of such measures and their $T \to \infty$ limits emerged only recently [38], and we take the opportunity to develop the theory a little further.

Let us give an outline of the thesis: Chapter 1 is devoted to a presentation of Nelson's model. We derive it from its classical counterpart on a heuristic level, give the well-established Fock representation, the representation in function space and the connection of the two. Extra care is taken to clarify this connection as well as its influence on the choice of the Gaussian measure in function space. While none of the results here is completely new, many of the proofs, especially in the function space context, are at least not easily available in the literature. Small improvements are made on some existing results like the Feynman-Kac-Nelson formula or the path continuity of the infinite dimensional Ornstein-Uhlenbeck process.

In Chapter 2, we study Gibbs measures relative to Brownian motion. We propose a framework of definitions and notions for these measures, systematizing and extending the approaches in [3, 4, 38]. We then give results about existence and uniqueness of infinite volume Gibbs measures in various contexts.

Chapter 3 resumes the study of Nelson's model, investigating properties of its ground state along the lines of (0.2) and (0.3). First, we give an overview over the current knowledge concerning the existence of the ground state. A general formula to calculate ground state expectations for many important operators is then established and applied to answer questions like a), b), c) above.

It may have become obvious from the preceding discussion that there have been two main objectives I had when writing this thesis: the first, of course, was the presentation of the main mathematical results, contained in Sections 2.3, 3.2 and

3.3. The second, and at least of equal importance to me, was to smoothen the learning curve for anybody who needs to work his way into the topics of Chapter 1, a learning curve that I experienced as being fairly steep. It is up to the reader to decide how much I succeeded.

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# Contents

# Chapter 1

# Nelson's model

## 1.1 A classical particle-field model

Nelson's model is a simplified description of a non-relativistic, charged quantum particle interacting with its own quantized radiation field. The simplification consists in ignoring the vector character of the Maxwell field and using a scalar field instead. One of the main aims of this work is to study properties of states of minimal energy, but before we describe the quantum model, it seems useful to take a look at the classical model and, at least on a heuristic level, to show how the quantum model can be derived from it. This will be done in the next two sections, where we restrict ourselves to three dimensional space for convenience. A reader who is only interested in the mathematical aspects of this work may safely skip the first two sections.

The scalar field is a function $\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, (x,t) \mapsto \phi(x,t)$, which has to fulfill the wave equation with inhomogeneity (= source) localized around the position $q \in \mathbb{R}^3$ of the particle. The corresponding equation reads

$$\frac{\partial^2}{\partial t^2}\phi(x,t) = \Delta_x \phi(x,t) - \varrho(x - q(t)). \tag{1.1}$$

The function $\varrho$ has support concentrated near 0 and is used to regularize the coupling of the particle with the field. The particle is assumed to feel the effect of an external potential $x \mapsto V(x)$ as well as a force due to its interaction with the field.

The equations of motion for such a system read

$$\begin{array}{ll} \dot{\phi}(x,t) = \pi(x,t), & \dot{\pi}(x,t) = \Delta_x \phi(x,t) - \varrho(x - q(t)), \\ \dot{q}(t) = p(t), & \dot{p}(t) = (-\nabla V)(q(t)) + \int \phi(x,t)(\nabla \varrho)(x - q(t))\, dx, \end{array} \tag{1.2}$$

where $(x,t) \mapsto \pi(x,t)$ and $p \in \mathbb{R}^3$ are the canonical conjugate (momentum) field and the momentum, respectively. Thus the Hamiltonian function

$$H_{\mathrm{cl}} : (\phi, q, \pi, p) \mapsto H_{\mathrm{cl}}(\phi, q, \pi, p) \in \mathbb{R}$$

is given by

$$H(\phi, q, \pi, p) = \frac{1}{2}p^2 + V(q) + \frac{1}{2}\int \left( (\pi(x))^2 + |\nabla\phi(x)|^2 \right) dx + \int \phi(x)\varrho(x-q)\, dx. \quad (1.3)$$

We are now looking for stationary solutions of (1.2) that minimize the Hamiltonian functional (1.3). Let us assume for simplicity that $V$ has a unique minimum at $q_0 \in \mathbb{R}^3$. In order to minimize (1.3), we then obviously have to take

$$p = 0, \quad q = q_0 \quad \text{and} \quad \pi(x) = 0.$$

This leaves us with the condition

$$\int \left( \frac{1}{2}|\nabla\phi(x)|^2 + \phi(x)\varrho(x - q_0) \right) dx = \text{ minimal}$$

and an application of variational calculus leads to

$$-\Delta\phi(x) + \varrho(x - q_0) = 0.$$

Thus, if we assume that $\varrho$ is in the range of the Laplace operator acting in $L^2(\mathbb{R}^3)$, we have the solution

$$\phi(x) = \left( \Delta^{-1}\varrho(. - q_0) \right)(x) = \frac{1}{4\pi}\int \frac{1}{|x-y|}\varrho(y - q_0)\, dy. \quad (1.4)$$

In sum, a configuration minimizing (1.3) consists of a particle sitting in the minimum $q_0$ of $V$, together with a field that decays like the inverse of the distance from the particle. One of the main objectives of this work is to obtain similar information about the state of minimal energy for the corresponding quantum system.

## 1.2 Quantizing the particle-field model

In this section we will give a purely heuristic outline of how one could guess the quantum system to be presented in Sections 1.3 and 1.4 from the classical system given in Section 1.1. No attempt at mathematical rigor is made.

There exists a standard recipe, known as canonical quantization, for deriving the quantum mechanical description of a given classical Hamiltonian particle system with canonical variables $p, q \in \mathbb{R}^n$. The procedure is to replace in the Hamiltonian function each occurrence of $q$ by the operator of multiplication with $q$ acting in $L^2(\mathbb{R}^n)$, and each occurrence of $p$ by the operator $-i\nabla_q$ acting in the same space. Then one has to symmetrize the resulting operator in order to get a formally self-adjoint operator $H$ and to prove that $H$ is in fact self-adjoint on a suitable domain. $H$ is then the Hamilton operator of the quantum system.

When we try to apply this procedure to the Hamilton functional (1.3), we have to decide what to do with the fields $\phi$ and $\pi$. An obvious choice is to discretize the integrals in (1.3) as follows: we take a centered cube $\Lambda \subset \mathbb{R}^3$ and divide it into $N^3$ small cubes of equal size. We fix the center $x_j$ of each of the small cubes, and put

$$\begin{aligned} H_{\mathrm{cl}}^{(N)}(\phi, q, \pi, p) &= \frac{1}{2}p^2 + V(q) + \frac{|\Lambda|}{2N^3}\sum_{j=1}^{N^3}\left((\pi(x_j))^2 + \sum_{x_i \sim x_j}\left(\frac{\phi(x_i)-\phi(x_j)}{\delta}\right)^2\right) + \\ &\quad + \frac{|\Lambda|}{N^3}\sum_{j=1}^{N^3}\phi(x_j)\varrho(x_j - q). \end{aligned} \tag{1.5}$$

Here, $x_i \sim x_j$ means that $x_i$ neighbors $x_j$ on the lattice, $|\Lambda|$ is the volume of $\Lambda$, and $\delta$ is the distance of the centers of two neighboring cubes. (1.3) is recovered from (1.5) by taking $N \to \infty$ and $\Lambda \to \infty$ in such a way that the volume of the small cubes goes to zero.

We now apply the canonical quantization procedure to (1.5) with the result

$$\begin{aligned} H^{(N)} &= -\frac{1}{2}\Delta_q + V(q) + \frac{|\Lambda|}{2N^3}\sum_{j=1}^{N^3}\left(-\Delta_{Q_{x_j}} + \sum_{x_i \sim x_j}\left(\frac{Q_{x_i}-Q_{x_j}}{\delta}\right)^2\right) + \\ &\quad + \frac{|\Lambda|}{N^3}\sum_{j=1}^{N^3}Q_{x_j}\varrho(x_j - q). \end{aligned} \tag{1.6}$$

$H^{(N)}$ is an operator in $L^2(\mathbb{R}^3 \times \mathbb{R}^{N^3})$ acting on functions of the variables $q$ and $(Q_{x_1}, \ldots, Q_{x_n})$. To undo the discretization, we now would like to take the $N \to \infty, |\Lambda| \to \infty$ limit in (1.6). However, this is a lot less straightforward than it was in (1.5), one obvious difficulty being that $H^{(N)}$ acts in a different Hilbert space for each $N$.

One way to take the limit in (1.6) is the Fock space approach. Only an outline of this extremely messy procedure is given here, somewhat more detail can be found in [32]. The first step is to transform $L^2(\mathbb{R}^3 \times \mathbb{R}^{N^3})$ in such a way that the coupling between the $Q_{x_j}$ for different $j$ disappears in the third term of (1.6). This leads to a system of $N^3$ independent, quantum mechanical harmonic oscillators. Using the representation in form of creation operators $a^*$ and annihilation operators $a$ for such oscillators [19], one arrives at

$$\begin{aligned} H_{\mathcal{F}}^{(N)} &= -\frac{1}{2}\Delta_q + V(q) + \frac{|\Lambda|}{N^3}\sum_{j=1}^{N^3}|k_j|\left(a_{k_j}^* a_{k_j} + \frac{1}{2}\right) + \\ &\quad + \frac{|\Lambda|}{N^3}\sum_{j=1}^{N^3}\frac{1}{\sqrt{2|k_j|}}\left(a_{k_j}^* e^{-ik_j q}\hat{\varrho}(k_j) + a_{k_j}e^{ik_j q}\hat{\varrho}(-k_j)\right). \end{aligned} \tag{1.7}$$

Here, $k_j \in \mathbb{R}^3$ with $k_j = x_j =$ the points of the original lattice, and the reason that we don't write $x_j$ any more is that the transformation which leads to (1.7) is essentially a discrete Fourier transformation on the lattice of the $x_j$. We now replace the terms $a^*_{k_j}a_{k_j} + 1/2$ in (1.7) by $a^*_{k_j}a_{k_j}$. This procedure, which is known as energy renormalization, is necessary because otherwise $\inf \mathrm{spec} H^{(N)}_{\mathcal{F}} \to \infty$ as $N, |\Lambda| \to \infty$. Fortunately, it is also not harmful since subtracting a constant from a Hamiltonian does not change the physics.

For each $k_j$, we now take the complete, orthonormal set of eigenfunctions $\{\psi^{(k_j)}_n : n \in \mathbb{N}\}$ belonging to the operator $a^*_{k_j}a_{k_j}$ and construct the Fock space over $L^2(\mathbb{R}^{N^3})$ with the help of these eigenfunctions as described in Appendix A.1. By formally taking the $n \to \infty, |\Lambda| \to \infty$ limit as described there also, we arrive at the Hamiltonian

$$H_{\mathcal{F}} = -\frac{1}{2}\Delta_q + V(q) + \int |k| a^*_k a_k \, d^3k + \int \frac{1}{\sqrt{2|k|}} \left( a^*_k e^{-ik\cdot q}\hat{\varrho}(k) + a_k \overline{e^{-ik\cdot q}\hat{\varrho}(k)} \right) d^3k \tag{1.8}$$

acting on the symmetric Fock space $\mathcal{F}(L^2(\mathbb{R}^3))$. A slight generalization of the operator (1.8) as well as symmetric Fock space will be defined rigorously and investigated in Section 1.3.

There is an alternative route for taking the limit in (1.6). The key observation is that the third term (= free field term) of (1.6),

$$H^{(N)}_{\mathrm{f}} = \frac{|\Lambda|}{2N^3} \sum_{j=1}^{N^3} \left( -\Delta_{Q_{x_j}} + \sum_{x_i \sim x_j} \left( \frac{Q_{x_i} - Q_{x_j}}{\delta} \right)^2 \right), \tag{1.9}$$

is, after the energy renormalization, unitarily equivalent to the generator of an $N^3$-dimensional Ornstein-Uhlenbeck process. To be more precise, let us denote by $A$ the positive, symmetric matrix generating the quadratic form in the second term of (1.9), i.e.

$$\left\langle \vec{Q}, A\vec{Q} \right\rangle_{\mathbb{R}^{N^3}} = \sum_{j=1}^{N^3} \sum_{x_i \sim x_j} \left( \frac{Q_{x_i} - Q_{x_j}}{\delta} \right)^2, \tag{1.10}$$

with $\vec{Q} = (Q_{x_1}, \ldots, Q_{x_{N^3}})$. Writing $\sqrt{A}$ for the nonnegative square root of $A$, it can be checked directly that

$$\psi^{(N)}_{\mathrm{f}} = \exp\left( -\frac{1}{2} \left\langle \vec{Q}, \sqrt{A}\vec{Q} \right\rangle_{\mathbb{R}^{N^3}} \right)$$

is an eigenfunction of $H^{(N)}_{\mathrm{f}}$ to the eigenvalue $\mathrm{tr}(\sqrt{A})/2$. Using the positivity of $\psi^{(N)}_{\mathrm{f}}$ and the strict positivity the kernel of of $e^{-H^{(N)}_{\mathrm{f}}}$, a Perron-Frobenius argument implies that $\psi^{(N)}_{\mathrm{f}}$ is the unique ground state, i.e. eigenfunction corresponding to the bottom

of the spectrum, of $H_{\rm f}^{(N)}$. When we subtract the ground state energy $\mathrm{tr}(\sqrt{A})/2$ from $H_{\rm f}^{(N)}$ and apply the ground state transformation to the resulting operator, we arrive at the generator of a stationary Ornstein-Uhlenbeck process with state space $\mathbb{R}^{N^3}$, mean value $0 \in \mathbb{R}^{N^3}$ and covariance

$$\mathbb{E}((X_s)_i(X_t)_j) = \frac{1}{2}\left(\sqrt{A}^{-1}e^{-|t-s|\sqrt{A}}\right)_{i,j}, \tag{1.11}$$

where the subscripts denote components of vectors resp. matrices. Details on the above procedure are given in Appendix A.3.

Note however that we cheated slightly above, and the resulting damage is that (1.11) does not make sense: $A$ as given in (1.10) is not invertible. This can be healed by replacing $A$ with $A+\mu I$, $\mu > 0$, eventually resulting in the Nelson model for massive bosons of mass $\mu$. More suitable for our purposes is to impose zero boundary conditions, i.e. to add a term $Q_{x_j}^2$ at the right hand side of (1.10) for each $x_j$ corresponding to a small cube at the boundary of $\Lambda$. This correction will then drop out upon taking the $|\Lambda| \to \infty$ limit.

Now we want to know what becomes of the Ornstein-Uhlenbeck process when $N \to \infty$ and $|\Lambda| \to \infty$. For this, it is better to write (1.11) in the form

$$\mathbb{E}\left(\langle f, X_s\rangle_{\mathbb{R}^{N^3}} \cdot \langle g, X_t\rangle_{\mathbb{R}^{N^3}}\right) = \left\langle f, \sqrt{A}^{-1}e^{-|t-s|\sqrt{A}}g\right\rangle_{\mathbb{R}^{N^3}}, \tag{1.12}$$

valid for all $f, g \in \mathbb{R}^{N^3}$. Now we identify vectors $\vec{Q} = (Q_{x_1}, \ldots, Q_{x_{N^3}}) \in \mathbb{R}^{N^3}$ with step functions $F^{(N)} : \mathbb{R}^3 \to \mathbb{R}$ which are constant on the small cubes corresponding to the $x_j$ and zero outside $\Lambda$. With this identification, the state space of the Ornstein-Uhlenbeck process is now a space of functions on $\mathbb{R}^3$, and the operator $-A$ is a discrete Laplacian on the lattice of the $x_i$. Thus in the limit, we expect that the actual Laplacian on $\mathbb{R}^3$ should appear, and $H_{\rm f}^{(N)}$ in (1.9) will become the generator of an infinite dimensional Ornstein-Uhlenbeck process with state space that is some space of (generalized) functions on $\mathbb{R}^3$, mean 0 and covariance

$$\mathbb{E}\left(\langle f, X_s\rangle_{L^2(\mathbb{R}^3)} \cdot \langle g, X_t\rangle_{L^2(\mathbb{R}^3)}\right) = \left\langle f, \sqrt{-\Delta}^{-1}e^{-|t-s|\sqrt{-\Delta}}g\right\rangle_{L^2(\mathbb{R}^3)}, \tag{1.13}$$

provided this expression makes sense. Turning to the coupling term on the right hand side of (1.6), we see that it is in the step function picture just the operator of multiplication with the linear functional

$$F^{(N)} \mapsto \sum_{j=1}^{N^3} F^{(N)}(x_j)\varrho(x_j - q).$$

So in the limit, the coupling will be the linear functional

$$X \mapsto \int X(x)\varrho(x-q)\, d^3x$$

acting in $L^2(\mathbb{R}^3)$. A rigorous treatment of the above structure is contained in Section 1.4.

## 1.3 Nelson's model in Fock space

Although it is not obvious from the way we derived the Fock space from the classical field in the previous section, the best way to think about it is as a model of indistinguishable quantum particles, where the number of particles is not conserved. We will say more on this after giving precise definitions.

**Definition 1.3.1** *For $n \in \mathbb{N}$, denote by $\mathcal{F}^{(n)} = L^2(\mathbb{R}^d)^{\hat{\otimes} n}$ the space of (complex-valued) $L^2(\mathbb{R}^{dn})$-functions $f$ which are symmetric in the sense that for each permutation $\pi$ of $\{1, \ldots, n\}$ and each $k_1, \ldots, k_n \in \mathbb{R}^d$, we have*

$$f(k_1, \ldots, k_n) = f(k_{\pi(1)}, \ldots, k_{\pi(n)}).$$

*Moreover, put $\mathcal{F}^{(0)} = \mathbb{C}$. $\mathcal{F}^{(n)}$ is called $n$-th Fock space component. The* **symmetric Fock space** *$\mathcal{F}$ is the set of all $F = (f_0, f_1, \ldots) \in \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}$ for which the norm*

$$\|F\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \|f_n\|_{L^2(\mathbb{R}^d)^{\hat{\otimes} n}}^2 , \tag{1.14}$$

*is finite. Let us write $P_n$ for the projection from $\mathcal{F}$ onto $\mathcal{F}^{(n)}$, and $F^{(n)}$ instead of $P_n F$ for $F \in \mathcal{F}$. We will also write $F^{(n)}$ for the $n$-th (and only nonzero) component of $P_n F$.*

The interpretation is that $P_n \mathcal{F}$ consists of the quantum mechanical states of the system with exactly $n$ particles. The function $F^{(n)} \in P_n \mathcal{F}$ then determines the exact behavior of these particles, while the symmetry condition corresponds to indistinguishability. In the model of an electron coupled to its radiation field, the 'particles' would be photons, but since in the Nelson model we have a scalar field instead of a vector field, they will just be called bosons in the following. If $\|F\|_{\mathcal{F}} = 1$, then $\langle F, P_n F \rangle_{\mathcal{F}}$ represents the probability for finding exactly $n$ bosons in the system described by $F$. Note that by (1.14), this probability must decay faster than $1/\sqrt{n}$ for large $n$ in order to have $F \in \mathcal{F}$. The reason for this constraint is mathematical convenience rather than physical necessity, a fact that we will encounter again when discussing the infrared divergence of Nelsons model.

Let us now introduce some important operators acting in $\mathcal{F}$. Obviously, it will be enough to specify the action of these operators on each $\mathcal{F}^{(n)}$, $n \in \mathbb{N}_0$. For this purpose, we will write $F = (F^{(0)}, F^{(1)}, \ldots)$ for $F \in \mathcal{F}$. $F^{(n)}$ will be denoted as a pointwisely defined function for convenience, but all of the following equalities are to be understood in $L^2$-sense.

**Definition 1.3.2** *Let $g : \mathbb{R}^d \to \mathbb{C}$ be a measurable function. The following operators in $\mathcal{F}$ are defined on functions $F \in \mathcal{F}$ such that the image of $F$ is again in $\mathcal{F}$.*

$$(\mathbf{N}F)^{(n)}(k_1,\ldots,k_n) = nF^{(n)}(k_1,\ldots,k_n), \tag{1.15}$$

$$(a^*(g)F)^{(n)}(k_1,\ldots,k_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(k_i)F^{(n-1)}(k_1,\ldots,\not{k}_i,\ldots,k_n), \tag{1.16}$$

$$(a(g)F)^{(n)}(k_1,\ldots,k_n) = \sqrt{n+1}\int g(k)F^{(n+1)}(k,k_1,\ldots,k_n)\,dk \tag{1.17}$$

$$(d\Gamma(g)F)^{(n)}(k_1,\ldots,k_n) = \sum_{i=1}^{n} g(k_i)F^{(n)}(k_1,\ldots,k_n) \tag{1.18}$$

*The $\not{k}_i$ in (1.16) means that the variable $k_i$ is omitted. Also, by the usual convention, the sums in (1.16) and (1.18) are zero for $n = 0$. $\mathbf{N}$ is called* **number operator**, *$a^*(g)$ is called* **creation operator** *associated with $g$, $a(g)$ is called* **annihilation operator** *associated with $g$, and $d\Gamma(g)$ is called* **differential second quantization** *of the operator of multiplication with $g$. Alternative notations are*

$$a^*(g) \equiv \int a_k^* g(k)\,dk, \qquad a(g) \equiv \int a_k g(k)\,dk, \quad d\Gamma(g) \equiv \int a_k^* a_k g(k)\,dk.$$

Note that $\mathbf{N} = d\Gamma(1)$. Except in the case of $d\Gamma(g)$ which will be treated later, the names of the operators above are easy to explain. $a^*$ takes $\mathcal{F}^{(n)}$ to $\mathcal{F}^{(n+1)}$, thus it creates a boson that is described by $g$. Similarly, $a(g)$ takes $\mathcal{F}^{(n)}$ to $\mathcal{F}^{(n-1)}$ and therefore destroys a boson. Finally, $\mathbf{N}$ counts the number of bosons in $F \in \mathcal{F}$ in the sense that $\langle F, \mathbf{N}F\rangle_{\mathcal{F}} / \|F\|_{\mathcal{F}}^2$ is the expected number of bosons in the state $F$.

In order to say a little more about the domains of the operators from Definition 1.3.2 as well as for later purposes, the following inequality is useful.

**Proposition 1.3.3** *Let $g, h : \mathbb{R}^d \to \mathbb{C}$ be measurable with $g \in L^2(\mathbb{R}^d)$ and $gh \in L^2(\mathbb{R}^d)$. Then for each $F \in \mathcal{F}$ such that $d\Gamma(|h|^2)F \in \mathcal{F}$, we have $a(gh)F \in \mathcal{F}$, $a^*(gh)F \in \mathcal{F}$, and*

$$\|a(gh)F\|_{\mathcal{F}}^2 \le \|g\|_{L^2(\mathbb{R}^d)}^2 \left\langle F, d\Gamma(|h|^2)F\right\rangle_{\mathcal{F}}, \tag{1.19}$$

$$\|a^*(gh)F\|_{\mathcal{F}}^2 \le \|g\|_{L^2(\mathbb{R}^d)}^2 \left\langle F, d\Gamma(|h|^2)F\right\rangle_{\mathcal{F}} + \|gh\|_{L^2(\mathbb{R}^d)}^2 \|F\|_{\mathcal{F}}^2 \tag{1.20}$$

*Proof:* Obviously it is enough to consider Fock space components. Note first that

$$\begin{aligned}\left\langle F^{(n)}, (d\Gamma(|h|^2)F)^{(n)}\right\rangle_{L^2(\mathbb{R}^{dn})} &= \sum_{i=1}^{n}\int |F^{(n)}(k_1,\ldots,k_n)|^2|h|^2(k_i)\,dk_1\cdots dk_n = \\ &= n\int |F^{(n)}(k_1,\ldots,k_n)|^2|h|^2(k_1)\,dk_1\cdots dk_n.\end{aligned}$$

Now,

$$\begin{aligned}
&\left\|(a(gh)F)^{(n-1)}\right\|^2_{L^2(\mathbb{R}^{d(n-1)})} = \\
&= n\int dk_2\cdots dk_n \left|\int g(k_1)h(k_1)F^{(n)}(k_1,\ldots,k_n)\,dk_1\right|^2 \leq \\
&\leq n\,\|g\|^2_{L^2(\mathbb{R}^d)}\int |h|^2(k_1)|F^{(n)}(k_1,\ldots,k_n)|^2\,dk_1\cdots dk_n = \\
&= \|g\|^2_{L^2(\mathbb{R}^d)}\left\langle F^{(n)},(d\Gamma(|h|^2)F)^{(n)}\right\rangle_{L^2(\mathbb{R}^{dn})},
\end{aligned}$$

and summation over $n$ gives (1.19). To get the other inequality, observe that

$$\left\|(a^*(gh)F)^{(n+1)}\right\|^2_{L^2(\mathbb{R}^{d(n+1)})} = \frac{1}{n+1}\sum_{i,j=1}^{n+1}\int \overline{g(k_i)h(k_i)}g(k_j)h(k_j)\times \tag{1.21}$$
$$\times\overline{F^{(n)}(k_1,\ldots,\not{k}_i,\ldots,k_{n+1})}F^{(n)}(k_1,\ldots,\not{k}_j,\ldots,k_{n+1})\,dk_1\cdots dk_{n+1}.$$

The $i=j$ terms of (1.21) now add up to $\|gh\|^2_{L^2(\mathbb{R}^d)}\left\|F^{(n)}\right\|_{L^2(\mathbb{R}^{dn})}$, while the $n(n+1)$ terms with $i\neq j$ are by symmetry all equal to

$$\begin{aligned}
&\int \overline{g(k_1)h(k_1)}g(k_2)h(k_2)\overline{F^{(n)}(k_1,k_3,k_4,\ldots,k_{n+1})}F^{(n)}(k_2,\ldots,k_{n+1})\,dk_1\cdots dk_{n+1} \leq \\
&\leq \int dk_3\cdots dk_{n+1}\left(\|g\|^2_{L^2(\mathbb{R}^d)}\int |h|^2(k)|F^{(n)}(k,k_3,\ldots,k_{n+1}|^2\,dk\right) = \\
&= \frac{1}{n}\|g\|^2_{L^2(\mathbb{R}^d)}\left\langle F^{(n)},(d\Gamma(h^2)F)^{(n)}\right\rangle_{L^2(\mathbb{R}^{dn})}.
\end{aligned}$$

Plugging this inequality into (1.21) and summing over $n\in\mathbb{N}$ gives (1.20). □

A first application of the above estimates is

**Proposition 1.3.4** *The number operator* $\mathbf{N}$ *is self-adjoint on the dense domain*

$$D(\mathbf{N}) = \left\{F\in\mathcal{F} : \sum_{n=0}^{\infty} n^2\left\|F^{(n)}\right\|^2_{L^2(\mathbb{R}^{dn})} < \infty\right\}.$$

*Moreover,* $a^*(g)$ *and* $a(g)$ *are well defined on this domain, with*

$$\langle F, a(g)G\rangle_{\mathcal{F}} = \langle a^*(\bar{g})F, G\rangle_{\mathcal{F}} \qquad \text{for all } F,G\in D(\mathbf{N}). \tag{1.22}$$

*Above,* $\bar{g}$ *denotes the complex conjugation of* $g$.

*Proof:* To show the assertions about $\mathbf{N}$ and $D(\mathbf{N})$ is a matter of checking definitions, as is (1.22). The assertion $a^*(g)F\in\mathcal{F}$ and $a(g)F\in\mathcal{F}$ for $F\in D(\mathbf{N})$ follows from Proposition 1.3.3 by putting $h=1$ there. □

We now introduce a set $\mathcal{E} \subset \mathcal{F}$ which is total in $\mathcal{F}$, i.e. linear combinations of elements from $\mathcal{E}$ are dense in $\mathcal{F}$. The purpose is that we will make definitions and statements about linear objects on $\mathcal{F}$ for the elements of $\mathcal{E}$ and extend by linearity. For $f_1, \ldots, f_n \in L^2(\mathbb{R}^d)$ write

$$(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n)(k_1, \ldots, k_n) = \frac{1}{n!} \sum_{\pi \in \Pi_n} f_1(k_{\pi(1)}) \cdots f_n(k_{\pi(n)}) \tag{1.23}$$

for the symmetric tensor product of $f_1, \ldots, f_n$. Above, $\Pi_n$ is the set of all permutations on $\{1, \ldots, n\}$. It is then proved by standard tools of integration theory that

**Proposition 1.3.5** *The set*

$$\mathcal{E}_n = \{f_1 \hat{\otimes} \ldots \hat{\otimes} f_n : f_1, \ldots, f_n \in L^2(\mathbb{R}^d)\}$$

*is total in* $\mathcal{F}^{(n)}$*. Consequently, the set*

$$\mathcal{E} = \{(0, \ldots, 0, F_n, 0, \ldots) \in \mathcal{F} : F_n \in \mathcal{E}_n, n \in \mathbb{N}_0\}$$

*is total in* $\mathcal{F}$*.*

By picking the $f_j$ from an orthonormal basis of $L^2(\mathbb{R}^d)$ and normalizing correctly, we can even obtain an orthonormal basis of $\mathcal{F}$ [22]. Moreover, the polarization identity for multilinear maps [37] implies that we also get a total set when requiring in addition $f_1 = \ldots = f_n$ in the definition of $\mathcal{E}_n$. We will not use any of these facts.

With the help of $\mathcal{E}$, we now study second quantization a little more closely. First, we give another definition which includes an extension of (1.18)

**Definition 1.3.6** *Let $A$ be an operator in $L^2(\mathbb{R}^d)$. For $f_1, \ldots, f_n \in D(A)$ define*

$$\Gamma(A)(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n) = Af_1 \hat{\otimes} \ldots \hat{\otimes} Af_n \tag{1.24}$$

$$d\Gamma(A)(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n) = \sum_{k=1}^{n} f_1 \hat{\otimes} \cdots \hat{\otimes} f_{k-1} \hat{\otimes} A f_k \hat{\otimes} f_{k+1} \hat{\otimes} \cdots \hat{\otimes} f_n \tag{1.25}$$

*and extend by linearity.* $\Gamma(A)$ *is called* **second quantization** *of $A$, and* $d\Gamma(A)$ *is called* **differential second quantization** *of $A$.*

By checking definitions, we may see that (1.25) is consistent with (1.18), i.e. for the special choice $F = f_1 \hat{\otimes} \ldots \hat{\otimes} f_n$ and $A =$ multiplication with $g$ , we get (1.25) from (1.18). In the same way, we find

$$a^*(g)(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n) = \sqrt{n+1}\, g \hat{\otimes} f_1 \hat{\otimes} \ldots \hat{\otimes} f_n,$$

$$a(g)(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \langle \bar{f}_i, g \rangle_{L^2(\mathbb{R}^d)} \hat{\bigotimes_{j \neq i}} f_j.$$

If $A$ is a bounded operator with $\|A\| \leq 1$, then also $\Gamma(A)$ is bounded with $\|\Gamma(A)\| \leq 1$. The proof can be found in [34]. However, in our case, namely when $A$ is the operator of multiplication with $g \in L^\infty$ and $\|g\|_\infty \leq 1$, the statement is obvious since then

$$(\Gamma(g)F)^{(n)}(k_1, \ldots, k_n) = \left( \prod_{i=1}^{n} g(k_i) \right) F^{(n)}(k_1, \ldots, k_n).$$

There is a natural connection between second quantization and differential second quantization: If $A$ is self-adjoint and $A \geq 0$, then $\Gamma(e^{-tA})$ is a strongly continuous semigroup of contractions on $\mathcal{F}$, and thus there exists the generator $B$ of this semigroup, a densely defined, self-adjoint operator in $\mathcal{F}$ with $\frac{d}{dt}\Gamma(e^{-tA})F|_{t=0} = BF$ for all $F \in D(B)$. A direct check for the elements of $\mathcal{E}$ and extension by linearity show that $B = d\Gamma(A)$, which also gives a reason for the word 'differential' in the name of $d\Gamma(A)$.

We are now prepared to give a rigorous definition of the operator we wrote down in (1.8) and to show that it is self-adjoint and bounded below. We will slightly generalize (1.8) by allowing for a more general dispersion relation. Let $\varrho : \mathbb{R}^d \to \mathbb{R}$ and $\omega : \mathbb{R}^d \to \mathbb{R}$ be measurable functions satisfying the following assumptions:

$$\omega(k) = \bar{\omega}(k) = \omega(-k), \quad \varrho(k) = \bar{\varrho}(k), \tag{1.26}$$

$$0 < \omega(k) \quad \text{except on a set of Lebesgue measure zero,} \tag{1.27}$$

$$\frac{\hat{\varrho}}{\sqrt{\omega}} \in L^2(\mathbb{R}^d), \quad \frac{\hat{\varrho}}{\omega} \in L^2(\mathbb{R}^d). \tag{1.28}$$

Define

$$H_{\mathrm{f}} = \int \omega(k) a_k^* a_k \, dk = d\Gamma(\omega) \tag{1.29}$$

$$\begin{aligned} H_{\mathrm{I}} &= \int \frac{1}{\sqrt{2\omega(k)}} \left( \hat{\varrho}(k) e^{-ikq} a_k^* + \overline{\hat{\varrho}}(k) e^{ikq} a_k \right) dk = \\ &= a^* \left( \frac{\hat{\varrho} e^{-iq\cdot}}{\sqrt{2\omega}} \right) + a \left( \overline{\frac{\hat{\varrho} e^{-iq\cdot}}{\sqrt{2\omega}}} \right). \end{aligned} \tag{1.30}$$

These are the last two terms of (1.8). $H_{\mathrm{f}}$ accounts for the energy contained in the field configuration, while $H_{\mathrm{I}}$ gives the interaction energy between field and particle. $H_{\mathrm{I}}$ is an operator in $L^2(\mathbb{R}^d) \otimes \mathcal{F}$, but since in the $L^2(\mathbb{R}^d)$ component it is just an operator of multiplication, we can (and will) also view it as an operator in $\mathcal{F}$ depending on the parameter $q \in \mathbb{R}^d$ whenever this is convenient.

The energy of the particle is described by

$$H_{\mathrm{p}} = -\frac{1}{2}\Delta + V \tag{1.31}$$

acting in $L^2(\mathbb{R}^d)$, where $V : \mathbb{R}^d \to \mathbb{R}$ is Kato-decomposable (see Appendix A.2) and $\Delta$ is the $d$-dimensional Laplace operator. Let us emphasize that in Appendix A.2 we make the convention that a constant is added to $V$ such that $\inf \mathrm{spec} H = 0$.

We now define the full Nelson Hamiltonian.

**Definition 1.3.7** *Let $H_\mathrm{f}, H_\mathrm{I}$ and $H_\mathrm{p}$ be as in (1.29), (1.30) and (1.31), respectively. The (formal) operator*

$$H_\mathcal{F} = H_\mathrm{p} \otimes 1 + 1 \otimes H_\mathrm{f} + H_\mathrm{I} \qquad \text{acting in } L^2(\mathbb{R}^d) \otimes \mathcal{F}$$

*is called Hamiltonian of Nelson's model in Fock space.*

We must say something about the domain of $H_\mathcal{F}$ in order to ensure that $H_\mathcal{F}$ is a densely defined, self-adjoint operator in $L^2(\mathbb{R}^d) \otimes \mathcal{F}$. Fortunately, most of the work has already been done in Proposition 1.3.4.

**Proposition 1.3.8** *The operator $H_0 = H_\mathrm{p} \otimes 1 + 1 \otimes H_\mathrm{f}$ is self-adjoint and nonnegative. Moreover, $H_\mathcal{F}$ is self-adjoint on $D(H_0)$ and bounded below. More precisely, $H_\mathrm{I}$ is infinitesimally Kato-bounded with respect to $H_0$.*

*Proof:* $H_0$ is self-adjoint and nonnegative because it is the sum of two commuting, self-adjoint, nonnegative operators. By Proposition 1.3.3 and (1.28), $D(H_\mathrm{I}) \supset D(H_0)$, and $H_I$ is seen to be symmetric on $D(1 \otimes H_\mathrm{f}) \supset D(H_0)$ by using (1.28) and proceeding as in Proposition 1.3.4. Let $\Psi \in D(H_0)$. We pick a version of $q \mapsto \Psi_q$ of $\Psi$, thus $\Psi_q \in \mathcal{F}$ for each $q \in \mathbb{R}^d$. Proposition 1.3.3 now gives for every $q \in \mathbb{R}^d$:

$$\begin{aligned}
\|H_\mathrm{I}\Psi_q\|_\mathcal{F}^2 &\leq 2\left(\left\|a^*\left(\frac{\hat{\varrho}e^{-ikq}}{2\omega}\sqrt{2\omega}\right)\Psi_q\right\|_\mathcal{F}^2 + \left\|a\left(\frac{\overline{\hat{\varrho}}e^{ikq}}{2\omega}\sqrt{2\omega}\right)\Psi_q\right\|_\mathcal{F}^2\right) \leq \\
&\leq 2\left(2\left\|\frac{\hat{\varrho}}{2\omega}\right\|_{L^2(\mathbb{R}^d)}^2 \langle\Psi_q, d\Gamma(2\omega)\Psi_q\rangle_\mathcal{F} + \left\|\frac{\hat{\varrho}}{\sqrt{\omega}}\right\|_{L^2}^2 \|\Psi_q\|_\mathcal{F}^2\right),
\end{aligned}$$

and integration over $q$ together with $H_\mathrm{p} \geq 0$ gives

$$\begin{aligned}
\|H_\mathrm{I}\Psi\|_{L^2(\mathbb{R}^d)\otimes\mathcal{F}}^2 &\leq 8\left\|\frac{\hat{\varrho}}{2\omega}\right\|_{L^2}^2 \langle\Psi, (1\otimes H_\mathrm{f})\Psi\rangle_{L^2(\mathbb{R}^d)\otimes\mathcal{F}} + 2\left\|\frac{\hat{\varrho}}{\sqrt{\omega}}\right\|_{L^2}^2 \|\Psi\|_{L^2(\mathbb{R}^d)\otimes\mathcal{F}}^2 \leq \\
&\leq 8\left\|\frac{\hat{\varrho}}{2\omega}\right\|_{L^2}^2 \langle\Psi, H_0\Psi\rangle_{L^2(\mathbb{R}^d)\otimes\mathcal{F}} + 2\left\|\frac{\hat{\varrho}}{\sqrt{\omega}}\right\|_{L^2}^2 \|\Psi\|_{L^2(\mathbb{R}^d)\otimes\mathcal{F}}^2 \leq \\
&\leq \varepsilon \|H_0\Psi\|_{L^2(\mathbb{R}^d)\otimes\mathcal{F}}^2 + \left(\frac{16\,\|\hat{\varrho}/2\omega\|_{L^2}^4}{\varepsilon} + 2\left\|\frac{\hat{\varrho}}{\sqrt{\omega}}\right\|_{L^2}^2\right)\|\Psi\|_{L^2(\mathbb{R}^d)\otimes\mathcal{F}}^2
\end{aligned}$$

for each $\varepsilon > 0$. Thus by the Kato-Rellich theorem ([41], Th. X.12), $H_\mathcal{F}$ is self-adjoint on $D(H_0)$ and bounded below. □

While the preceding proposition ensures that $H_{\mathcal{F}}$ is a well-behaved Hamiltonian, it says nothing about the existence of eigenfunctions for $H_{\mathcal{F}}$. In fact it it possible that $H_{\mathcal{F}}$ has no eigenfunctions in $L^2(\mathbb{R}^d) \otimes \mathcal{F}$ [30]. We will come back to this theme later on, cf. 1.5.15 and Section 3.1.

## 1.4 An infinite dimensional Ornstein-Uhlenbeck process

In this section we provide the probabilistic preparations which are necessary to give an alternative description of Nelson's model. We start by giving some basic facts about Gaussian measures.

**Definition 1.4.1** *Let $X$ be a locally convex vector space over $\mathbb{R}$, and let $\mathcal{X}$ be the $\sigma$-field generated by the continuous linear functionals $f \in X'$. A* **Gaussian measure** *$\gamma$ on $(X, \mathcal{X})$ is a probability measure on $(X, \mathcal{X})$ such that for each $f \in X'$, the image $\gamma \circ f^{-1}$ is a Gaussian measure on $\mathbb{R}$.*

It is important to keep in mind that by this very definition, Gaussian measures can only be defined naturally on a *real* vector space. Thus to make the connection with Fock space (which is a complex vector space), we will later study complex-valued functionals on $X$ instead of complexifying $X$.

The basic existence theorem for Gaussian measures on Hilbert spaces is [8]

**Theorem 1.4.2** *Let $\Phi$ be a separable, real Hilbert space, $a \in \Phi$ and $A : \Phi \supset D(A) \to \Phi$ a self-adjoint operator in $\Phi$ with $A > 0$.*

*a) If $A^{-1}$ is a Hilbert-Schmidt operator, then there exists a Gaussian measure $\gamma$ on $\Phi$ uniquely characterized by its Fourier transform*

$$\int e^{i\langle \phi, f\rangle_\Phi} \, d\gamma(\phi) = e^{i\langle a, f\rangle_\Phi} e^{-\frac{1}{2}\left\|A^{-1} f\right\|_\Phi^2}. \tag{1.32}$$

*b) If $A^{-1}$ is not a Hilbert-Schmidt operator, then a Gaussian measure on $\Phi$ with Fourier transform (1.32) does not exist.*

In case a) of the above theorem, $a$ is called the mean of $\gamma$ and the bilinear form

$$\Phi \times \Phi \to \mathbb{R}, \qquad \phi, \psi \mapsto \left\langle A^{-1}\phi, A^{-1}\psi \right\rangle_\Phi$$

is called the covariance of $\gamma$. Sometimes also $A^{-2}$ is called the covariance operator of $\gamma$.

Since the identity is not a Hilbert-Schmidt operator, Theorem 1.4.2 b) implies that a Gaussian measure on a Hilbert space with covariance given by the inner

product of the Hilbert space can not exist. However, given any operator $A > 0$ with Hilbert-Schmidt inverse, it is possible to construct a larger Hilbert space carrying the desired measure:

**Theorem 1.4.3** *Let $K$ be a separable real Hilbert space, and let $A > 0$ in $K$ such that $A^{-1}$ is Hilbert-Schmidt. Let $\Phi$ be the completion of $K$ with respect to the Hilbert norm*

$$\|\phi\|_\Phi := \left\|A^{-1}\phi\right\|_K \qquad \forall \phi \in K.$$

*Then for each $a \in \Phi$ there exists a unique Gaussian measure $\gamma$ on $\Phi$ such that for all $f \in D(A)$,*

$$\int_\Phi e^{i\phi(f)}\, d\gamma(\phi) = e^{ia(f)} e^{-\frac{1}{2}\|f\|_K^2}. \tag{1.33}$$

*Here, for $\phi \in \Phi$ we defined*

$$\phi(f) \equiv \langle f, \phi\rangle_K = \lim_{n\to\infty} \left\langle Af, A^{-1}\phi_n\right\rangle_K \tag{1.34}$$

*where $(\phi_n) \subset K$ is any sequence converging to $\phi$ in the norm of $\Phi$, similarly for $a(f)$.*

*Proof:* First, for $f \in D(A)$, the limit in (1.34) exists, is independent of the chosen sequence, and $\phi \mapsto \phi(f)$ is continuous with respect to the norm of $\Phi$. This follows from the Cauchy-Schwarz inequality in $K$.

Now, $A^{-1}$ is a Hilbert-Schmidt operator also on $\Phi$. To see this, note that for an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $K$ consisting of eigenvectors of $A^{-1}$, the sequence $(Ae_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $\Phi$, and

$$\sum_{i=1}^\infty \left\langle Ae_i, A^{-1}(Ae_i)\right\rangle_\Phi^2 = \sum_{i=1}^\infty \left\langle e_i, A^{-1}e_i\right\rangle_K^2 = \left\|A^{-1}\right\|_{\text{H.S.},K}^2 < \infty.$$

Now by Theorem 1.4.2 there exists a Gaussian measure $\gamma$ on $\Phi$ with

$$\int e^{i\langle\phi,\psi\rangle_\Phi}\, d\gamma(\phi) = e^{i\langle a,\psi\rangle_\Phi} e^{-\frac{1}{2}\left\|A^{-1}\psi\right\|_\Phi^2}$$

for each $\psi \in \Phi$. From (1.34) it can be easily deduced that for each $f \in D(A)$, the continuous linear form $\phi \mapsto \phi(f)$ is represented by the element $A^2 f \in \Phi$, i.e.

$$\phi(f) = \left\langle A^2 f, \phi\right\rangle_\Phi \qquad \forall \phi \in \Phi.$$

Here, $A^2$ is defined via the Riesz theorem and approximation like in (1.34). We now have

$$\begin{aligned}\int e^{i\phi(f)}\, d\gamma(\phi) &= \int e^{i\left\langle A^2 f,\phi\right\rangle_\Phi}\, d\gamma(\phi) = e^{i\left\langle A^2 f, a\right\rangle_\Phi} e^{-\frac{1}{2}\left\|A^{-1}A^2 f\right\|_\Phi^2} = \\ &= e^{ia(f)} e^{-\frac{1}{2}\|f\|_K^2},\end{aligned}$$

as desired. □

In case $f \in K \setminus D(A)$, $\phi \mapsto \phi(f)$ is not a bounded linear functional on $\Phi$. However, it has a chance of being an $L^p(\gamma)$-function for $p < \infty$.

**Lemma 1.4.4** *Consider the setting of Theorem 1.4.3 and the natural embedding $K \to \Phi$. If the mean $a$ of $\gamma$ is an element of $K$, then for each $f \in K$, the map $\phi \mapsto \phi(f)$ exists in $L^p(\gamma)$ for each $p$ with $1 \leq p < \infty$. In particular,*

$$\int \phi(f)^2 \, d\gamma(\phi) = a(f)^2 + \|f\|_K^2 \,. \tag{1.35}$$

*Proof:* Let $f_n \to f$ in $K$ with $f_n \in D(A)$ for all $n$. Then (1.33) implies that the image of $\gamma$ under $\phi(f_n - f_m)$ is the Gaussian measure with mean $a(f_n - f_m)$ and variance $\|f_n - f_m\|_K$. Explicit integration with respect to the image measure shows that $\phi \mapsto \phi(f_n)$ is an $L^p$-Cauchy sequence for each $1 \leq p < \infty$, implying existence of $\phi(f)$ in $L^p$ as well as (1.35). □

We are now ready to construct the infinite-dimensional Ornstein-Uhlenbeck process. For this purpose we need to make some extra assumptions on the Hilbert space $K$, and indeed we will specialize right away to the case of the Nelson model. Greater generality can be achieved, but it does not make things more transparent.

**Definition 1.4.5** *From now on, we will work with the following setup:*

*(i):* $\omega : \mathbb{R}^d \to \mathbb{R}$ *is measurable with* $0 \leq \omega(k) = \omega(-k)$, *and* $\omega(k) = 0$ *only on a set of Lebesgue-measure zero. This is the same* $\omega$ *we defined in (1.26) and (1.27).*

*(ii):* $\mathbf{B}$ *is the operator in* $L^2(\mathbb{R}^{d+1}, \mathbb{C})$ *given by*

$$(\mathbf{B}f)(\mathbf{k}) = \frac{\hat{f}(\mathbf{k})}{\sqrt{\omega(k)^2 + \kappa^2}} \qquad \forall f \in D(\mathbf{B}),$$

*with* $\mathbf{k} = (\kappa, k), \kappa \in \mathbb{R}, k \in \mathbb{R}^d$. *By (i),* $\mathbf{B}^*\mathbf{B}$ *maps real-valued functions into real-valued functions, and thus for real-valued* $f, g \in D(\mathbf{B})$,

$$\langle f, g \rangle_{\mathbf{K}} = \langle \mathbf{B}f, \mathbf{B}g \rangle_{L^2} = \int \overline{\hat{f}(\mathbf{k})} \frac{1}{\omega(k)^2 + \kappa^2} \hat{g}(\mathbf{k}) \, d\mathbf{k}$$

*defines an inner product on* $L^2(\mathbb{R}^{d+1}, \mathbb{R}) \cap D(\mathbf{B})$. *The real vector space obtained by completing this space with respect to* $\langle ., . \rangle_{\mathbf{K}}$ *will be denoted by* $\mathbf{K}$.

*(iii):* $\mathbf{\Phi}$ *is the Hilbert space constructed from* $\mathbf{K}$ *and a positive operator* $\mathbf{A}$ *with Hilbert-Schmidt inverse as described above.* $\mathbf{K}$ *is considered as a subspace of* $\mathbf{\Phi}$ *in the natural way.* $\gamma$ *is the Gaussian measure on* $\mathbf{\Phi}$ *with mean* 0 *and covariance given by the inner product in* $\mathbf{K}$ *as described in Theorem 1.4.3.*

*(iv): $B$ is the operator in $L^2(\mathbb{R}^d, \mathbb{C})$ with*

$$Bf(k) = \frac{\hat{f}(k)}{\sqrt{2\omega(k)}} \qquad \forall f \in D(B).$$

*As above, the map*

$$\langle f, g\rangle_K = \langle Bf, Bg\rangle_{L^2} = \int \overline{\hat{f}(k)} \frac{1}{2\omega(k)} \hat{g}(k)\, dk$$

*defines an inner product both on $D(B)$ and on the real vector space $L^2(\mathbb{R}^d, \mathbb{R}) \cap D(B)$. $K_{\mathbb{C}}$ will denote the completion of $D(B)$ with respect to this inner product, while $K$ will denote the completion of $L^2(\mathbb{R}^d, \mathbb{R}) \cap D(B)$. We will also employ the letter $B$ to denote the Hilbert space isomorphism $K_{\mathbb{C}} \to L^2(\mathbb{R}^d)$, $f \mapsto Bf$.*

*(v): $\Phi$ is the Hilbert space constructed from $K$ and a positive operator $A$ with Hilbert-Schmidt inverse, i.e. the completion of $K$ with respect to the norm*

$$\|g\|_\Phi = \left\|A^{-1} g\right\|_K, \qquad (g \in K).$$

*$K$ is considered as a subspace of $\Phi$ in the natural way.*

**Proposition 1.4.6** *For $t \in \mathbb{R}$ let $\delta_t$ denote the Dirac-distribution at $t$, i.e. $\delta_t(f) = f(t)$ for all continuous functions $f : \mathbb{R} \to \mathbb{R}$. If $g \in K$, then $\delta_t \otimes g \in \mathbf{K}$. Moreover,*

$$\langle \delta_s \otimes f, \delta_t \otimes g\rangle_{\mathbf{K}} = \int \overline{\hat{f}} \frac{1}{2\omega} e^{-|t-s|\omega} \hat{g}\, dk \tag{1.36}$$

*for all $f, g \in K$.*

*Proof:* Since $\widehat{\delta_t \otimes g}(\mathbf{k}) = \frac{1}{\sqrt{2\pi}} e^{it\kappa} \hat{g}(k)$ with $\mathbf{k} = (\kappa, k)$ as above, we have

$$\begin{aligned} \langle \delta_s \otimes f, \delta_t \otimes g\rangle_{\mathbf{K}} &= \int \frac{e^{i(t-s)\kappa}}{2\pi(\omega^2(k) + \kappa^2)} \overline{\hat{f}(k)} \hat{g}(k)\, d\mathbf{k} \\ &= \int \frac{e^{-|t-s|\omega(k)}}{2\omega(k)} \overline{\hat{f}(k)} \hat{g}(k)\, dk \leq \|f\|_K \|g\|_K . \end{aligned} \tag{1.37}$$

□

From this Proposition and Lemma 1.4.4 it follows that for each $t \in \mathbb{R}$ and each $g \in K$,

$$\mathbf{\Phi} \to \mathbb{R}, \qquad \phi \mapsto \phi_t(g) := \phi(\delta_t \otimes g)$$

exists as a $L^2(\gamma)$-function. The $\phi_t$ form a Gaussian Markov process, a fact that was already shown by Nelson [35]. Our aim is to show some sample path properties for this process, therefore we take a more concrete approach than he did. In particular, we will specify the state space of the process.

**Theorem 1.4.7** *Consider the setup from Definition 1.4.5. For each $n \in \mathbb{N}$ and $t_1 < t_2 < \ldots < t_n \in \mathbb{R}$ there exists a unique Gaussian measure $\mathcal{G}_{t_1,\ldots,t_n}$ on $\Phi^n$ such that*

$$\int F\left(\phi_1(g_1), \ldots, \phi_n(g_n)\right) \, d\mathcal{G}_{t_1,\ldots,t_n}(\vec{\phi}) = \int F\left(\phi(\delta_{t_1} \otimes g_1), \ldots, \phi(\delta_{t_n} \otimes g_n)\right) \, d\gamma(\phi) \tag{1.38}$$

*for all $g_1, \ldots, g_n \in D(A)$ and all integrable $F$. In the above formula, we wrote $\vec{\phi}$ for $(\phi_1, \ldots, \phi_n) \in \Phi^n$.*

*Proof:* By definition,

$$\int \exp\left(i \sum_{j=1}^{n} \phi(\delta_{t_j} \otimes g_j)\right) \, d\gamma(\phi) = \exp\left(-\frac{1}{2} \sum_{j,l=1}^{n} \left\langle \delta_{t_j} \otimes g_j, \delta_{t_l} \otimes g_l \right\rangle_{\mathbf{K}}\right). \tag{1.39}$$

What we have to show is that the right hand side of (1.39) is the Fourier transform of a (unique) Gaussian measure $\mathcal{G}_{t_1,\ldots,t_n}$ on $\Phi^n$. Once this is established, it will follow from the characterization theorem of the Fourier transform that the images of $\gamma$ under $\phi \mapsto (\phi(\delta_{t_1} \otimes g_1), \ldots, \phi(\delta_{t_n} \otimes g_n)$ and of $\mathcal{G}_{t_1,\ldots,t_n}$ under $(\phi_1, \ldots, \phi_n) \mapsto (\phi_1(g_1), \ldots, \phi_n(g_n))$ are one and the same Gaussian measure, implying (1.38) for general integrable $F$. To prove that (1.39) is indeed the Fourier transform of a Gaussian measure on $\Phi^n$, we have to show that there exists a Hilbert-Schmidt operator $\sqrt{L} > 0$ on $\Phi^n$ with

$$\left\langle \vec{\phi}, L\vec{\psi} \right\rangle_{\Phi^n} = \sum_{j,l=1}^{n} \left\langle \delta_{t_j} \otimes A^{-2}\phi_j, \delta_{t_l} \otimes A^{-2}\psi_l \right\rangle_{\mathbf{K}}.$$

Since we found $\phi(g) = \langle A^2 g, \phi \rangle_{\Phi}$ in the proof of Theorem 1.4.3, this will be sufficient. To find $\sqrt{L}$, let $g, h \in K$ and $s, t \in \mathbb{R}$, then by by (1.37)

$$\left\langle \delta_t \otimes g, \delta_s \otimes h \right\rangle_{\mathbf{K}} \leq \|g\|_K \, \|h\|_K,$$

and thus

$$\begin{aligned} &\sum_{j,l=1}^{n} \left\langle \delta_{t_j} \otimes A^{-2}g_j, \delta_{t_l} \otimes A^{-2}h_l \right\rangle_{\mathbf{K}} \leq \\ &\leq \sum_{j,l=1}^{n} \left\|A^{-1}g_j\right\|_{\Phi} \left\|A^{-1}h_l\right\|_{\Phi} \leq n \left\|\vec{A}^{-1}\vec{g}\right\|_{\Phi^n} \left\|\vec{A}^{-1}\vec{h}\right\|_{\Phi^n}, \end{aligned} \tag{1.40}$$

where $\vec{g}, \vec{h} \in \Phi^n$, $\vec{A}^{-1}\vec{g} = (A^{-1}g_1, \ldots, A^{-1}g_n)$, $\Phi^n$ is equipped with its Hilbert norm.

(1.40) implies the existence of a bounded operator $L : \Phi^n \to \Phi^n$ with

$$\left\langle \vec{\phi}, L\vec{\psi} \right\rangle_{\Phi^n} = \sum_{j,l=1}^{n} \left\langle \delta_{t_j} \otimes A^{-2}\phi_j, \delta_{t_l} \otimes A^{-2}\psi_l \right\rangle_{\mathbf{K}}.$$

Obviously, $L$ is symmetric and therefore self-adjoint. Moreover,

$$\mathbb{C} \ni (z_1, \ldots, z_n) \mapsto \sum_{j,l=1}^{n} z_j \bar{z}_l e^{-|t_j - t_l|}$$

is positive definite (see e.g. [45], Lemma 4.4), and thus $L > 0$ and $\sqrt{L}$ exists and is strictly positive. Furthermore, (1.40) shows that $\left\|\sqrt{L}\vec{\phi}\right\|_{\Phi^n} \leq \sqrt{n}\left\|\vec{A}^{-1}\vec{\phi}\right\|_{\Phi^n}$, and since $\vec{A}^{-1}$ is Hilbert-Schmidt on $\Phi^n$, $\sqrt{L}$ is also a Hilbert-Schmidt operator. □

The preceding theorem enables us to make the following

**Definition 1.4.8** *The $\Phi$-valued Ornstein-Uhlenbeck process with mean 0 and covariance*

$$\mathrm{Cov}_{s,t}(f,g) = \int \overline{\hat{f}(k)} \frac{1}{2\omega(k)} e^{-|t-s|\omega(k)} \hat{g}(k)\, dk \qquad (f, g \in K)$$

*is the Gaussian measure $\mathcal{G}$ on $\Phi^{\mathbb{R}}$ with mean 0 and*

$$\int \phi_s(f)\phi_t(g)\, d\mathcal{G}(\phi) = \int \overline{\hat{f}(k)} \frac{1}{2\omega(k)} e^{-|t-s|\omega(k)} \hat{g}(k)\, dk \qquad (f, g \in K).$$

*Here, $\phi_t$ denotes the projection of $\phi \in \Phi^{\mathbb{R}}$ onto the $t$-th factor.*
*The stationary measure of $\mathcal{G}$, i.e. the image of $\mathcal{G}$ under the projection $\phi \mapsto \phi_t$, is denoted by $\mathsf{G}$.*

Indeed, Theorem 1.4.7 ensures the existence of the "finite"-dimensional distributions in $\Phi^n$, while consistency follows directly from (1.38), and thus $\mathcal{G}$ exists according to Kolmogorov's existence theorem.

We called $\mathcal{G}$ an Ornstein-Uhlenbeck process, thus suggesting that it is a Markov process. This is indeed the case, as we will see now. We start with a formula describing certain conditional expectations under $\gamma$.

**Lemma 1.4.9** *Let $\mathbf{K}_0 \subset \mathbf{K}$ be a closed subspace, let $P : \mathbf{K} \to \mathbf{K}_0$ be the corresponding orthogonal projection, and let $\mathcal{F}_P$ be the $\sigma$-field generated by $\{\phi \mapsto \phi(f) : f \in \mathbf{K}_0\}$. Then for each $\alpha \in \mathbb{C}, f \in \mathbf{K}$,*

$$\mathbb{E}_\gamma(e^{\alpha\phi(f)}|\mathcal{F}_P)(\bar{\phi}) = e^{\alpha\bar{\phi}(Pf)} e^{\frac{\alpha^2}{2}\|f-Pf\|^2_{\mathbf{K}}},$$

*where equality is to be understood in $L^1(\gamma)$.*

*Proof:* First of all, indeed $\phi \mapsto e^{\alpha\phi(f)}$ is in $L^1(\gamma)$ for each $f \in \mathbf{K}$, with integral equal to $\exp(\alpha^2 \|f\|^2_{\mathbf{K}}/2)$. Now

$$\begin{aligned} \mathbb{E}_\gamma\left(e^{\alpha\phi(f)}|\mathcal{F}_P\right)(\bar{\phi}) &= \mathbb{E}_\gamma\left(e^{\alpha\phi(Pf)}e^{\alpha\phi(f-Pf)}|\mathcal{F}_P\right)(\bar{\phi}) = e^{\alpha\bar{\phi}(Pf)}\mathbb{E}_\gamma\left(e^{\alpha\phi(f-Pf)}|\mathcal{F}_P\right)(\bar{\phi}) \\ &= e^{\alpha\bar{\phi}(Pf)}\mathbb{E}_\gamma\left(e^{\alpha\phi(f-Pf)}\right) = e^{\alpha\bar{\phi}(Pf)}e^{\frac{\alpha^2}{2}\|f-Pf\|^2_{\mathbf{K}}}. \end{aligned} \tag{1.41}$$

Equalities in the above equation are in $L^2(\gamma)$, and the third equality is due to the fact that independence with respect to $\gamma$ is equivalent with orthogonality in $L^2(\gamma)$. □

We now fix a notation that we will use throughout: For an interval $I \subset \mathbb{R}$, $\mathcal{F}_I$ is the $\sigma$-field over $\Phi^{\mathbb{R}}$ generated by $\{\phi_t(f) : t \in I, f \in K\}$.

**Proposition 1.4.10** *$\mathcal{G}$ is the path measure of a Markov process with state space $\Phi$, where the Markov property holds with respect to the canonical filtration $(\mathcal{F}_{[-\infty,t]})_{t\in\mathbb{R}}$.*

*Proof:* Denote by $\mathbf{K}_{]-\infty,0]}$ the closed subspace of $\mathbf{K}$ generated by $\{\delta_t \otimes f : t \leq 0, f \in K\}$, and by $\mathbf{K}_{\{0\}}$ the closed subspace generated by $\{\delta_0 \otimes f : f \in K\}$. Write $P_{]-\infty,0]}$ and $P_{\{0\}}$ for the corresponding projections. We claim that for each $t \geq 0$ and each $g \in K$,

$$P_{]-\infty,0]}(\delta_t \otimes g) = \delta_0 \otimes e^{-\frac{t}{2}|B|^{-2}} g = P_{\{0\}}(\delta_t \otimes g). \tag{1.42}$$

Indeed, $\delta_0 \otimes e^{-\frac{t}{2}|B|^{-2}} g \in \mathbf{K}_{]-\infty,0]}$,

$$(\widehat{|B|^{-2} f}) = ((B^*B)^{-1} f)\hat{} = 2\omega \hat{f},$$

consequently $(\exp(-\frac{t}{2}|B|^{-2})f)\hat{} = e^{-t\omega}\hat{f}$, and for each $s \leq 0$,

$$\begin{aligned}
\langle \delta_s \otimes f, \delta_t \otimes g \rangle_{\mathbf{K}} &= \int \frac{1}{2\omega} e^{-|t-s|\omega} \overline{\hat{f}} \hat{g}\, dk = \\
&= \int \frac{1}{2\omega} e^{-|s|\omega} \overline{\hat{f}} \left(e^{-\frac{t}{2}|B|^{-2}} g\right)\hat{}\, dk = \\
&= \left\langle \delta_s \otimes f, \delta_0 \otimes e^{-\frac{t}{2}|B|^{-2}} g \right\rangle_{\mathbf{K}}.
\end{aligned}$$

From this we get

$$\langle F, \delta_t \otimes g \rangle_{\mathbf{K}} = \left\langle F, \delta_0 \otimes e^{-\frac{t}{2}|B|^{-2}} g \right\rangle_{\mathbf{K}} \qquad \forall F \in \mathbf{K}_{]-\infty,0]}$$

by linearity and approximation, and thus the first equality in (1.42) is shown. The second equality there now follows from the fact that $\delta_0 \otimes e^{-\frac{t}{2}|B|^{-2}} g$ is not only in $\mathbf{K}_{]-\infty,0]}$ but even in $\mathbf{K}_{\{0\}}$.

Now by Lemma 1.4.9 we have

$$\mathbb{E}_\gamma\left(e^{\alpha\phi(\delta_t\otimes g)}\middle|\mathcal{F}_{P_{]-\infty,0]}}\right)(\bar{\phi}) = e^{\alpha\bar{\phi}(P_{]-\infty,0]}\delta_t\otimes g)} \exp\left(\frac{\alpha^2}{2}\left\|(1-P_{]-\infty,0]})\delta_t \otimes g\right\|^2_{\mathbf{K}}\right),$$

and (1.38) gives

$$\mathbb{E}_{\mathcal{G}}\left(e^{\alpha\phi_t(g)}\middle|\mathcal{F}_{]-\infty,0]}\right)(\bar{\phi}) = e^{\bar{\phi}_0(e^{-\frac{t}{2}|B|^{-2}}g)} \exp\left(\frac{\alpha^2}{2}\left\langle g, (1-e^{-t|B|^{-2}}g\right\rangle_K\right) \in \mathcal{F}_{\{0\}}$$

for each $t \geq 0$. $\mathcal{F}_{\{0\}}$ and $\mathcal{F}_{]-\infty,0]}$ below denote the $\sigma$-fields generated by the point evaluations for the corresponding points $t$, cf. Chapter 2 for more explanation. Again by approximation, we find

$$\mathbb{E}_{\mathcal{G}}\left(F(\phi)\middle|\mathcal{F}_{]-\infty,0]}\right) \in \mathcal{F}_{\{0\}}$$

for all integrable, $\mathcal{F}_{[0,\infty[}$-measurable functions, proving our claim. □

The method used in the above proof to calculate conditional expectations will be used again in Chapter 3. We should also note that the Markov property alternatively follows directly from (1.38) and the results in [35].

The last task in this section is to establish continuity of sample paths for $\mathcal{G}$. The following constitutes a refinement of results in [30].

**Proposition 1.4.11** *In addition to the setup from Definition 1.4.5, assume that $|B|^{-1}A^{-1}$ is a Hilbert-Schmidt operator on $K$. Then for $\mathcal{G}$-almost every $\phi \in \Phi^{\mathbb{R}}$, the path $\mathbb{R} \to \Phi, t \mapsto \phi_t$ is continuous.*

*Proof:* Using Kolmogorov's criterion for continuity (see e.g. [26]) we only need to show that

$$\int \|\phi_s - \phi_t\|_\Phi^4 \, d\mathcal{G}(\phi) \leq C|t-s|^2 \tag{1.43}$$

for some $C > 0$. Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of $K$ consisting of eigenvectors of $A$, with $Ae_i = \lambda_i e_i$. Then $(\lambda_i e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $\Phi$, and thus for $\phi_t \in \Phi$,

$$\begin{aligned}\|\phi_t\|_\Phi^4 &= \left(\sum_{i=1}^\infty \lambda_i^2 \langle e_i, \phi_t\rangle_\Phi^2\right)^2 = \\ &= \sum_{i,j=1}^\infty \lambda_i^2\lambda_j^2\phi_t(A^{-2}e_i)^2\phi_t(A^{-2}e_j)^2 = \sum_{i,j=1}^\infty \lambda_i^{-2}\lambda_j^{-2}\phi_t(e_i)^2\phi_t(e_j)^2.\end{aligned}$$

Note that $\phi_t \mapsto \phi_t(e_i) = \langle A^2 e_i, \phi_t\rangle_\Phi$ is defined for every $\phi_t \in \Phi$ since $e_i \in D(A^2)$. By monotone convergence and the Cauchy-Schwarz inequality,

$$\int \|\phi_s - \phi_t\|_\Phi^4 \, d\mathcal{G}(\phi) \leq \left(\sum_{i=1}^\infty \lambda_i^{-2}\left(\int ((\phi_t - \phi_s)(e_i))^4 \, d\mathcal{G}(\phi)\right)^{1/2}\right)^2 = (*).$$

By (1.38) and integration with respect to the Gaussian image measure,

$$\begin{aligned}\left(\int ((\phi_t - \phi_s)(e_i))^4 \, d\mathcal{G}(\phi)\right)^{1/2} &= \left(\int (\phi((\delta_t - \delta_s)\otimes e_i))^4 \, d\gamma(\phi)\right)^{1/2} = \\ = \sqrt{3}\,\|(\delta_t - \delta_s)\otimes e_i\|_{\mathbf{K}}^2 &= 2\sqrt{3}\int \frac{1 - e^{-|t-s|\omega(k)}}{2\omega(k)}|\hat{e}_i(k)|^2\,dk = (**).\end{aligned}$$

Since $|B|^{-1}A^{-1}$ is bounded, $e_i \in D(|B|^{-1})$ for all $i$, and

$$(**) = \sqrt{3}\int \frac{1-e^{-|t-s|\omega(k)}}{\omega(k)} \frac{|\widehat{|B|^{-1}e_i}(k)|^2}{2\omega(k)}\, dk \leq \sqrt{3}|t-s| \left\||B|^{-1}e_i\right\|_K^2 .$$

The last inequality holds because the first fraction in the integral is bounded by $|t-s|$. We conclude

$$\begin{aligned}(*) &\leq 3|t-s|^2 \left(\sum_{j=1}^{\infty} \lambda_j^{-2} \left\||B|^{-1}e_j\right\|_K^2\right)^2 = \\ &= 3|t-s|^2 \left(\sum_{j=1}^{\infty} \left\||B|^{-1}A^{-1}e_j\right\|_K^2\right)^2 = 3|t-s|^2 \left\||B|^{-1}A^{-1}\right\|_{\text{H.S.}}^4 ,\end{aligned}$$

finishing the proof. □

Since we have considerable freedom in choosing our operator $A$, it is no problem to find $A$ such that both $A$ and $A|B|$ have Hilbert-Schmidt inverse. Indeed, take an orthonormal basis $(e_j)$ of $K$ contained in $D(|B|^{-1})$ and define

$$Ae_j = j \max\{\left\||B|^{-1}e_j\right\|_K, 1\}e_j.$$

Then $\||B|^{-1}A^{-1}e_j\|_K \leq 1/j$ and $\|A^{-1}e_j\|_K \leq 1/j$, showing that both $A$ and $A|B|$ have Hilbert-Schmidt inverse.

## 1.5 Nelson's model in function space

We will now use the Gaussian process from the previous section to obtain a unitarily equivalent representation of $H_{\mathcal{F}}$ as an operator in $L^2(\mathsf{P}_0 \otimes \mathsf{G})$. To do so, we will use an isomorphism between $\mathcal{F}$ and $L^2(\mathsf{G})$, and since $\mathcal{F}$ is a complex vector space, we also need to consider $L^2(\mathsf{G})$ as a complex space. For $f \in K_{\mathbb{C}}$, $f = g + ih$ with $g, h \in K$, we define

$$\phi(f) := \phi(g) + i\phi(h),$$

where $\phi(g)$ and $\phi(h)$ are as in Lemma 1.4.4. For $f, g \in K_{\mathbb{C}}$, we find

$$\langle \phi(f), \phi(g)\rangle_{L^2(\mathsf{G})} = \int \overline{\phi(f)}\phi(g)\, d\mathsf{G}(\phi) = \int \overline{Bf}Bg\, dk = \langle f, g\rangle_{K_{\mathbb{C}}} . \tag{1.44}$$

To introduce the isomorphism mentioned above, we need the following functions:

**Definition 1.5.1** *Let $f, f_1, \ldots, f_n \in K_{\mathbb{C}}$. The* **Wick polynomial** *of order $n$ is the $L^2(\mathsf{G})$-function defined recursively by*

$$
\begin{aligned}
:\phi(f)^0: &= 1, && (1.45)\\
:\phi(f): &= \phi(f), && (1.46)\\
:\phi(f_1)\cdots\phi(f_n): &= :\phi(f_1)\cdots\phi(f_{n-1}):\ \phi(f_n) - \\
&\quad - \sum_{i=1}^{n-1} \left( \int \phi(f_i)\phi(f_n)\, d\mathsf{G}(\phi) \right) :\prod_{j\neq i}^{n-1} \phi(f_j): \ . && (1.47)
\end{aligned}
$$

**1.5.2 Remark:**

a) Definition 1.5.1 has the virtue of being fairly explicit, and we will need especially (1.47) later on. However, more insight in the significance of Wick polynomials is gained by defining them as follows: Let $\mathrm{Pol}(n)$ be the closed subspace of $L^2(\mathsf{G})$ spanned by the polynomials in $\phi(f), f \in K$ of degree $\leq n$. Let $\Pi(0) = \mathrm{Pol}(0) = 1$, and let $\Pi(n)$ be the $L^2(\mathsf{G})$-orthogonal complement of $\Pi(n-1)$ in $\mathrm{Pol}(n)$, i.e. $\mathrm{Pol}(n) = \bigoplus_{k=0}^{n} \Pi(k)$ for each $n$. The Wick polynomial $:\phi(f_1)\ldots\phi(f_n):$ is then defined as the orthogonal projection in $L^2(\mathsf{G})$ of $\phi(f_1)\ldots\phi(f_n)$ onto $\Pi(n)$. Thus the Wick polynomials arise from the ordinary polynomials by an Gram-Schmidt orthonormalization procedure. Equivalence of the above description to Definition 1.5.1 can be shown by using Hermite polynomials [22].

b) Of course, the $\mathsf{G}$ we use in Definition 1.5.1 plays no special role there and can be replaced by another Gaussian measure. However, the coefficients of the Wick polynomials do depend on the Gaussian measure that is used, a dependence that is suppressed in the notation.

c) In (1.47), $\int \phi(f_i)\phi(f_j)\, d\mathsf{G}$ and not $\langle \phi(f), \phi(g) \rangle_{L^2(\mathsf{G})}$ is used. At first glance, the latter choice might seem more natural, but since the scalar product is antilinear in the first factor, the Wick polynomials would then not be $n$-linear in the $f_i$, i.e. would not even be polynomials any more.

d) It is also possible to express (1.47) using the scalar product in $K_{\mathbb{C}}$, namely

$$
:\phi(f_1)\cdots\phi(f_n): = :\phi(f_1)\cdots\phi(f_{n-1}):\ \phi(f_n) - \sum_{i=1}^{n-1} \left\langle \bar{f}_i, f_n \right\rangle_{K_{\mathbb{C}}} :\prod_{j\neq i}^{n-1} \phi(f_j): \ . \tag{1.48}
$$

Note that the usual complex conjugation $f \mapsto \bar{f}$ is in fact a conjugation in $K_{\mathbb{C}}$, since $\left\langle \bar{f}, g \right\rangle_{K_{\mathbb{C}}} = \overline{\langle f, \bar{g} \rangle_{K_{\mathbb{C}}}}$. This can be seen either by using (1.44) with $\bar{f}$ instead of $f$, or directly by virtue of the fact that $B^*B$ maps real

functions into real functions. Another conjugation on $K_{\mathbb{C}}$ is given by $f^* = B^{-1}\overline{Bf}$, i.e. $\langle f^*, g\rangle_{K_{\mathbb{C}}} = \langle \overline{Bf}, Bg\rangle_{L^2(\mathbb{R}^d)}$. In contrast to $\bar{f}$, $f^*$ continues to be a conjugation when we drop the condition that $B^*B$ must map real functions into real functions, and for this reason may be more natural at first glance. However, for our purposes $f^*$ is the wrong choice.

The following proposition is essentially a restatement of Remark 1.5.2 a). A proof can be found in [22].

**Proposition 1.5.3**

*a) For $n, m \in \mathbb{N}$ and $f_1, \ldots, f_n, g_1, \ldots, g_n \in K_{\mathbb{C}}$, we have*

$$\int :\phi(f_1)\ldots\phi(f_n): :\phi(g_1)\ldots\phi(g_m): d\mathsf{G}(\phi) = \delta_{n,m} \sum_{\pi\in\Pi(n)} \langle \bar{f}_1, g_{\pi(1)}\rangle_{K_{\mathbb{C}}} \cdots \langle \bar{f}_n, g_{\pi(n)}\rangle_{K_{\mathbb{C}}}, \tag{1.49}$$

*where $\Pi(n)$ is the set of all permutations on $\{1, \ldots, n\}$.*

*b) The set $\{ :\phi(f_1)\cdots\phi(f_n): \, | f_1, \ldots, f_n \in D(A), n \in \mathbb{N}\}$ is total in $L^2(\mathsf{G})$.*

Now if we remember Proposition 1.3.5 and compare (1.49) with (1.14) and (1.23), we are led to

**1.5.4 The Wiener-Itô-Segal isomorphism:** Define

$$\theta : L^2(\mathsf{G}) \ni :\phi(f_1)\ldots\phi(f_n): \mapsto \sqrt{n!}(Bf_1 \hat{\otimes} \ldots \hat{\otimes} Bf_n) \in \mathcal{F}^{(n)},$$

and extend by linearity and density. Then $\theta : L^2(\mathsf{G}) \to \mathcal{F}$ is an isomorphism, the Wiener-Itô-Segal isomorphism.

We now investigate how the various Fock space operators transform under $\theta$. By Proposition 1.5.3 b), we may restrict our attention to their action on Wick polynomials.

**Proposition 1.5.5** *Let $f_1, \ldots, f_n \in K$, $g \in L^2$. Then*

$$\theta^{-1}\mathbf{N}\theta \; :\phi(f_1)\ldots\phi(f_n): \;=\; n \; :\phi(f_1)\ldots\phi(f_n):, \tag{1.50}$$

$$\theta^{-1}a^*(g)\theta \; :\phi(f_1)\ldots\phi(f_n): \;=\; :\phi(f_1)\ldots\phi(f_n)\phi(B^{-1}g):, \tag{1.51}$$

$$\theta^{-1}a(g)\theta \; :\phi(f_1)\ldots\phi(f_n): \;=\; \sum_{i=1}^{n} \langle B^{-1}\bar{g}, f_i\rangle_{K_{\mathbb{C}}} \; :\prod_{j\neq i}^{n-1} \phi(f_j): \; . \tag{1.52}$$

*Let $L$ be an operator in $L^2(\mathbb{R}^d, \mathbb{C})$ such that $Bf_1, \ldots, Bf_n \in D(L)$. Then the Wick polynomial $:\phi(f_1)\ldots\phi(f_n):$ is in the domain of the operators below, and*

$$\theta^{-1}\Gamma(L)\theta \; :\phi(f_1)\ldots\phi(f_n): \;=\; :\phi(B^{-1}LBf_1)\ldots\phi(B^{-1}LBf_n):, \tag{1.53}$$

$$\theta^{-1}d\Gamma(L)\theta \; :\phi(f_1)\ldots\phi(f_n): \;=\; \sum_{i=1}^{n} :\phi(B^{-1}LBf_i)\prod_{j\neq i}^{n} \phi(f_j): \; . \tag{1.54}$$

*Proof:* All the proofs are straightforward chasing of definitions. As an example, we show (1.52):

$$\begin{aligned}\theta^{-1}a(g)\theta\ {:}\phi(f_1)\dots\phi(f_n){:} &= \sqrt{n!}a(g)(Bf_1\hat{\otimes}\dots\hat{\otimes}Bf_n) = \\ &= \sqrt{n!}\theta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\langle\bar{g},Bf_i\rangle_{L^2}\widehat{\bigotimes_{j\neq i}}Bf_j = \\ &= \sum_{i=1}^{n}\left\langle B^{-1}\bar{g},f_i\right\rangle_{K_{\mathbb{C}}}{:}\prod_{j\neq i}^{n}\phi(f_j){:}\ .\end{aligned}$$

□

**Corollary 1.5.6** *Let $g\in L^2(\mathbb{R}^d,\mathbb{C})$ be the $B$-image of a real-valued function, i.e. et $B^{-1}g$ be real valued. Then*

$$\theta^{-1}(a(\bar{g})+a^*(g))\theta\ {:}\phi(f_1)\dots\phi(f_n){:}={:}\phi(f_1)\dots\phi(f_n){:}\ \phi(B^{-1}g).$$

*Proof:* By assumption, $B^{-1}\bar{\bar{g}}=B^{-1}g=\overline{B^{-1}g}$ and the claim now follows from (1.51), (1.52) and (1.47). □

In order to write down the Nelson Hamiltonian in function space, we need one last ingredient: $H_{\mathcal{F}}$ is an operator in $L^2(\mathbb{R}^d)\otimes\mathcal{F}$, and since we up to now only have the isomorphism $\theta:\mathcal{F}\to L^2(\mathsf{G})$, we still need to transform the first factor. We could of course use the identity there, but will chose the ground state transform from Appendix A.2 instead. This has the benefit that the transformed Hamiltonian will be the sum of the generator of a stochastic process and a multiplication operator.

Recall that the ground state transform is given by

$$\psi_0: L^2(\mathsf{N}_0)\equiv L^2(\psi_0^2\,dx)\to L^2(\mathbb{R}^d),\qquad f\mapsto\psi_0 f,$$

where $\psi_0$ is the strictly positive, $L^2$-normalized ground state of $H_{\mathrm{p}}$. From now on, we will always use $\mathsf{N}_0$ to denote the measure $\psi_0^2dx$. Moreover, the measure $\mathsf{N}_0\otimes\mathsf{G}$ will be denoted by $\mathsf{P}_0$.

We write

$$\Theta\equiv\psi_0\otimes\theta: L^2(\mathsf{P}_0)\to L^2(\mathbb{R}^d)\otimes\mathcal{F},$$

and define the Nelson Hamiltonian in function space by

$$H:=\Theta^{-1}H_{\mathcal{F}}\Theta.$$

By construction, $H$ is unitarily equivalent to $H_{\mathcal{F}}$, and like the latter is a sum of three terms. We now give the action of $H$ on the Wick polynomials, investigating each term separately.

**Proposition 1.5.7** *Let $f_1, \dots, f_n \in K, g \in L^2(\mathsf{N}_0)$. Whenever the $f_i$ and $g$ are in the domains of the operators appearing below,*

$$(\Theta^{-1}(H_{\rm p} \otimes 1)\Theta)(g \otimes :\phi(f_1)\dots\phi(f_n):) = Lg \otimes :\phi(f_1)\dots\phi(f_n):, \tag{1.55}$$

$$(\Theta^{-1}(1 \otimes H_{\rm f})\Theta)(g \otimes :\phi(f_1)\dots\phi(f_n):) = g \otimes \sum_{i=1}^{n} :\phi\left(\frac{1}{2}|B|^{-2}f_i\right)\prod_{j\neq i}^{n}\phi(f_j):, \tag{1.56}$$

$$(\Theta^{-1}H_{\rm I}\Theta)(g \otimes :\phi(f_1)\dots\phi(f_n):)(q) = g(q)\ :\phi(f_1)\dots\phi(f_n):\ \phi(\varrho_q), \tag{1.57}$$

*where $\varrho_q = \varrho(.-q)$ and $Lg = -\frac{1}{2}\Delta g - \left\langle \frac{\nabla\psi_0}{\psi_0}, \nabla g\right\rangle_{\mathbb{R}^d}$.*

*Proof:* (1.55) comes from Appendix A.2. (1.56) follows from (1.29), (1.54) and the equalities $2\omega(k)f(k) = (BB^*)^{-1}f$ and $B^{-1}(BB^*)^{-1}B = |B|^{-2}$. For (1.57), note that $H_{\rm I} = a^*(B\varrho_q) + a(\overline{B\varrho_q})$ and that $B\varrho_q$ is the $B$-image of the real-valued function $\varrho_q$, and use Corollary 1.5.6. □

In [4], $\phi * \varrho$ is used instead of $\phi(\varrho_q)$. We avoid this notation, because $f * g$ usually denotes convolution with respect to Lebesgue measure, i.e. $f * g(x) = \int f(x-y)g(y)\,dy$, while $\phi(\varrho_q) = \langle \phi, \varrho_q\rangle_K$.

To complete the comparison between this section and the previous one, note that condition (1.28) in the language of this section reads $\varrho \in K$ and $|B|\varrho \in K$, and that by the choice of $K$ we have made, this is also true for $\varrho_q$ with arbitrary $q \in \mathbb{R}^d$.

The next step in our program is to show that $\theta^{-1}H_{\mathcal{F}}\theta$ is the generator of the Ornstein-Uhlenbeck process $\mathcal{G}$. For this purpose, it is convenient to introduce yet another class of functions in $L^2(\mathsf{G})$.

**Definition 1.5.8** *The* **Wick exponential** *corresponding to $f \in K_{\mathbb{C}}$ is given by*

$$:\exp(\phi(f)): = \sum_{n=0}^{\infty}\frac{1}{n!}\ :\phi(f)^n:\ .$$

Like Wick polynomials, Wick exponentials have many nice properties.

**Proposition 1.5.9** *Let $f, g \in K_{\mathbb{C}}$. Then*

*a)* $:\exp(\phi(f)): = \exp(\phi(f))e^{-\frac{1}{2}\langle \bar{f}, f\rangle_{K_{\mathbb{C}}}}$,

*b)* $\langle :\exp(\phi(f)):, :\exp(\phi(g)):\rangle_{L^2(\mathsf{G})} = e^{\langle f,g\rangle_{K_{\mathbb{C}}}}$.

*c) For a self-adjoint operator $L$ in $L^2(\mathbb{R}^d)$ with $f \in D(B^{-1}LB)$,*

$$\theta^{-1}\Gamma(L)\theta\ :\exp(\phi(f)): = :\exp(\phi(B^{-1}LBf)):\ .$$

*d) For a self-adjoint operator $L$ in $L^2(\mathbb{R}^d)$ with $g \in D(B^{-1}LB)$,*

$$\left\langle :\exp(\phi(f)):, \theta^{-1}d\Gamma(L)\theta \ :\exp(\phi(g)):\right\rangle_{L^2(\mathsf{G})} = \left\langle f, B^{-1}LBg\right\rangle_{K_{\mathbb{C}}} e^{\langle f,g\rangle_{K_{\mathbb{C}}}}.$$

*e) The set $\{ :\exp(\phi(f)): : f \in D(A)\}$ is total in $L^2(\mathsf{G})$.*

*Proof:* b), c) and d) follow by checking definitions. a) is a consequence of the relationship of Wick polynomials with Hermite polynomials, and e) is an application of the Stone-Weierstrass theorem. For details see [22]. □

**Theorem 1.5.10** *The operator $\theta^{-1}H_{\mathrm{f}}\theta$ is the generator of the process $\mathcal{G}$.*

*Proof:* According to Proposition 1.5.9 a), for $f, g \in K$ and $t > 0$ we have

$$\begin{aligned}\left\langle :\exp(\phi_0(f)):, :\exp(\phi_t(g)):\right\rangle_{L^2(\mathcal{G})} &= e^{-\frac{1}{2}(\|f\|_K^2+\|g\|_K^2)} \int \exp(\phi_0(f)+\phi_t(g))\, d\mathcal{G} = \\ = \ e^{-\frac{1}{2}(\|f\|_K^2+\|g\|_K^2)} &\exp\left(\frac{1}{2}\left\|\delta_0\otimes f + \delta_t \otimes g\right\|_{\mathbf{K}}^2\right) = \exp\left(\left\langle f, e^{-\frac{t}{2}|B|^{-2}}g\right\rangle_K\right).\end{aligned}$$

Thus the action of the Dirichlet form $\mathcal{E}$ corresponding to $\mathcal{G}$ on the Wick exponentials is given by

$$\begin{aligned}\mathcal{E}(:\exp(\phi_0(f)):, :\exp(\phi_0(g)):) &= \frac{d}{dt}\left\langle :\exp(\phi_0(f)):, :\exp(\phi_t(g)):\right\rangle_{L^2(\mathcal{G})}\Big|_{t=0} = \\ &= \frac{d}{dt}\exp\left(\left\langle f, e^{-\frac{1}{2}t|B|^{-2}}g\right\rangle_K\right)\Big|_{t=0} = \\ &= -\left\langle f, \frac{1}{2}|B|^{-2}g\right\rangle_K e^{\langle f,g\rangle_K},\end{aligned}$$

By 1.5.9 d), also

$$\left\langle :\exp(\phi_0(f)):, \theta^{-1}H_{\mathrm{f}}\theta \ :\exp(\phi_0(g)):\right\rangle_{L^2(\mathsf{G})} = \left\langle f, \frac{1}{2}|B|^{-2}g\right\rangle_K e^{\langle f,g\rangle_K}.$$

Since the span of the Wick exponentials is dense, it follows that $\theta^{-1}H_{\mathrm{f}}\theta$ is the generator of the Dirichlet form corresponding to $\mathcal{G}$, hence the generator of the process $\mathcal{G}$. □

The proof of the last theorem also allows us to calculate the action of the transition semigroup $P_t$ corresponding to $\mathcal{G}$ on the Wick exponentials as

$$P_t \ :\exp(\phi(f)):=:\exp(\phi(e^{-\frac{t}{2}|B|^{-2}}f)): \ .$$

Let us pause to summarize what we have achieved so far. Write $\mathcal{N}_0$ for the path measure of the $P(\phi)_1$-process corresponding to $H_{\rm p}$ and put

$$\begin{aligned} \mathcal{P}_0 &= \mathcal{N}_0 \otimes \mathcal{G}, \\ \mathsf{P}_0 &= \mathsf{N}_0 \otimes \mathsf{G}, \\ H_0 &= \Theta^{-1}(H_{\rm p} \otimes 1 + 1 \otimes H_{\rm f})\Theta, \\ H_\varrho &= \Theta^{-1} H_{\rm I} \Theta. \end{aligned}$$

We have seen that $H_0$ is the generator of the Markov process with path measure $\mathcal{P}_0$ and stationary measure $\mathsf{P}_0$ on the state space $\mathbb{R}^d \times \Phi$. $H_\varrho$ is a multiplication operator in $L^2(\mathsf{P}_0)$, and by Proposition 1.3.8, $H_\varrho$ is infinitesimally bounded with respect to $H_0$. These are the ingredients needed to prove

**Theorem 1.5.11 The Feynman-Kac-Nelson formula:** *Let $\mathcal{V}$ be a real valued element of $L^2(\mathsf{P}_0)$, and write $\mathcal{V}$ for the operator of multiplication with $\mathcal{V}$ as well. Suppose that $\mathcal{V}$ is Kato-bounded with respect to $H_0$ with bound $< 1$, i.e. that there exist $a < 1, b > 0$ such that $\|\mathcal{V}F\|_{L^2(\mathsf{P}_0)} \leq a \|H_0 F\|_{L^2(\mathsf{P}_0)} + b \|F\|_{L^2(\mathsf{P}_0)}$. Then for all $F, G \in L^2(\mathsf{P}_0), t \geq 0$,*

$$\left\langle F, e^{-t(H_0+\mathcal{V})} G \right\rangle_{L^2(\mathsf{P}_0)} = \int \overline{F(q_0,\phi_0)} e^{-\int_0^t \mathcal{V}(q_s,\phi_s)\,ds} G(q_t,\phi_t)\, d\mathcal{P}_0(q,\phi). \tag{1.58}$$

*Proof:* Our proof closely follows the one given in [45] for the finite dimensional version of (1.58), the Feynman-Kac formula. By the Kato-Rellich theorem, $H_0 + \mathcal{V}$ is self-adjoint on $D(H_0)$ and bounded below, hence $e^{-t(H_0+\mathcal{V})}$ exists. To prove (1.58), we will use Trotters formula on the left hand side and integral convergence theorems on the right hand side. Explicitly, for $n \in \mathbb{N}$ we have

$$\left\langle F, \left(e^{-\frac{t}{n}H_0} e^{-\frac{t}{n}\mathcal{V})}\right)^n G \right\rangle_{L^2(\mathsf{P}_0)} = \int \overline{F(q_0,\phi_0)} \exp\left(-\frac{t}{n}\sum_{j=1}^n \mathcal{V}(q_{\frac{tj}{n}}, \phi_{\frac{tj}{n}})\right) G(q_t,\phi_t)\, d\mathcal{P}_0, \tag{1.59}$$

where on the right hand side of (1.59) we used the fact that $\exp(-tH_0)$ is the transition semigroup of $\mathsf{P}_0$. Let us now for the moment assume that $\mathcal{V}$ is continuous from $\mathbb{R}^d \times \Phi$ to $\mathbb{R}$ and bounded. By path continuity, it follows that also $t \mapsto \mathcal{V}(q_t, \phi_t)$ is continuous for $\mathcal{P}_0$-almost every path, and thus the Riemann sum in (1.59) converges to the integral in (1.58) $\mathcal{P}_0$-almost everywhere. Moreover, for $F, G \in L^\infty$, the integrands in (1.59) are bounded by $\|F\|_\infty \|G\|_\infty \exp(t \|\mathcal{V}\|_\infty)$, and (1.58) follows by Trotters formula on the left hand side and dominated convergence on the right hand side of (1.59).

Now consider $\mathcal{V} \in L^\infty$. We approximate $\mathcal{V}$ pointwise $\mathsf{P}_0$-almost everywhere by bounded continuous functions $\mathcal{V}_n$. (1.58) holds for each of the approximating functions, and we only have to show that both sides converge as $n \to \infty$. For the

left hand sides, note that $H_0 + \mathcal{V}_n$ converges to $H_0 + \mathcal{V}$ in strong resolvent sense. Indeed, $\mathcal{V}_n F \to \mathcal{V} F$ in $L^2(\mathsf{P}_0)$ for each $F \in L^2(\mathsf{P}_0)$ by dominated convergence, and thus

$$\begin{aligned} &((H_0 + \mathcal{V}_n + i)^{-1} - (H_0 + \mathcal{V} + i)^{-1})F = \\ &= \ (H_0 + \mathcal{V}_n + i)^{-1}(\mathcal{V}_n - \mathcal{V})(H_0 + \mathcal{V} + i)^{-1}F \overset{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{1.60}$$

Here we used that $(H_0 + \mathcal{V}_n + i)^{-1}$ is bounded uniformly in $n$. This implies strong convergence of all bounded continuous functions of the corresponding operators. Now $H_0 + \mathcal{V}_n$ and $H_0 + \mathcal{V}$ are bounded below, say $H_0 + \mathcal{V}_n \geq -c$, $H_0 + \mathcal{V} \geq -c$. Thus

$$e^{-t(H_0+\mathcal{V}_n)} = e^{tc} e^{-t|H_0+\mathcal{V}_n+c|} \overset{n\to\infty}{\longrightarrow} e^{tc} e^{-t|H_0+\mathcal{V}+c|} = e^{-t(H_0+\mathcal{V})}$$

in the sense of strong convergence. This shows convergence of the left hand side in (1.58). For the right hand side, first note that for each fixed $t \geq 0$,

$$\mathcal{P}_0\Big(\mathcal{V}_n(q_t, \phi_t) \not\to \mathcal{V}(q_t, \phi_t)\Big) = 0$$

by assumption. Two applications of Fubini's theorem imply that for $\mathcal{P}_0$-almost all $(q, \phi)$,

$$\mathcal{V}_n(q_s, \phi_s) \to \mathcal{V}(q_s, \phi_s) \quad \text{Lebesgue-almost everywhere on } [0, t].$$

By dominated convergence we now have

$$\int_0^t \mathcal{V}_n(q_s, \phi_s)\, ds \to \int_0^t \mathcal{V}(q_s, \phi_s)\, ds$$

$\mathcal{P}_0$-almost everywhere, and another application of dominated convergence ensures convergence at the right hand side of (1.58).

In the last step, we approximate general $\mathcal{V}$ by $\mathcal{V}_{n,m} = \min\{\max\{\mathcal{V}, -n\}, m\}$. Now we let first $n$ and then $m$ go to $\infty$ and again prove convergence of both sides of (1.58). The only differences to the previous step is that we now always use monotone convergence instead of dominated convergence, and that we now use $(H_0 + \mathcal{V} + i)^{-1}F \in D(H_0 + \mathcal{V}) \subset D(\mathcal{V})$ in (1.60). Finally, general $F, G \in L^2(\mathsf{P}_0)$ are obtained from bounded ones by monotone convergence together with positivity properties of $e^{t(H_0+\mathcal{V})}$ and the exponential function. □

**Corollary 1.5.12** *For $F, G \in L^2(\mathsf{P}_0), t > 0$, we have*

$$\big\langle F, e^{-tH} G \big\rangle_{L^2(\mathsf{P}_0)} = \int \overline{F(q_0, \phi_0)} e^{-\int_0^t \phi_s(\varrho_{q(s)})\, ds} G(q_t, \phi_t)\, d\mathcal{P}_0(q, \phi). \tag{1.61}$$

*Here, we wrote $q(s)$ instead of $q_s$ in the exponent for the sake of notational aesthetics.*

*Proof:* $(\phi_s, q_s) \mapsto \phi_s(\varrho_{q(s)})$ is real-valued and by Proposition 1.3.8 fulfills the hypothesis of Theorem 1.5.11. □

(1.61) is the key to the strategy described in the introduction: If we suppose that $H$ has a unique ground state $\Psi$, i.e. an eigenvector whose eigenvalue $\lambda$ is the bottom of the spectrum of $H$, then

$$\Psi_T := e^{-T(H-\lambda)}F \overset{T\to\infty}{\longrightarrow} \langle \Psi, F\rangle_{\mathsf{P}_0}\, \Psi.$$

(1.61) describes $\Psi_T$ in a fairly explicit way, and thus properties of $\Psi$ can be studied by investigating $\Psi_T$ via (1.61) and taking the limit. This is the strategy of our estimates in Chapter 3.

An especially interesting feature of (1.61) is that the exponent on the right hand side is linear in $\phi$. If $F$ and $G$ are independent of $\phi$, this enables us to carry out the $\mathcal{G}$-integration explicitly, as stated in

**Proposition 1.5.13** *Let $f, g \in L^2(\mathsf{N}_0)$. We also write $f, g$ for the elements $f \otimes 1, g \otimes 1 \in L^2(\mathsf{P}_0)$. For each $T > 0$,*

$$\left\langle f, e^{-TH} g\right\rangle_{L^2(\mathsf{P}_0)} = \int \overline{f(q_0)} \exp\left(-\int_0^T ds \int_0^T dt W(q_s - q_t, s-t)\right) g(q_T)\, d\mathcal{N}_0(q), \tag{1.62}$$

*where*

$$W(q,t) = -\frac{1}{2}\int \frac{|\hat{\varrho}(k)|^2}{2\omega(k)} \cos(k\cdot q) e^{-\omega(k)|t|}\, dk \tag{1.63}$$

*and $k \cdot q$ denotes scalar product in $\mathbb{R}^d$.*

*Proof:* For each path $q = \{q_t : t \in \mathbb{R}\} \in C(\mathbb{R}, \mathbb{R}^d)$, the map

$$I_q(\phi) = \int_0^T \phi_s(\varrho_{q(s)})\, ds$$

is in $L^2(\mathcal{G})$ and

$$\begin{aligned}\int (I_q(\phi))^2\, d\mathcal{G}(\phi) &= \int_0^T ds \int_0^T dt \int \phi_s(\varrho_{q(s)})\phi_t(\varrho_{q(t)})\, d\mathcal{G}(\phi) = \\ &= \int_0^T ds \int_0^T dt \int \frac{\overline{\hat{\varrho}_{q(s)}}\hat{\varrho}_{q(t)}}{2\omega} e^{-|t-s|\omega(k)}\, dk = -2\int_0^T ds \int_0^T dt W(q_t - q_s, t-s).\end{aligned}$$

Moreover, $I_q$ is the $L^2(\mathcal{G})$-limit of continuous linear functionals on $C(\mathbb{R}, \Phi)$, thus the image of $\mathcal{G}$ under $I_q$ is the Gaussian measure on $\mathbb{R}$ with mean 0 and covariance given by the above equation. It follows that

$$\int e^{-I_q(\phi)}\, d\mathcal{G}(\phi) = \exp\left(\frac{1}{2}\int I_q^2\, d\mathcal{G}\right) = \exp\left(-\int_0^T ds \int_0^T dt W(q_s - q_t, s-t)\right)$$

for each path $q$, and the fact that $\mathcal{P}_0 = \mathsf{N}_0 \otimes \mathcal{G}$ and Fubini's Theorem imply the claim. □

In the above form, the Feynman-Kac-Nelson formula was proven by Nelson in [32] by direct approximation of the interaction.

Like many models of quantum field theory, Nelson's model possesses some uncomfortable features concerning existence and well-definedness of certain objects. If we did not encounter any of these problems so far, this is simply because we ruled them out from the start by imposing conditions on $\varrho$. We will now discuss them, restricting to the physically important case $\omega(k) = |k|$ and $d = 3$.

**1.5.14 Ultraviolet divergence:** This type of divergence appears when we try to put $\varrho = \delta_0$, i.e. to make $q$ a point charge instead of a smeared-out charge distribution. Physically, a point charge is desirable in order to obtain Lorentz invariance. Moreover, often many explicit calculations can be done that were impossible or much more difficult before. In the point charge case $\hat{\varrho} = 1$, and obviously neither $\varrho \in K$ nor $|B|\varrho \in K$, since neither $1/|k|$ nor $1/|k|^2$ are integrable on $\mathbb{R}^3$. Thus $H_{\mathrm{I}}$ resp. $H_\varrho$ can not be defined in any rigorous way.
However, in a famous article Nelson [33] showed how to deal with the situation. First, an ultraviolet cutoff $\kappa$ is introduced, i.e. one puts $\hat{\varrho}_\kappa = 1_{\{|k| \leq \kappa\}}$. With this $\hat{\varrho}_\kappa$, (1.28) is fulfilled, and one writes $H_\kappa$ for the respective Hamiltonian. Nelson then shows the existence of an operator $\hat{H}$ and numbers $E_\kappa$ with $E_\kappa \to \infty$ as $\kappa \to \infty$ such that $e^{it(H_\kappa - E_\kappa)} \to e^{it\hat{H}}$ for all $t > 0$ as $\kappa \to \infty$. Thus although $\lim_{\kappa \to \infty} H_\kappa$ does not exist, the reason is just the divergence of $\inf \mathrm{spec} H_\kappa$ to $-\infty$. Once this is repaired by subtracting the constants $E_\kappa$, the divergence disappears.
The name ''ultraviolet divergence' comes from the Fock space picture. Since the non-integrability of $1/|k|$ is due to the large $|k|$ behavior of $1/|k|$, it is the bosons with high energies which cause the trouble. When making the ultraviolet cutoff, we suppress (cut off) the interaction of the charge with such high-energy (ultra-violet) bosons.

**1.5.15 Infrared divergence:** Just like ultraviolet divergence comes from the interaction of the charge with high energy bosons, infrared divergence comes from interaction with low energy bosons. However, the problems here are slightly more subtle than they were above. In particular, it is not necessary to cut off the interaction with low energy bosons in order to get a well-defined Hamiltonian, since $1/|k|$ and $1/|k|^2$ are integrable around $k = 0$ in three dimensions. However, if we take e.g. $\hat{\varrho}_\kappa = 1_{\{|k| \leq \kappa\}}$ as above, then under some technical assumptions it has been shown in [30] that $H_{\mathcal{F}}$ has no ground state in $L^2(\mathbb{R}^3) \otimes \mathcal{F}$. This means there exists no stationary state at all. On the other hand, if we assume $\hat{\varrho}/|k|^{3/2} \in L^2(\mathbb{R}^d)$ resp. $|B|^2\varrho \in K$ in addition to our other conditions, then there does exist such a ground state [48]. This means that we have to cut off the interaction of the particle with low-energy photons to obtain a ground state in Fock space. We have a strong interest in having

such a ground state, since the whole Chapter 3 deals with its properties. We will thus assume an infrared cutoff condition in Chapter 3.
One consequence of the infrared cutoff is that we now assume $0 = \hat{\varrho}(0) = \int \varrho(x)\, dx$. This means that we only deal with particles that have total charge zero, a physically slightly unsatisfactory situation. However, the reason for the infrared divergence lies more in the choice of the mathematical description than in the physics of the model. When we introduced the Fock space, we 'arbitrarily' imposed conditions on the number of bosons for all the vectors in Fock space in order to define a norm on $\mathcal{F}$. When we introduced the Hamiltonian in function space, we 'arbitrarily' chose the mean zero for the Ornstein-Uhlenbeck process $\mathcal{G}$. This now turns out to have been the wrong choice: although $H$ does not have a ground state in $L^2(\mathsf{N}_0 \otimes \mathsf{G})$, it does have a ground state in $L^2(\mathsf{N}_0 \otimes \tilde{\mathsf{G}})$, where $\tilde{\mathsf{G}}$ is a Gaussian measure with mean sufficiently different from zero to be singular with respect to $\mathsf{G}$. Details can be found in [31] and in Section 3.1

# Chapter 2

# Gibbs measures relative to Brownian motion

## 2.1 Definitions, Examples and finite volume Gibbs measures

The right hand sides of (1.58), (1.61) and (1.62) have a structure similar to that of a Gibbs measure: a (reference-) measure is modified by a density given by the exponential of some (energy-) functional. Gibbs measures play a central role in statistical mechanics, and a well-established theory exists [18]. However, there is one crucial difference between the cases covered by that theory and our context: while normally, the reference measure is a countable product of measures on the state space of the system, the reference measures in (1.58), (1.61) and (1.62) are measures on spaces of continuous functions and can not be written as product measures.

In order to discuss similarities and differences of our measures to the ones covered by the classical theory of Gibbs measures, we introduce an example from the latter theory that has the virtue of being fairly close to our models.

**2.1.1 A one-dimensional system of unbounded spins:** Let $\tilde{\nu}_0$ be a probability measure on $\mathbb{R}^d$, and $\tilde{\nu} = \tilde{\nu}_0^{\otimes\mathbb{Z}}$ the countable product on $\Omega = (\mathbb{R}^d)^{\mathbb{Z}}$. Consider $V : \mathbb{R}^d \to \mathbb{R}$ and $W : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ and suppose that

$$Z_N = \int \exp\left( - \sum_{i=-N}^{N} V(x_i) - \sum_{i,j=-N}^{N} W(x_i, x_j, |i-j|) \right) d\tilde{\nu}(x) < \infty. \tag{2.1}$$

Then

$$d\tilde{\nu}_N(x) = \frac{1}{Z_N} \exp\left( - \sum_{i=-N}^{N} V(x_i) - \sum_{i,j=-N}^{N} W(x_i, x_j, |i-j|) \right) d\tilde{\nu}(x) \tag{2.2}$$

is a probability measure on $\Omega$.

A good way to think about this measure is to view it as describing a chain of particles with positions in $\mathbb{R}^d$: Imagine $\mathbb{Z}$ as a straight line of equidistant dots, and attach one $\mathbb{R}^d$ to each of them. In each of the $\mathbb{R}^d$'s, there is exactly one particle at position $x_i$. Since the summations in (2.1) and (2.2) only go from $-N$ to $N$, it is even better to think of just $2N+1$ particles, the rest of the $\mathbb{R}^d$'s being added only for later mathematical convenience. The particles are at random positions, and the joint distribution of their positions is $\tilde{\nu}_N$. $\tilde{\nu}$ is called the reference measure of the system. Without interaction (i.e. for $V = W = 0$), each of the particles would just be distributed according to $\tilde{\nu}_0$ independently. $V$ only plays a trivial role in this model: $e^{-V}$ could as well have been incorporated into the reference measure from the very beginning. $W$, on the other hand, is a more interesting object, leading to an interaction between different particles in the chain. For example, $W(x, y, |t|) = |x-y|^2 e^{-|t|}$ would encourage particles from sites close to each other to remain in the same region of $\mathbb{R}^d$, while hardly affecting particles that are on distant sites of the lattice $\mathbb{Z}$.

A system like (2.2) is called one-dimensional system of unbounded spins. Here, 'one-dimensional' refers to the lattice $\mathbb{Z}$ (as opposed to $\mathbb{Z}^n$), and the reasons for calling the positions $x_i$ 'spins' are historical. $V$ is called single site potential, and $W$ pair potential. In the theory of Gibbs measures, the potentials are regarded as the central objects, and $\tilde{\nu}_N$ is called finite volume Gibbs measure for $V$ and $W$. 'Volume' here refers to $[-N, N] \subset \mathbb{Z}$.

If we want to describe a chain of infinite length, the most natural thing is to take $N$ to infinity in (2.2) and hope for some type of convergence. However, there is considerable freedom in the way we can try to take this limit. (2.2) is made so that the ends $x_{-N}$ and $x_N$ of the chain are just left to float around freely, but instead we could also have forced them to stay e.g. in $\bar{x} \in \mathbb{R}^d$. Different ways of fixing the end of the chain might result in different limiting measures, especially when the interaction $W$ is very long-ranged. It has turned out that instead of investigating all possible limits, it is advantageous to characterize the set of limiting measures in a direct way involving conditional expectations. This characterization is known as DLR (Dobrushin-Lanford-Ruelle) equations. We will not write down the corresponding formulas here: they will appear later in the context of (1.58), (1.61) and (1.62), and one-dimensional systems of unbounded spins are only used for comparison with those cases anyway. A comprehensive account of lattice Gibbs measures is [18], which in the introduction also contains a nice discussion on motivations for studying Gibbs measures and on their physical significance, going far beyond what we have done here.

Formally, we can try to derive systems like (1.62) from those given in (2.2) by simply replacing $\mathbb{Z}$ with $\mathbb{R}$ and all the sums with integrals. As a reference measure, however, a product measure on $(\mathbb{R}^d)^{\mathbb{R}}$ is not a very sensible object to look at. With

its manifestly discontinuous paths, even the existence of the integrals in the exponent would in general fail. An alternative choice for a reference measure that still retains a fair amount of independence between different sites is the measure of a reversible Markov process; in some cases, we will also allow the (infinite mass) Wiener measure as a reference measure, and it was this choice that motivated the name 'Gibbs measures relative to Brownian motion'.

An interesting consequence of our choice of a Markov process as a reference measure is that now we have to deal with something like an 'infinitesimal potential' in addition to $V$ and $W$. It is this potential that forces the paths to be continuous. If we think in the picture of the chain of particles developed above, the chain now has become a continuous string, and the infinitesimal potential prevents the string from breaking apart.

In the same way as Brownian motion arises from the random walk by a scaling limit, Gibbs measures relative to Brownian motion arise from lattice Gibbs measures when the lattice spacing goes to zero. The infinitesimal potential is then the limit of a nearest neighbor potential, a fact that is used for proving many of the results in [38].

Let us now give some precise definitions. In the view of (1.58), $\mathbb{R}^d$ as a state space will not be large enough.

**Definition 2.1.2** *Let $X$ be a separable Hilbert space, equipped with the $\sigma$-field generated by the continuous linear forms. A* **reference measure** *is the measure $\nu$ of a reversible Markov process with continuous paths and state space $X$. In case $X = \mathbb{R}^d$, we also allow the (infinite mass) Wiener measure as a reference measure. Wiener measure is the unique measure $\mathcal{W}$ on $C(\mathbb{R}, \mathbb{R}^d)$ with*

$$\int f_1(q_{t_1}) \cdots f_n(q_{t_n}) d\mathcal{W}(q) = \left\langle f_1, e^{-|t_2-t_1|H_{\mathcal{W}}} f_2 \dots e^{-|t_n-t_{n-1}|H_{\mathcal{W}}} f_n \right\rangle_{L^2(\mathbb{R}^d)}$$

*for all $f_1, \dots, f_n \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $t_1 < \dots < t_n$, and where $H_{\mathcal{W}} = -\frac{1}{2}\Delta$.*

We assumed some regularity of the state space $X$ as well as path continuity in order to ensure the existence of regular conditional expectations with respect to the natural $\sigma$-fields induced by the point evaluations [13]. While it might be possible to live without regular conditional expectations, it would make life (and our formulas) much more inconvenient and is thus best avoided.

We need to introduce some more

**2.1.3 Notation:** For $I \subset \mathbb{R}$, $\mathcal{F}_I$ is the $\sigma$-field over $C(\mathbb{R}, X)$ generated by the point evaluations with points in $I$. For $T > 0$, we write $\mathcal{F}_T$ instead of $\mathcal{F}_{[-T,T]}$ and $\mathcal{T}_T$ instead of $\mathcal{F}_{[-T,T]^c}$. Beware that $\mathcal{F}_T$ is used differently by many authors, especially in the context of Markov processes, where it denotes $\mathcal{F}_{[0,T]}$ or $\mathcal{F}_{]-\infty,T]}$. We sometimes write $f \in \mathcal{F}_I$ to state that $f$ is $\mathcal{F}_I$-measurable, and write $\nu|_{\mathcal{F}_I}$ to denote the restriction of $\nu$ to $\mathcal{F}_I$, which is regarded as a measure on $C(I, X)$.

Let us introduce a special version of the regular conditional probability $\nu(.|\mathcal{T}_T)$. For $f \in \mathcal{F}_T$, $g \in \mathcal{T}_T$ and $\bar{x} \in C(\mathbb{R}, X)$ we define

$$\int f g \, d\nu^{T,\bar{x}} = \mathbb{E}_\nu(f|\mathcal{F}_{\{-T,T\}})(\bar{x}) g(\bar{x}), \tag{2.3}$$

which is unique up to a set from $\mathcal{F}_{\{-T,T\}}$ of $\nu$-measure zero. For general $f \in \mathcal{F}$, $\int f \, d\nu^{T,\bar{x}}$ is obtained from (2.3) by approximation of $f$. In order to make this precise, one would have to define $\nu^{T,\bar{x}}$ as a product measure on $C([-T,T],X) \times C([-T,T]^c, X)$ of the regular conditional probability inside $[-T,T]$ and the Dirac measure outside $[-T,T]$. Intuitively, $\nu^{T,\bar{x}}$ is the measure we get from $\nu$ when the path is fixed to $\bar{x}$ outside $[-T,T]$ and left free inside $[-T,T]$. The conditioning on $\mathcal{F}_{\{-T,T\}}$ reflects the path continuity of $\nu$, and is necessary to have $\int f \, d\nu^{T,\bar{x}}$ well-defined for general $f$.

It is clear that $\nu^{T,\bar{x}}$ is indeed a version of $\nu(.|\mathcal{T}_T)$, and it is probably the most natural one. But we want to emphasize that we did have a lot of choice here. Namely, by a law of the iterated logarithm-type argument, measures of Markov processes are usually concentrated on a set of paths that is characterized by the asymptotic behavior of $x_t$ as $|t| \to \infty$ (see e.g. [39]). Thus $\nu(.|\mathcal{T}_T)$ can be defined arbitrarily on the set $N \subset C(\mathbb{R}, X)$ of measure zero that consists of all paths not having this precise asymptotic behavior. Our definition of $\nu^{T,\bar{x}}$ reduces this arbitrariness considerably, and we will absolutely need this when we define Gibbs measures. In fact, then even the (in general unavoidable) set from $\mathcal{F}_{\{-T,T\}}$ of $\nu$-measure zero will cause some trouble.

For finite dimensional $X$, however, it is often possible to avoid using conditional expectations altogether in the definition of $\nu^{T,\bar{x}}$. This is e.g. the case if $\nu = \mathcal{W}$, the $d$-dimensional Wiener measure. We may then define $\mathcal{W}^{T,\bar{x}}$ by

$$\int f g \, d\mathcal{W}^{T,\bar{x}} = \int f \, d\mathcal{W}^{\bar{x}_{-T},\bar{x}_T}_{[-T,T]} g(\bar{x}), \tag{2.4}$$

where $\mathcal{W}^{\xi,\eta}_{[a,b]}$ denotes the conditional Wiener measure starting in $\xi \in \mathbb{R}^d$ at time $a$ and ending in $\eta \in \mathbb{R}^d$ at time $b$. This is the unique finite measure on $C([a,b],\mathbb{R}^d)$ with

$$\begin{aligned} \int f_1(x_{t_1}) \cdots f_n(x_{t_n}) \, d\mathcal{W}^{\xi,\eta}_{[a,b]}(x) = \int K_{t_1-a}(\xi,q_1) f_1(q_1) \times \\ \times K_{t_2-t_1}(q_1,q_2) f_2(q_2) \cdots K_{t_n-t_{n-1}}(q_{n-1},q_n) f_n(q_n) K_{b-t_n}(q_n,\eta) \, dq_1 \cdots dq_n \end{aligned} \tag{2.5}$$

for all $a \le t_1 < \ldots < t_n \le b$. Here, $K_t(\xi,\eta) = (2\pi t)^{-d/2} \exp(-|\xi-\eta|^2/2t)$ denotes the integral kernel of $e^{t\Delta/2}$ at points $\xi$ and $\eta$ in case $t > 0$, and is $\delta(\xi-\eta)$ in case $t = 0$. By using the Feynman-Kac formula, a similar definition can be made when we replace $\mathcal{W}$ by a $P(\phi)_1$-process, but we will not need this. Note that $\mathcal{W}^{\xi,\eta}_{[a,b]}$ is not a probability measure, but rather one of mass $K_{b-a}(\xi,\eta)$.

**Definition 2.1.4** *A measurable function $V : X \to \mathbb{R}$ is called ($\nu$-admissible)* **single site potential** *if it fulfills the following assumption:*

*a) In case $\nu$ is the measure of a Markov process, we must have*

$$0 < \int \exp\left(-\int_{-T}^{T} V(x_s)\,ds\right)\,d\nu(x) < \infty \tag{2.6}$$

*for all $T > 0$.*

*b) In case $\nu = \mathcal{W}$, $V$ must be Kato-decomposable (cf. Appendix A.2).*

*A measurable function $W : X \times X \times \mathbb{R}^+ \to \mathbb{R}$ is called (admissible)* **pair potential** *if there exists $C_\infty < \infty$ with*

$$\int_{-\infty}^{\infty} \sup_{x,y\in X} |W(x, y, |s|)|\,ds < C_\infty. \tag{2.7}$$

Write

$$\begin{aligned} \Lambda(S,T) &= ([-S,S]\times[-T,T]) \cup ([-T,T]\times[-S,S]) \subset \mathbb{R}^2, \\ \Lambda(T) &= (\mathbb{R}\times[-T,T]) \cup ([-T,T]\times\mathbb{R}) \subset \mathbb{R}^2, \end{aligned}$$

where $S, T > 0$. For $\nu$-admissible potentials $V$ and $W$ and $T \le S$ we will use the notation

$$\mathcal{H}_{\Lambda(S,T)}(x) = \int_{-T}^{T} V(x_s)\,ds + \iint_{\Lambda(S,T)} W(x_t, x_s, |t-s|)\,ds\,dt \qquad (x \in C(\mathbb{R}, X)), \tag{2.8}$$

and define $\mathcal{H}_{\Lambda(T)}$ by replacing $\Lambda(S,T)$ with $\Lambda(T)$ in (2.8). We will also write $\mathcal{H}_T$ instead of $\mathcal{H}_{\Lambda(T,T)}$.

The notion of potentials above is not the most general possible; apart from the fact that we only consider shift-invariant potentials, we could also have required (2.7) to hold only pointwise on a subset of $C(\mathbb{R}, X)$ instead of uniformly [21]. We then would need to suppose that

$$\int e^{-\mathcal{H}_{\Lambda(T)}(x)}\,d\nu^{T,\bar{x}}(x) < \infty \tag{2.9}$$

for almost every $\bar{x}$. In our case, which is general enough to cover all situations we are interested in, (2.9) follows from the admissibility of $V$ and the fact that (2.7) implies

$$\left|\iint_{\Lambda(T)} W(x_t, x_s, |t-s|)\,ds\,dt\right| \le 4C_\infty T. \tag{2.10}$$

The validity of estimates like (2.10) is often expressed by saying that the left hand side of (2.10) is an extensive quantity.

### 2.1.5 Examples:

a) In (1.58), $X = \mathbb{R}^d \times \Phi, \nu = \mathcal{P}_0, V = \mathcal{V}$, and $W = 0$. To check admissibility, put $F = G = 1$ in (1.58). Finiteness of the partition function is then immediate. To see that it is also greater than zero, apply Jensen's inequality to the right hand side of (1.58) and use the stationarity of $\mathcal{P}_0$ to obtain

$$\int e^{-\int_0^t \mathcal{V}(q_s,\phi_s)\,ds}\,d\mathcal{P}_0(q,\phi) \geq \exp\left(-t\int \mathcal{V}\,d\mathsf{P}_0\right) \geq$$
$$\geq \exp\left(-t\,\|\mathcal{V}\|_{L^2(\mathsf{P}_0)}\right) \geq e^{-tb},$$

where $b$ is the second constant from the Kato-Rellich bound in the assumptions of Theorem 1.5.11.

b) (1.61) is a special case of a) with $\mathcal{V}(q,\phi) = \phi(\varrho_q)$.

c) In (1.62), $X = \mathbb{R}^d, \nu = \mathcal{N}_0$ and

$$W(x,y,s) = -\frac{1}{2}\int \frac{|\hat{\varrho}(k)|^2}{2\omega(k)}\cos(k\cdot(x-y))e^{-\omega(k)|s|}\,dk. \tag{2.11}$$

$W$ is admissible since

$$\sup_{q\in C(\mathbb{R},\mathbb{R}^d)}\left|\int_{-\infty}^{\infty} W(q_t,q_s,|t-s|))\,ds\right| = -\int_{-\infty}^{\infty} !W(0,0,|t-s|)\,ds = \int \frac{|\hat{\varrho}(k)|^2}{2\omega^2(k)}\,dk$$

is finite by the assumption $|B|\varrho \in K$.

d) Using the Feynman-Kac formula (A.2) and (A.5), we find for each $\mathcal{F}_T$-measurable, bounded $f$ that

$$\int f(q)e^{-\int_{-T}^{T} ds\int_{-T}^{T} dt\, W(q_s,q_t,|t-s|)}\,d\mathcal{N}_0(q) = \tag{2.12}$$
$$= \int \psi_0(q_{-T})f(q)e^{-\int_{-T}^{T} V(q_s)\,ds}e^{-\int_{-T}^{T} ds\int_{-T}^{T} dt\, W(q_s,q_t,|t-s|)}\psi_0(q_T)\,d\mathcal{W}(q),$$

with $\mathcal{W}$ denoting the Wiener measure.

We are now ready to give the central definition of this chapter.

**Definition 2.1.6** *Let $\nu$ be a reference measure and $V, W$ be $\nu$-admissible potentials. A probability measure $\nu_\infty$ on $C(\mathbb{R}, X)$ is called (infinite volume)* **Gibbs measure** *for the potentials $V$ and $W$ if for each $T > 0$,*

*(i): $\nu_\infty|_{\mathcal{F}_{\{-T,T\}}} \ll \nu|_{\mathcal{F}_{\{-T,T\}}}$, and*

*(ii): for each bounded, $\mathcal{F}$-measurable $f$,*

$$\mathbb{E}_{\nu_\infty}(f|\mathcal{T}_T)(\bar{x}) = \frac{\mathbb{E}_{\nu^{T,\bar{x}}}\left(fe^{-\mathcal{H}_{\Lambda(T)}}\right)}{\mathbb{E}_{\nu^{T,\bar{x}}}\left(e^{-\mathcal{H}_{\Lambda(T)}}\right)} \tag{2.13}$$

*$\nu_\infty$-almost surely.*

*A probability measure $\nu_T$ on $C(\mathbb{R},\mathbb{R}^d)$ is called* **finite volume Gibbs measure** *for the volume $T$ if (i) above holds, and*

*(ii)': for all $0 < S < T$ and all bounded $f \in \mathcal{F}$,*

$$\mathbb{E}_{\nu_T}(f|\mathcal{T}_S)(\bar{x}) = \frac{\mathbb{E}_{\nu^{S,\bar{x}}}\left(fe^{-\mathcal{H}_{\Lambda(S,T)}}\right)}{\mathbb{E}_{\nu^{S,\bar{x}}}\left(e^{-\mathcal{H}_{\Lambda(S,T)}}\right)} \tag{2.14}$$

*$\nu_T$-almost surely.*

(ii) in the above definition corresponds to the DLR equations in the lattice context. To understand why (i) is necessary, let us first investigate why we needed to consider the special version $\nu^{T,\bar{x}}$ of $\mathbb{E}_\nu(.|\mathcal{T}_T)(\bar{x})$. Writing the (much more natural) expression

$$\mathbb{E}_{\nu_\infty}(f|\mathcal{T}_T) = \frac{\mathbb{E}_\nu(fe^{-\mathcal{H}_{\Lambda(T)}}|\mathcal{T}_T)}{\mathbb{E}_\nu(e^{-\mathcal{H}_{\Lambda(T)}}|\mathcal{T}_T)} \tag{2.15}$$

instead of (2.13) is certainly visually more appealing, but unfortunately meaningless in most cases. The problem is that the left hand side of the above equation is only defined uniquely outside a set $N$ of $\nu_\infty$-measure zero, while the right hand side is only defined uniquely outside a set $M$ of $\nu$-measure zero. In many cases of interest, $\nu$ and $\nu_\infty$ are mutually singular on $C(\mathbb{R},X)$, and then (2.15) is no condition at all. Thus the general attitude that sets of measure zero do not matter leads to undesirable results here.

The reason why we need to require (i) explicitly is the same: Without it, the two sides of (ii) could possibly speak about functions that are uniquely defined only on disjoint subsets of the probability space. In case of a finite dimensional state space and a Feller Markov process as reference measure, this inconvenience can be avoided, as discussed in the paragraph before Definition 2.1.4. Then (i) is unnecessary. However, for a large state space like $\Phi$ it is often not possible to sensibly pick a fixed version of $\mathbb{E}_\nu(.|\mathcal{F}_{\{-T,T\}})$.

Finding (or proving existence of) an infinite volume Gibbs measure for potentials $V$ and $W$ can be quite difficult, but for a finite volume Gibbs measure it is easy.

**Proposition 2.1.7** *Let $\nu$ be a reference measure, and $V, W$ be admissible potentials. Then for each $T > 0$ with $0 < \mathbb{E}_\nu(\exp(-\mathcal{H}_T)) < \infty$,*

$$d\nu_T(x) = \frac{1}{\mathbb{E}_\nu(e^{-\mathcal{H}_T})} e^{-\mathcal{H}_T(x)}\, d\nu(x)$$

*is a finite volume Gibbs measure to the volume $T$.*

*Proof:* By assumption $e^{-\mathcal{H}_T}$ is $\nu$-integrable. Thus $\nu_T \ll \nu$, and in particular $\nu_T|_{\mathcal{F}_{\{-T,T\}}} \ll \nu|_{\mathcal{F}_{\{-T,T\}}}$. To check (ii), let $f, g \in L^\infty(C(\mathbb{R}, X))$, and suppose $g$ is $\mathcal{T}_S$-measurable. Then for $T > S$,

$$\begin{aligned}
&\mathbb{E}_\nu(e^{-\mathcal{H}_T})\mathbb{E}_{\nu_T}\left(g\frac{\mathbb{E}_{\nu^{S,\cdot}}\left(fe^{-\mathcal{H}_{\Lambda(S,T)}}\right)}{\mathbb{E}_{\nu^{S,\cdot}}\left(e^{-\mathcal{H}_{\Lambda(S,T)}}\right)}\right) = \\
&= \mathbb{E}_\nu\left(\mathbb{E}_\nu\Big(g\frac{\mathbb{E}_{\nu^{S,\cdot}}\left(fe^{-\mathcal{H}_{\Lambda(S,T)}}\right)}{\mathbb{E}_{\nu^{S,\cdot}}\left(e^{-\mathcal{H}_{\Lambda(S,T)}}\right)}e^{-\mathcal{H}_T}\Big|\mathcal{T}_S\Big)\right) = \\
&= \mathbb{E}_\nu\left(g\frac{\mathbb{E}_{\nu^{S,\cdot}}\left(fe^{-\mathcal{H}_{\Lambda(S,T)}}\right)}{\mathbb{E}_{\nu^{S,\cdot}}\left(e^{-\mathcal{H}_{\Lambda(S,T)}}\right)}e^{-\mathcal{H}_{[-T,T]^2\setminus\Lambda(S,T)}}\mathbb{E}_\nu\Big(e^{-\mathcal{H}_{\Lambda(S,T)}}\Big|\mathcal{T}_S\Big)\right) = \\
&= \mathbb{E}_\nu(e^{-\mathcal{H}_T})\mathbb{E}_{\nu_T}(fg).
\end{aligned}$$

The last equality is due to the fact that $\mathbb{E}_{\nu^{S,\cdot}}\left(e^{-\mathcal{H}_{\Lambda(S,T)}}\right)$ is a version of the conditional expectation $\mathbb{E}_\nu(e^{-\mathcal{H}_{\Lambda(S,T)}}|\mathcal{T}_S)$. Dividing by $\mathbb{E}_\nu(e^{-\mathcal{H}_T})$ shows (ii). □

From this proposition it follows that the measures on the right hand sides of (1.58), (1.61) and (1.62) all are finite volume Gibbs measures. The right hand side of (2.12) is slightly different, since there we have the $\psi_0$'s sitting at $-T$ and $T$. However, this difference is inessential, as the following remark shows.

**2.1.8 Remark:** Let $h : C(\mathbb{R}, X) \to \mathbb{R}$ be a $\mathcal{T}_T$-measurable, nonnegative function such that $0 < \mathbb{E}_\nu(he^{-\mathcal{H}_T}) < \infty$. Then the same proof as in Proposition 2.1.7 shows that

$$d\nu_T^h(x) = \frac{1}{\mathbb{E}_\nu(he^{-\mathcal{H}_T})}h(x)e^{-\mathcal{H}_T(x)}\,d\nu(x)$$

is a finite volume Gibbs measure to the volume $T$. $h$ can be regarded as a 'boundary distribution', and formally replacing $h$ with a delta function on $C([-T,T]^c, X)$ gives a (sharp) boundary condition.

**2.1.9 Notation:** The finite volume Gibbs measures for the potentials from Example 2.1.5 b) and c) will reappear a lot in this work, so we give them special names here. We write

$$d\mathcal{P}_T(q,\phi) = \frac{1}{Z_T}\exp\left(-\int_{-T}^{T}\phi_s(\varrho_{q(s)})\,ds\right)\,d\mathcal{P}_0(q,\phi), \tag{2.16}$$

and

$$d\mathcal{N}_T(q) = \frac{1}{Z_T}\exp\left(-\int_{-T}^{T}ds\int_{-T}^{T}dt\,W(q_s,q_t,|s-t|)\right)\,d\mathcal{N}_0(q). \tag{2.17}$$

$Z_T$ is the normalizing constant, and as usual, the dependence of $Z_T$ on the potential is suppressed from the notation; but note that here, 'by chance', $Z_T$ has the same

value in both (2.16) and (2.17). The reader who takes the effort to look up the definition of $W$ in (2.11) will notice that $W$ carries a minus sign there, which makes it look more natural to drop both that minus sign and the one in the exponent of (2.17). We do not do so, following the convention that the energy $\mathcal{H}_T$ must alway enter with a minus sign into the exponent in a Gibbs measure.

In Example 2.1.1 we claimed that taking limits of finite volume Gibbs measures is the most natural way to construct infinite volume Gibbs measures. On the other hand, Definition 2.1.6 has nothing to do with taking limits. The final statement of this section will make the connection between the two concepts, but before we can do so, we must specify the type of convergence we want to consider.

**Definition 2.1.10** *Let $(\nu_N)$ be a sequence of probability measures on $C(\mathbb{R}, X)$. We say that $\nu_N$* **converges locally** *to a probability measure $\nu$ if for each $0 < T < \infty$ and each $A \in \mathcal{F}_T$, $\nu_N(A) \to \nu(A)$ as $N \to \infty$.*

**2.1.11 Remarks:**

(i): Typically infinite volume Gibbs measures are mutually singular with respect to their reference measures, but finite volume Gibbs measures are absolutely continuous with respect to them. Thus local convergence is the best we can hope for. In particular, we will usually find a $\mathcal{T}$-measurable function $f$ (with $\mathcal{T} = \bigcap_{N\in\mathbb{N}} \mathcal{T}_N$) such that $\int f \, d\nu_N = 0$ for all $N$ but $\int f \, d\nu = 1$.

(ii): Setwise convergence usually is a much too strong concept for the convergence of measures. In fact, would we insist on convergence on all sets of $\mathcal{F}$, we would be in trouble due to (i). Since we however only have to check convergence on sets from $\mathcal{F}_T$ for finite $T$, the special structure of Gibbs measures will allow us to show local convergence of finite volume Gibbs measures in all cases we consider. Also in the context of Gibbs measures on a lattice, local convergence is the most useful concept of convergence [18].

**Proposition 2.1.12** *Let $\nu$ be a reference measure,and $V, W$ be admissible potentials. Let $(\nu_n)$ be a sequence of finite volume Gibbs measures for $V$ and $W$ to the volumes $T_n$ with $T_n \to \infty$ as $n \to \infty$. Suppose that there exists a probability measure $\nu_\infty$ on $C(\mathbb{R}, X)$ such that $\nu_n \to \nu_\infty$ in the topology of local convergence, and suppose that $\nu_\infty$ fulfills (i) of Definition 2.1.6 with respect to $\nu$. Then $\nu_\infty$ is an infinite volume Gibbs measure for $V$ and $W$.*

*Proof:* Since we supposed (i), only (ii) from Definition 2.1.6 remains to be shown. Each $\nu_n$ is a finite volume Gibbs measure, thus for $f, g \in L^\infty$ with $\mathcal{T}_S$-measurable $g$ we have

$$\mathbb{E}_{\nu_n}\left(g\frac{\mathbb{E}_{\nu^{S,\cdot}}\left(fe^{-\mathcal{H}_{\Lambda(S,T_n)}}\right)}{\mathbb{E}_{\nu^{S,\cdot}}\left(e^{-\mathcal{H}_{\Lambda(S,T_n)}}\right)}\right) = \mathbb{E}_{\nu_n}(fg) \tag{2.18}$$

for each $S < T_n$. All we have to show is that (2.18) survives the $n \to \infty$ limit. By a monotone class argument [13], we may assume that both $f$ and $g$ are $\mathcal{F}_R$-measurable for some $R > 0$. In this case, the right hand side of (2.18) converges by definition of local convergence. On the left hand side, (2.7) implies that

$$\iint_{\Lambda(S,T_n)} W(x_s, x_t, |t-s|)\, ds\, dt \overset{n\to\infty}{\longrightarrow} \iint_{\Lambda(S)} W(x_s, x_t, |t-s|)\, ds\, dt$$

uniformly in $x \in C(\mathbb{R}, X)$, and thus

$$F_n(x) = \frac{\mathbb{E}_{\nu^{S,x}}\big(f e^{-\mathcal{H}_{\Lambda(S,T_n)}}\big)}{\mathbb{E}_{\nu^{S,x}}\big(e^{-\mathcal{H}_{\Lambda(S,T_n)}}\big)} \overset{n\to\infty}{\longrightarrow} \frac{\mathbb{E}_{\nu^{S,x}}\big(f e^{-\mathcal{H}_{\Lambda(S)}}\big)}{\mathbb{E}_{\nu^{S,x}}\big(e^{-\mathcal{H}_{\Lambda(S)}}\big)} = F(x)$$

uniformly in $x$. Thus for each $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\|F_n - F\|_\infty < \varepsilon$ whenever $n > N$. By the triangle inequality,

$$|\mathbb{E}_{\nu_n}(gF_n) - \mathbb{E}_\nu(gF)| \le |\mathbb{E}_{\nu_n}(gF_N) - \mathbb{E}_\nu(gF_N)| + 3\,\|g\|_\infty \varepsilon \overset{n\to\infty}{\longrightarrow} 3\,\|g\|_\infty \varepsilon;$$

since $\varepsilon$ was arbitrary, convergence of the left hand side in (2.18) is shown. □

## 2.2 Infinite volume Gibbs measures - the Markovian case

In this section we investigate existence of the infinite volume Gibbs measure for the special case that the pair potential $W$ is zero. In this case, one infinite volume Gibbs measure is usually the measure of a stationary Markov process. If there is more than one Gibbs measure, also non-stationary and even non-Markovian processes are possible, as we will see later. The same phenomenon occurs in lattice Gibbs measures, cf. [18, Ch. 10 and 11].

We will always write $H_0$ for the generator of the reference measure $\nu$, or alternatively $H_0 = -\frac{1}{2}\Delta$ in case $\nu = \mathcal{W}$. For the single site potential, we will need different assumptions in different situations. However, since most of them are needed more often than not, we list them all here for later reference.

### 2.2.1 Assumptions on $V$:

(V1): $V$ is $\nu$-admissible (cf. Definition 2.1.4) and a Feynman-Kac formula with respect to $\nu$ holds, i.e. for all $f, g \in L^2(\nu_0) \cap L^\infty(\nu_0)$ we have

$$\int \overline{f(x_0)} e^{-\int_0^t V(x_s)\,ds} g(x_t)\, d\nu(x) = \left\langle f, e^{-t(H_0+V)} g \right\rangle_{L^2(\nu_0)},$$

where $\nu_0$ is the stationary measure of $\nu$, or Lebesgue measure in case $\nu = \mathcal{W}$.

(V2): The operator $H = H_0 + V$ has a unique, positive ground state $\psi_0 \in L^2(\nu_0)$. This means that $E_0 = \inf \mathrm{spec}(H)$ is an eigenvalue of multiplicity one, with eigenfunction $\psi_0$ that can be chosen such that $\psi_0 > 0$. We will always assume $\psi_0$ to be normalized, i.e. $\|\psi_0\|_{L^2(\nu)} = 1$.

(V3): The ground state $\psi_0$ is not only in $L^2(\nu)$, but also in $L^1(\nu)$.

(V4): The lowest eigenvalue $E_0$ of $H$ is separated from the rest of the spectrum of $H$ by a spectral gap $\gamma > 0$.

Let us discuss these assumptions for the two cases we are most interested in: the case $\nu = \mathcal{W}$, and the case $\nu = \mathcal{P}_0$ with $H$ being the Nelson Hamiltonian.

### 2.2.2 Remarks:

a) In case $\nu = \mathcal{W}$, the assumption that $V$ is Kato-decomposable (= admissible) in (V1) already implies the Feynman-Kac-formula [46]. In case $\nu = \mathcal{P}_0$, (V1) is fulfilled if $V$ satisfies the assumptions of Theorem 1.5.11, which is in particular the case if $H$ is the Nelson Hamiltonian.

b) (V2) is the crucial assumption that will allow us to solve the problem of existence of an infinite volume Gibbs measure with relatively little effort using spectral theory. Unfortunately, it is also a nontrivial assumption, especially if the state space $X$ is infinite dimensional. In case of the Nelson Hamiltonian, existence of a ground state is proven in [48] under certain conditions on the coupling $\varrho$. We will discuss these conditions in detail at the beginning of Section 3.1. In case $\nu = \mathcal{W}$, the problem of existence of a ground state for Schrödinger operators is well studied [42]. In particular, a ground state exists as soon as $V(x) \to \infty$ as $|x| \to \infty$, but also for many other potentials like the Coulomb potential $V(x) = -1/|x|$ in dimension $d \geq 3$. However, for $V = 0$ we have, of course, no ground state. In all of the above situations, existence of the ground state already implies uniqueness and positivity by a Perron-Frobenius argument [19].

c) Assumption (V3) is very weak. Except in case $\nu = \mathcal{W}$, it is automatically fulfilled, since then $\nu_0$ is a probability measure. In case $\nu = \mathcal{W}$, the ground state $\psi_0$ is known to decay exponentially for large $|x|$ in most cases of interest [10], and thus in these cases (V3) also holds.

d) In case of a finite dimensional state space, most Schrödinger operators that have a ground state also have a spectral gap. In particular, this is the case for all of the potentials mentioned in b) (except $V = 0$, of course). For the Nelson Hamiltonian, however, an argument [16] along the lines of the KVZ theorem [42] shows that there exists no spectral gap.

The key result for proving existence of an infinite volume Gibbs measure is

**Theorem 2.2.3** *Assume (V1) and (V2). Then for each $g \in L^2(\nu_0)$ with $g \geq 0$ and $\langle \psi_0, g \rangle_{L^2(\nu_0)} \neq 0$, the sequence*

$$d\nu_T(x) = \frac{1}{Z_T} g(x_{-T}) e^{-\int_{-T}^{T} V(x_s)\, ds} g(x_T)\, d\nu(x)$$

*converges locally. The limiting measure $\nu_\infty$ is the stationary Markov process with transition probabilities*

$$\mathbb{E}_{\nu_\infty}(f(x_t)|\mathcal{F}_{\{0\}})(x_0) = \frac{1}{\psi_0(x_0)} \mathbb{E}_\nu \left( e^{-\int_0^t (V(x_s)-E_0)\, ds} f(x_t)\psi_0(x_t) \middle| \mathcal{F}_{\{0\}} \right)(x_0) \quad (2.19)$$

*and stationary measure $\psi_0^2 d\nu_0$.*

*Proof:* By the Feynman-Kac formula and the reversibility of $\nu$,

$$\mathbb{E}_\nu \left( e^{-\int_{-T}^{-S} V(x_s)\, ds} g(x_{-T}) \middle| \mathcal{F}_{\{-S\}} \right) = \mathbb{E}_\nu \left( e^{-\int_S^T V(x_s)\, ds} g(x_T) \middle| \mathcal{F}_{\{S\}} \right) = e^{-(T-S)H} g.$$

Since the multiplicity of the eigenvalue $E_0$ is one by assumption, spectral theory implies

$$e^{(T-S)E_0} e^{-(T-S)H} g \overset{T\to\infty}{\longrightarrow} \langle \psi_0, g \rangle_{L^2(\nu_0)}\, \psi_0$$

in $L^2(\nu_0)$. Moreover, $Z_T = \left\| e^{-TH} g \right\|^2_{L^2(\nu_0)}$, and thus for $\mathcal{F}_S$-measurable $F \in L^\infty$, we have

$$\begin{aligned}
\int F(x) d\nu_T(x) = \frac{1}{Z_T} \mathbb{E}_\nu \Big( & \mathbb{E}_\nu \left( e^{-\int_{-T}^{-S} V(x_s)\, ds} g(x_{-T}) \middle| \mathcal{F}_{\{-S\}} \right) \times \\
& \times \mathbb{E}_\nu \left( e^{-\int_{-S}^{S} V(x_s)\, ds} F(x) \middle| \mathcal{F}_{\{-S,S\}} \right) \mathbb{E}_\nu \left( e^{-\int_S^T V(x_s)\, ds} g(x_T) \middle| \mathcal{F}_{\{S\}} \right) \Big) = \\
= \quad & \frac{1}{\| e^{-TH} g \|^2_{L^2(\nu_0)}} \mathbb{E}_\nu \left( \left( e^{-(T-S)H} g \right)(x_{-S}) \times \right. \\
& \left. \times \mathbb{E}_\nu \left( e^{-\int_{-S}^{S} V(x_s)\, ds} F(x) \middle| \mathcal{F}_{\{-S,S\}} \right) \left( e^{-(T-S)H} g \right)(x_S) \right) \overset{T\to\infty}{\longrightarrow} \\
\overset{T\to\infty}{\longrightarrow} \quad & \frac{1}{\langle \psi_0, g \rangle^2_{L^2(\nu_0)}} \mathbb{E}_\nu \left( \langle \psi_0, g \rangle^2_{L^2(\nu_0)}\, \psi_0(x_{-S}) e^{-\int_{-S}^{S} V(x_s)\, ds} F(x) \psi_0(x_S) \right) e^{2SE_0} = \\
= \quad & \mathbb{E}_\nu \left( \psi_0(x_{-S}) e^{-\int_{-S}^{S} (V(x_s)-E_0)\, ds} F(x) \psi_0(x_S) \right) = \mathbb{E}_{\nu_\infty}(F).
\end{aligned}$$

□

**Corollary 2.2.4** *With the assumptions and notations of Theorem 2.2.3, $\nu_\infty$ is an infinite volume Gibbs measure for the potential $V$.*

*Proof:* By Remark 2.1.8, each $\nu_T$ is a finite volume Gibbs measure to the volume $T$. By Proposition 2.1.12, we only need to check local absolute continuity, i.e. (i) from Definition 2.1.6. By (2.19), for $F \in \mathcal{F}_S$ we have

$$\mathbb{E}_{\nu_\infty}(F) = \mathbb{E}_\nu \left( \psi_0(x_{-S}) e^{-\int_{-S}^{S} (V(x_s) - E_0)\, ds} F(x) \psi_0(x_S) \right).$$

This shows $\nu_\infty|_{\mathcal{F}_S} \ll \nu|_{\mathcal{F}_S}$, which implies $\nu_\infty|_{\mathcal{F}_{\{-S,S\}}} \ll \nu|_{\mathcal{F}_{\{-S,S\}}}$. □

The immediate consequence for the measures $\mathcal{N}_T$ and $\mathcal{P}_T$ from 2.1.9 is

**Corollary 2.2.5** *Assume that the Hamiltonian $H$ of Nelson's model has a (unique, positive) ground state $\Psi$ in $L^2(\mathsf{P}_0)$ with $H\Psi = E_0\Psi$. Then the infinite volume Gibbs measure $\mathcal{P}$ to the reference measure $\mathcal{P}_0$ and potential $(q, \phi) \mapsto \phi(\varrho_q)$ exists and is the measure on $C(\mathbb{R}, \mathbb{R}^d \times \Phi)$ uniquely determined by*

$$\int F(q,\phi)\, d\mathcal{P}(q,\phi) = \int \Psi(q_{-S}, \phi_{-S}) e^{-\int_{-S}^{S} (\phi_s(\varrho_{q(s)}) - E_0)\, ds} F(q,\phi) \Psi(q_S, \phi_S)\, d\mathcal{P}_0(q,\phi)$$

*for each $\mathcal{F}_S$-measurable, bounded $F$. Moreover, in this case the local limit of the measures $\mathcal{N}_T$ exists, hence there exists an infinite volume Gibbs measure to the reference measure $\mathcal{N}_0$, the pair potential $W$ and the state space $\mathbb{R}^d$. This measure will be denoted by $\mathcal{N}$. We have*

$$\int f\, d\mathcal{N} = \int f \otimes 1\, d\mathcal{P}$$

*for all $f \in L^1(\mathcal{N})$.*

This Corollary is basically all we will need from this section in Chapter 3. It also shows a first example where we are able to prove existence of a Gibbs measure with a nonzero pair potential. More general results on this situation are contained in the next section. We caution the reader that, althogh $\mathcal{P}$ is uniquely *determined* by the formula given in the above Corollary, it is not necessarily the one and only *Gibbs measure* for the given potential. The question of uniqueness in this case is open.

We now investigate uniqueness of the infinite volume Gibbs measure in a restricted set-up. For the rest of this section, which is taken mainly from [5], we let $X = \mathbb{R}^d, \nu = \mathcal{W}$, and use $\mathcal{W}^{T,\bar{x}}$ as defined in (2.4). For a shorter notation, we will also write $\Omega = C(\mathbb{R}, \mathbb{R}^d)$. For $\bar{x} \in \Omega$, denote by $\nu_T^{\bar{x}}$ the finite volume Gibbs measure to the volume $T$ with sharp boundary condition $\bar{x}$, i.e. the probability measure on $\Omega$ with

$$\mathbb{E}_{\nu_T^{\bar{x}}}(f) = \frac{1}{Z_T^{\bar{x}}} \int e^{-\int_{-T}^{T} V(x_s)\, ds} f(x) d\mathcal{W}^{T,\bar{x}}(x),$$

where

$$Z_T^{\bar{x}} = \int e^{-\int_{-T}^{T} V(x_s)\, ds} d\mathcal{W}^{T,\bar{x}}(x).$$

In this setting, the stationary Gibbs measure from Corollary 2.2.4 is the $P(\phi)_1$-process to the Schrödinger operator $H = H_0 + V$ from Appendix A.2. As mentioned there, we always add a constant to $V$ in order to achieve $\inf \operatorname{spec}(H_0 + V) = 0$ in this context. Thanks to the normalizing constant, this does not change our finite volume Gibbs measures.

The following example shows that uniqueness of the Gibbs measure can not be expected to hold in general. A similar example in the lattice context has been given in [2].

**2.2.6 Example:** Let $d = 1$ and $V = \frac{1}{2}(x^2 - 1)$. Then the ground state of $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$ is $\psi_0(x) = \pi^{-1/4} e^{-x^2/2}$, and the one-dimensional Ornstein-Uhlenbeck process is the stationary infinite volume Gibbs measure. However, it is not the only Gibbs measure. Denote by $K_t$ the integral kernel of $e^{-tH}$. By Mehler's formula,

$$K_t(\xi, \eta) = \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \exp\left( \frac{4\xi\eta e^{-t} - (\xi^2 + \eta^2)(1 + e^{-2t})}{2(1 - e^{-2t})} \right). \tag{2.20}$$

Now fix $\alpha, \beta \in \mathbb{R}$ and define for $s, \xi \in \mathbb{R}$

$$\psi_s^l(\xi) := \pi^{-1/4} \exp\left( -\frac{1}{2}(\xi + \alpha e^{-s})^2 \right) \exp\left( \frac{\alpha e^{-s}}{2} \right)^2,$$
$$\psi_s^r(\xi) := \pi^{-1/4} \exp\left( -\frac{1}{2}(\xi + \beta e^{+s})^2 \right) \exp\left( \frac{\beta e^{+s}}{2} \right)^2.$$

An explicit calculation using (2.20) shows that

$$e^{-tH}\psi_s^l = \psi_{s+t}^l, \quad e^{-tH}\psi_s^r = \psi_{s-t}^r, \quad \text{and} \quad \left\langle \psi_s^l, \psi_s^r \right\rangle = e^{\alpha\beta/2}. \tag{2.21}$$

Therefore for $F(x) = f_1(x_{t_1}) \cdots f_n(x_{t_n})$, we have

$$\mathbb{E}_{\nu_{\alpha,\beta}}(F) := e^{-\alpha\beta/2} \left\langle \psi_{t_1}^l f_1, e^{-(t_2 - t_1)H} f_2 \dots e^{-(t_n - t_{n-1})H} f_n \psi_{t_n}^r \right\rangle_{L^2}. \tag{2.22}$$

Equations (2.22) define a consistent family of probability measures, and hence they are the finite dimensional distributions of a probability measure $\nu_{\alpha,\beta}$ on $C(\mathbb{R}, \mathbb{R})$. With (2.21) we also see that the family

$$d\nu_{\alpha,\beta,T}(x) = \frac{1}{Z_T} \psi_{-T}^l(x_{-T}) e^{-\int_{-T}^{T} V(x_s)\, ds} \psi_T^r(x_T)$$

of finite volume Gibbs measures converges locally to $\nu_{\alpha,\beta}$. Thus for each $\alpha$ and each $\beta$, $\nu_{\alpha,\beta}$ is an infinite volume Gibbs measure for $V$. Certainly, none of these measures except the one for $\alpha = \beta = 0$ is stationary, but they are still all Markov processes. To get a Gibbs measure that is not a Markov process, we can e.g. take a random variable $\omega$ taking value 1 and $-1$ both with probability $1/2$, and take the local limit of $\nu_{\omega,\omega,T}$. For this measure, $\mathbb{P}(\limsup_{t\to\infty} x_t = +\infty | \mathcal{F}_{\{0\}}) = 1/2$ almost surely, while the behavior of a path at $t \to +\infty$ can be read off by its behavior at $t \to -\infty$, thus $\mathbb{P}(\limsup_{t\to\infty} x_t = +\infty | \mathcal{F}_{]-\infty,0]})$ is a random variable taking the values 0 and 1.

The Gibbs measures with $\alpha, \beta \neq 0$ from the previous example all have exponentially growing paths. We will now see that once we rule out measures with quickly growing paths from our considerations, we can show uniqueness. We start by giving a result that is connected with Proposition 2.1.12. There we saw that each $\bar{x}$ for which $(\nu_T^x)_{T>0}$ converges (at least along a subsequence) gives rise to an infinite volume Gibbs measure. Here we will see that if all the measures obtained like this are identical, then there is no other infinite volume Gibbs measure.

We say that a measure $\mu$ on $\Omega$ is supported on a measurable set $\Omega^* \subset \Omega$ if $\mu(\Omega^*) = 1$.

**Lemma 2.2.7** *Let $\Omega^* \subset \Omega$ be measurable and let $\nu_\infty$ be an infinite volume Gibbs measure for the potential $V$ supported on $\Omega^*$. If $\nu_N^{\bar{x}} \to \nu_\infty$ as $\mathbb{N} \ni N \to \infty$ locally for each $\bar{x} \in \Omega^*$, then $\nu_\infty$ is the only infinite volume Gibbs measure for $V$ supported on $\Omega^*$.*

*Proof:* Let $\tilde{\nu}$ be any infinite volume Gibbs measure supported by $\Omega^*$. For each $A \in \mathcal{F}_S$ $(S > 0)$, $\bar{x} \mapsto \tilde{\nu}(A|\mathcal{T}_N)(\bar{x})$ is a backward martingale in $N$, thus convergent almost everywhere to $\tilde{\nu}(A|\mathcal{T})(\bar{x})$. By the DLR equations (2.13) , $\tilde{\nu}(A|\mathcal{T}_N)(\bar{x}) = \nu_N^{\bar{x}}(A)$ $\tilde{\nu}$-almost surely, and thus for $\tilde{\nu}$-almost every $\bar{x} \in \Omega^*$, we find

$$\tilde{\nu}(A|\mathcal{T})(\bar{x}) = \lim_{N\to\infty} \tilde{\nu}(A|\mathcal{T}_N)(\bar{x}) = \lim_{N\to\infty} \nu_N^{\bar{x}}(A) = \nu_\infty(A).$$

Here we put $\mathcal{T} = \bigcap_{n\in\mathbb{N}} \mathcal{T}_n$. Taking $\tilde{\nu}$-expectations on both sides of the above equation shows $\tilde{\nu}(A) = \nu_\infty(A)$. Since this is true for each $A \in \mathcal{F}_S$ and each $S > 0$, $\tilde{\nu} = \nu_\infty$. □

Arguments of this kind are quite standard in the theory of Gibbs measures. See Theorem 7.12 of [18] for a related result.

Our uniqueness theorem now reads

**Theorem 2.2.8** *Assume (V1), (V2), (V3) and (V4). Recall that $\gamma$ is the spectral gap of the Schrödinger operator $H = H_0 + V$ with ground state $\psi_0$. Define*

$$\Omega^* := \left\{ x \in \Omega : \lim_{\mathbb{N}\ni N\to\infty} \frac{e^{-\gamma N}}{\psi_0(x(\pm N))} = 0 \right\}. \tag{2.23}$$

*Then the unique Gibbs measure for $V$ supported by $\Omega^*$ is the $P(\phi)_1$-measure $\mu$ corresponding to $V$.*

*Proof:* Pick $\bar{x} \in \Omega^*$. In order to apply Lemma 2.2.7 we have to show

$$\lim_{N\to\infty} \nu_N^{\bar{x}}(A) = \mu(A). \tag{2.24}$$

for each $\mathcal{F}_S$-measurable set $A \subset \Omega$, and since we already know the limit is a probability measure and $\mathcal{W}$ is $\sigma$-finite, it will be sufficient to show (2.24) for $A$ with

$\mathcal{W}(A) < \infty$. To do so, first note that by the Markov property of Brownian motion, for $N > S$ we have

$$\nu_N^{\bar{x}}(A) = \frac{1}{Z_N(\bar{x})} \iint K_{N-S}(\bar{x}_{-N}, \xi) \int e^{-\int_{-S}^{S} V(x_s)\, ds} 1_A(x)\, d\mathcal{W}_{[-S,S]}^{\xi,\eta}(x) K_{N-S}(\eta, \bar{x}_N)\, d\xi d\eta, \tag{2.25}$$

Here,

$$K_t(\xi, \eta) = \int e^{-\int_0^t V(x(s))\, ds}\, d\mathcal{W}_{[0,t]}^{\xi,\eta}(x)$$

is the integral kernel of $e^{-tH}$ by the virtue of the Feynman-Kac formula, and $\mathcal{W}_{[a,b]}^{\xi,\eta}$ is conditional Wiener measure starting in $\xi$ at time $a$ and ending in $\eta$ at time $b$, cf. (2.4). Since $Z_N(\bar{x}) = K_{2N}(\bar{x}_{-N}, \bar{x}_N)$, and since $\int \exp(-\int_{-S}^{S} V(x_s)\, ds) 1_A(x)\, d\mathcal{W}_{[-S,S]}^{\xi,\eta}$ is an integrable function on $\mathbb{R}^{2d}$, it will be sufficient if we show that

$$\frac{K_{N-S}(\bar{x}_{-N}, \xi) K_{N-S}(\eta, \bar{x}_N)}{K_{2N}(\bar{x}_{-N}, \bar{x}_N)} \overset{N\to\infty}{\longrightarrow} \psi_0(\xi)\psi_0(\eta) \tag{2.26}$$

uniformly in $\xi, \eta \in \mathbb{R}^d$.

Let $P_{\psi_0} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the projection onto the one-dimensional subspace spanned by $\psi_0$, and put

$$L_t := e^{-tH} - P_{\psi_0}.$$

$L_t$ is an integral operator with kernel $\tilde{K}_t(\xi, \eta) = K_t(\xi, \eta) - \psi_0(\xi)\psi_0(\eta)$. By the assumption $\gamma > 0$ we have

$$\|L_t\|_{2,2} = e^{-\gamma t}. \tag{2.27}$$

Here and below $\|.\|_{p,q}$ denotes the norm of an operator from $L^p$ to $L^q$. For estimating $\tilde{K}_t$ note that

$$\sup_{\xi,\eta\in\mathbb{R}^d} \left|\tilde{K}_t(\xi, \eta)\right| = \sup_{f\in L^1, \|f\|_{L^1}=1} \left\| \int \tilde{K}_t(\xi, \eta) f(\eta)\, d\eta \right\|_\infty = \|L_t\|_{1,\infty}.$$

Since $e^{-tH} P_{\psi_0} = P_{\psi_0} e^{-tH} = P_{\psi_0}$, for all $t > 2$ we have

$$\|L_t\|_{1,\infty} = \left\|e^{-H}(e^{-(t-2)H} - P_{\psi_0})e^{-H}\right\|_{1,\infty} \le \left\|e^{-H}\right\|_{2,\infty} \|L_{t-2}\|_{2,2} \left\|e^{-H}\right\|_{1,2}.$$

By (S1) from Appendix A.2, both $\left\|e^{-H}\right\|_{2,\infty}$ and $\left\|e^{-H}\right\|_{1,2}$ are finite. It thus follows that for every $(N - S) \ge 2$,

$$|K_{N-S}(\xi, \eta) - \psi_0(\xi)\psi_0(\eta)| \le C_S e^{-\gamma N}, \tag{2.28}$$

where $C_S = \left\|e^{-H}\right\|_{2,\infty} \left\|e^{-H}\right\|_{1,2} e^{\gamma(2+S)}$ is independent of $x, y$ and $N$. Writing

$$Q_N(\xi, \eta) = \frac{K_N(\xi, \eta)}{\psi_0(\xi)\psi_0(\eta)},$$

we have

$$\left|\frac{K_{N-S}(\bar{x}_{-N},\xi)K_{N-S}(\eta,\bar{x}_N)}{K_{2N}(\bar{x}_{-N},\bar{x}_N)} - \psi_0(\xi)\psi_0(\eta)\right| = \psi_0(\xi)\psi_0(\eta)\left|\frac{Q_{N-S}(\bar{x}_{-N},\xi)Q_{N-S}(\eta,\bar{x}_N)}{Q_{2N}(\bar{x}_{-N},\bar{x}_N)} - 1\right|.$$

(2.28) now just states that $|Q_{N-S}(\xi,\eta)-1| \le C_S e^{-\gamma N}/(\psi_0(\xi)\psi_0(\eta))$, and thus for all $\bar{x} \in \Omega^*$, we find

$$\begin{aligned} |Q_{N-S}(\bar{x}_{-N},\xi)-1|\psi_0(\xi) &\le C_S e^{-\gamma N}/\psi_0(\bar{x}_N) \to 0, \\ |Q_{N-S}(\eta,\bar{x}_N)-1|\psi_0(\eta) &\le C_S e^{-\gamma N}/\psi_0(\bar{x}_{-N}) \to 0, \qquad \text{and} \\ |Q_{2N}(\bar{x}_{-N},\bar{x}_N)-1| &\le e^{-2\gamma N}/(\psi_0(\bar{x}_{-N})\psi_0(\bar{x}_N)) \to 0 \end{aligned}$$

as $N \to \infty$, proving (2.26). It remains to show that the $P(\phi)_1$-measure is actually supported on $\Omega^*$. By time reversibility of $\mu$, it will be enough to show

$$\mu\left(\limsup_{N\to\infty} \frac{e^{-\gamma N}}{\psi_0(x_N)} > q\right) = 0 \tag{2.29}$$

for each $q > 0$. To prove (2.29), note that by the stationarity of $\mu$,

$$\begin{aligned} \mu\left(\frac{e^{-\gamma N}}{\psi_0(x_N)} \ge q\right) &= \mu\left(\psi_0(x_0) \le \frac{\exp(-\gamma N)}{q}\right) = \\ &= \int 1_{\{\psi_0 \le \exp(-\gamma N)/q\}} \psi_0^2 \, d\xi \le \\ &\le \frac{\exp(-\gamma N)}{q} \|\psi_0\|_{L^1}. \end{aligned}$$

The right hand side of the last expression is summable in $N$ for each $q$, and so the Borel-Cantelli lemma proves (2.29), finishing the proof. □

Of course we are interested in how large the set $\Omega^*$ from the above theorem actually is. The size of $\Omega^*$ depends on the decay of $\psi_0$ at infinity, which in many cases can be read off the growth of $V$ at infinity. One of the strongest results in this direction is due to R. Carmona [10] and requires some mild additional restrictions on $V$.

**Definition 2.2.9** *$V : \mathbb{R}^d \to \mathbb{R}$ is in the* **Carmona class** *if there exists a breakup $V = V_1 - V_2$, such that*

$$\begin{aligned} &V_1 \in L^{d/2+\varepsilon}_{\text{loc}} \text{ for some } \varepsilon > 0, \text{ and } V_1 \text{ is bounded below,} \\ &V_2 \in L^p \text{ for some } p > \max\{1, d/2\}, \text{ and } V_2 \ge 0. \end{aligned}$$

**Theorem 2.2.10** *[10] Assume that $V$ is from the Carmona class and that $H$ has a ground state $\psi_0$.*

*a) If there exist* $m > 0, C > 0$ *such that* $V(x) \geq C|x|^{2m}$*, then there exist* $D > 0, \delta > 0$ *with*

$$|\psi_0(x)| \leq D \exp(-\delta |x|^{m+1}) \qquad \forall x \in \mathbb{R}^d.$$

*b) If there exist* $m > 0, C > 0$ *such that* $V(x) \leq C|x|^{2m}$*, then there exist* $D > 0, \delta > 0$ *with*

$$|\psi_0(x)| \geq D \exp(-\delta |x|^{m+1}) \qquad \forall x \in \mathbb{R}^d.$$

*c) If* $\liminf_{|x|\to\infty} V(x) > 0 = E_0$*, then there exist* $D > 0, \delta > 0$ *with*

$$|\psi_0(x)| \leq D \exp(-\delta |x|) \qquad \forall x \in \mathbb{R}^d.$$

As an example for the application of this theorem, consider a potential $V$ growing at infinity like $|x|^{2m}$, $m > 0$. The ground state $\psi_0$ of the corresponding Schrödinger operator can then be estimated as

$$Ce^{-a|x|^{m+1}} \leq \psi_0(x) \leq De^{-b|x|^{m+1}}$$

with suitable constants $C, D, a, b$, and consequently in this case $\Omega^*$ consists of functions that grow at most like $T^{1/(m+1)}$ at infinity. Hence $\Omega^*$ becomes smaller for more confining potentials. This is quite the opposite of what one would expect to be true since intuitively a more confining potential brings a path back to the origin more quickly and should therefore allow more rapidly growing boundary conditions. In fact, by using completely different methods than ours, [25] shows uniqueness on the set $\{\omega \in \Omega : \lim_{|T|\to\infty} \omega(T)e^{-a|T|} = 0 \ \forall a > 0\}$ under the restriction that $x \mapsto V(x) - \kappa |x|^2$ is convex for some $\kappa > 0$. This is much closer to what we would expect to be true, but the restriction on the potentials is severe. The reason why our subset of uniqueness is relatively small and even shrinks for more confining potentials is that in the proof of Theorem 2.2.8, we used the rather crude estimate $K_{2T}(\bar{x}(-T), \bar{x}(T)) = \psi_0(\bar{x}(-T))\psi_0(\bar{x}(T)) + O(e^{-2\gamma T})$ in the denominator, and hence a fast decay of $\psi_0$ had to be compensated by a slow growth of $\bar{x}$. However, from the point of view of Gibbs measures, $\Omega^*$ is already large enough since it carries full $P(\phi)_1$-measure.

## 2.3 Infinite volume Gibbs measures - the non-Markovian case

We now investigate existence of an infinite volume Gibbs measure in case $W \neq 0$. In this case, the Gibbs measure will not be a Markov process, and the powerful spectral theory we used in the previous section is no longer available. The contents of this section are taken from [3].

Several authors have by now studied the problem. All of them assume (2.10) in some form, but also need additional restrictions on $V$ and $W$. In [38], the first mathematical account on the subject, correlation inequalities are used, and consequently the potentials $V$ and $W$ have to fulfill certain convexity assumptions. In [29], a cluster expansion method is applied, requiring a small parameter (coupling constant) in front of $W$ as well as a $V$ that is growing faster than quadratically at infinity. Recently, [21] used an integration by parts formula. His restrictions on $W$ are weak, but strong assumptions on the asymptotic behavior of $V$ are needed. In particular, $V$ has to grow at least quadratically at infinity.

Our method relies on a compactness argument, using the strong Markov property of the reference measure and a stopping time estimate. The main advantage over the existing approaches is that our restrictions on $V$ are weak: only (V1), (V2) and (V3) from 2.2.1 are needed. This covers all cases from [38, 29, 21], and in addition allows for $V$'s which do not grow at infinity. For the pair potential $W$, we need (uniform) integrability conditions that are stronger than the integrability conditions imposed in [38], [29] or [21]. On the other hand, we neither need the convexity assumed in [38], nor the small parameter of [29], nor the differentiability needed in [21].

All existing works treating the case $W \neq 0$ only consider a finite dimensional state space, i.e. take $X = \mathbb{R}^d$, and so do we. We will take Wiener measure $\mathcal{W}$ as a reference measure, but will use (V2) to get rid of the single site potential $V$, at the same time replacing $\mathcal{W}$ by a $P(\phi)_1$-process. To be precise, let us write $\mu_0$ for the measure of the $P(\phi)_1$-process arising from the Schrödinger operator $H = -\frac{1}{2}\Delta + V$. Then in analogy to (2.12), the Feynman-Kac formula implies

$$\int \psi_0(x_{-T}) f(x) e^{-\int_{-T}^{T} V(x_s)\,ds} e^{-\int_{-T}^{T} ds \int_{-T}^{T} dt\, W(x_s, x_t, |t-s|)} \psi_0(x_T)\, d\mathcal{W}(x) =$$
$$= \int f(x) e^{-\int_{-T}^{T} ds \int_{-T}^{T} dt\, W(x_s, x_t, |t-s|)}\, d\mu_0(x) \tag{2.30}$$

for each $\mathcal{F}_T$-measurable, $\mu_0$-integrable function $f$. According to Remark 2.1.8, both sides of (2.30) display integration of $f$ with respect to a finite volume Gibbs measure. On the left hand side, it is the one with boundary condition $\psi_0$, potentials $V$ and $W$ and reference measure $\mathcal{W}$, while on the right hand side, we have no boundary condition, no single site potential and reference measure $\mu_0$.

We are only going to prove existence, not uniqueness, of the infinite volume Gibbs measure. For this purpose we will employ Proposition 2.1.12 and thus are looking for a local limit of any sequence of finite volume Gibbs measures. Choosing the ones with boundary condition $\psi_0$ like we do on the left hand side of (2.30), we may (and will) as well treat the right hand side of (2.30) instead.

Thus from now on we assume:

(i): The reference measure is the measure $\mu_0$ of a $P(\phi)_1$-process arising from the Schrödinger operator $H = -\frac{1}{2}\Delta + V$, where $V$ fulfills (V1) and (V2).

(ii): The single site potential (relative to $\mu_0$) is equal to zero. We will continue to refer to (V1)-(V3), but this will mean the potential in the Schrödinger operator from now on.

Taking (ii) into account, (2.8) reads

$$\mathcal{H}_\Lambda(x) = \iint_\Lambda W(x_t, x_s, |t-s|)\, ds\, dt \qquad (x \in C(\mathbb{R}, \mathbb{R}^d)) \tag{2.31}$$

with $\Lambda \subset \mathbb{R}^2$, and the finite volume Gibbs measures we consider are

$$d\mu_T(x) = \frac{1}{Z_T} e^{-\mathcal{H}_T(x)}\, d\mu_0(x), \tag{2.32}$$

where $Z_T = \int \exp(-\mathcal{H}_T(x))\, d\mu_0(x)$ as usual.

In order to state our assumptions on $W$, we need the following

**Definition 2.3.1** *Let $C^{(0)}(\mathbb{R}, \mathbb{R}^d)$ denote the space of functions which are continuous with the possible exception of the point 0 but have left and right hand side limits there. For $\tau > 0$ we define the map*

$$\theta_\tau^{(0)} : C(\mathbb{R}, \mathbb{R}^d) \to C^{(0)}(\mathbb{R}, \mathbb{R}^d), \quad (\theta_\tau^{(0)} x)_t = \begin{cases} x_{t+\tau} & \text{if } t \geq 0, \\ x_{t-\tau} & \text{if } t < 0. \end{cases} \tag{2.33}$$

### 2.3.2 Assumptions on $W$:

(W1): $W$ is admissible in the sense of Definition 2.1.4.

(W2): Define

$$\alpha = \liminf_{|x| \to \infty} V(x) - E_0 \leq \infty, \tag{2.34}$$

where $E_0 = \inf \mathrm{spec} H$. We assume that there exist $D \geq 0$ and $0 \leq C < \alpha$ such that

$$-\mathcal{H}_T(x) \leq -\mathcal{H}_T(\theta_\tau^{(0)} x) + C\tau + D \tag{2.35}$$

for all $T, \tau > 0$ and all $x \in C(\mathbb{R}, \mathbb{R}^d)$.

### 2.3.3 Remarks:

a) An immediate consequence of (W1) is

$$\left|\mathcal{H}_{\mathbb{R}\times[-S,S]}(x)\right| \leq 2C_\infty S, \quad \text{and} \quad \left|\mathcal{H}_{[-S,S]\times\mathbb{R}}(x)\right| \leq 2C_\infty S. \tag{2.36}$$

(2.36) will be used frequently below.

b) In contrast to (2.36), the quantity

$$I = 2 \sup_{x\in C(\mathbb{R},\mathbb{R}^d)} \left| \int_{-\infty}^{0} ds \int_0^\infty dt\, W(x_t, x_s, |t-s|) \right| \tag{2.37}$$

is not controlled by (W1) (and also not by (W2)). $I$ is the interaction energy between the left and right half line. According to the folklore, $I < \infty$ is a sufficient condition for uniqueness of the infinite volume Gibbs measure. Such a strong result is not available at present, but [38] and [29] have some results about uniqueness, and [38] gives an example where uniqueness fails when (2.37) is not finite.

c) In the context of Nelson's model, i.e. with

$$W(x,y,t) = -\frac{1}{2}\int \frac{|\hat{\varrho}(k)|^2}{2\omega(k)} e^{ik(x-y)} e^{-\omega(k)|t|}\, dk,$$

$I$ is closely connected with infrared divergence, cf. 1.5.15. Indeed, in this case

$$I = \int \frac{|\hat{\varrho}(k)|^2}{2\omega(k)^3}\, dk = \left\| |B|^2 \varrho \right\|_K^2,$$

which is finite if and only if the infrared cutoff is made. One important feature of our method below is that it also works in cases where $I = \infty$.

d) (W2) looks a little mysterious at first, but the proof of Theorem 2.3.7 will show how it comes about naturally. To see when (W2) is fulfilled, note that by (2.7),

$$\begin{aligned} -\int_0^T ds \int_0^T dt\, W(x_t, x_s, |t-s|) &\leq 4C_\infty\tau - \int_\tau^{T+\tau} ds \int_\tau^{T+\tau} dt\, W(x_t, x_s, |t-s|) = \\ &= 4C_\infty\tau - \int_0^T ds \int_0^T dt W(x_{t+\tau}, x_{s+\tau}, |t-s|), \end{aligned}$$

and similarly for the region $[-T, 0]^2$. Thus, if we suppose

$$\tilde{I} = \sup_{x\in C(\mathbb{R},\mathbb{R}^d)} \int_{-\infty}^{0} ds \int_0^\infty dt\, |W(x_t, x_s, |t-s|)| < \infty,$$

then $8C_\infty < \alpha$ is a sufficient condition for (W2). In case $\tilde{I} = \infty$, it is not hard to see that if there exist $L, M > 0$ with

$$\int_{-T}^{0} ds \int_{0}^{T} dt \big(W(x_s, x_t, |s-t| + 2\tau) - W(x_s, x_t, |s-t|)\big) \leq L\tau + M \quad (2.38)$$

uniformly in $x \in C(\mathbb{R}, \mathbb{R}^d)$ and $T > 0$, then $16C_\infty + 2L < \alpha$ is a sufficient condition for (W2). (2.38) can be checked directly for many choices of $W$, and is in particular true in the case of Nelson's model without infrared cutoff. This is seen from the estimate

$$\begin{aligned}
&\left| \int_{-T}^{0} ds \int_{0}^{T} dt \big(W(x_s, x_t, |s-t| + 2\tau) - W(x_s, x_t, |s-t|)\big) \right| \leq \\
&\leq \int_{-T}^{0} ds \int_{0}^{T} dt \int dk \, \frac{|\hat{\varrho}(k)|^2}{2\omega(k)} e^{-\omega(k)|t-s|} \left(1 - e^{-2\omega(k)\tau}\right) \leq \\
&\leq \int_{-T}^{0} ds \int_{0}^{T} dt \int dk \, \frac{|\hat{\varrho}(k)|^2}{2\omega(k)} e^{-\omega(k)|t-s|} |2\omega(k)\tau| = \tau \int \frac{|\hat{\varrho}(k)|^2}{\omega(k)^2} \, dk.
\end{aligned}$$

On the other hand, for

$$W(x, y, |t|) = \begin{cases} -\frac{1}{|t|^2+1} & \text{if } |x-y| \leq 2t \\ 0 & \text{otherwise} \end{cases}$$

$(x, y \in \mathbb{R})$ together with the path $x_t = t$, we find that $\int_{-T}^{0} ds \int_{0}^{T} dt \, W(x_s, x_t, |t-s|)$ diverges as $T \to \infty$, but e.g. $\int_{-T}^{0} ds \int_{0}^{T} dt \, W(x_{s-1}, x_{t+1}, |t-s|) = 0$. Thus (W2) need not hold in general.

In order to show existence of an infinite volume Gibbs measure, due to Proposition 2.1.12 it will be sufficient to pick a sequence $(T_n)$ with $T_n \to \infty$ and to find a cluster point of the sequence $(\mu_{T_n})_{n \in \mathbb{N}}$. To do so, we will use the following concept ...

**Definition 2.3.4** *A family $(\nu_T)$ of probability measures on $C(\mathbb{R}, \mathbb{R}^d)$ is said to be* **locally uniformly dominated** *if for each $S < \infty$ there exists a finite measure $\bar{\nu}_S$ on $\mathcal{F}_S$ with the property that for each $\varepsilon > 0$, one can find a $\delta > 0$ such that*

$$\bar{\nu}_S(A) < \delta \quad \textit{implies} \quad \limsup_{T \to \infty} \nu_T(A) < \varepsilon \quad \textit{for each } A \in \mathcal{F}_S.$$

... and the following result:

**Proposition 2.3.5** *Every locally uniformly dominated family of probability measures on $C(\mathbb{R}, \mathbb{R}^d)$ has at least one cluster point, which itself is a probability measure on $C(\mathbb{R}, \mathbb{R}^d)$.*

*Proof:* Since $C(\mathbb{R},\mathbb{R}^d)$ is a standard Borel space (cf. [18] for the notion), Proposition (4.9) of [18] and the comment thereafter imply the claim. □

We thus have to show local uniform domination of the family $(\mu_T)$. A first step is

**Proposition 2.3.6** *Define $(\mu_T)_{T>0}$ as in (2.32), and assume (V1), (V2) and (W1). Further suppose that*

*(U): For each $S>0$ and each $\varepsilon>0$, there exists $R>0$ with*

$$\mu_T(|x_s|>R)<\varepsilon \qquad \text{uniformly in } |s|<S \text{ and } T>0.$$

*Then $(\mu_T)$ is locally uniformly dominated by the restrictions of $\mu_0$ to the $\sigma$-fields $\mathcal{F}_S$.*

*Proof:* Fix $S>0$ and $\varepsilon>0$. Using (U), choose $R>0$ so large that $\mu_T(|x_s|>R)<\varepsilon/8$ uniformly in $|s|<S+1$ and $T>0$. Then with

$$B=\{x\in C(\mathbb{R},\mathbb{R}^d):|x_{-S-1}|<R,|x_{-S}|<R,|x_S|<R,|x_{S+1}|<R\},$$

we have $\mu_T(B^c)<\varepsilon/2$ uniformly in $T$. Now for arbitrary $A\in\mathcal{F}_S$,

$$\mu_T(A)=\mu_T(A\cap B^c)+\mu_T(A\cap B)\le \varepsilon/2+\mu_T(\mu_T(A\cap B|\mathcal{T}_{S+1})). \tag{2.39}$$

We are only interested in the lim sup as $T\to\infty$, so let $T>S+1$. Then the fact that $\mu_T$ is a finite volume Gibbs measure gives

$$\begin{aligned}\mu_T(A\cap B|\mathcal{T}_{S+1})(\bar x) &= \frac{1}{Z_T^S(\bar x)}\int e^{-\mathcal{H}_{\Lambda(S+1,T)}(x)}1_{A\cap B}(x)\,d\mu_0^{S+1,\bar x}(x)\le\\ &\le e^{8C_\infty(S+1)}\int 1_{A\cap B}(x)\,d\mu_0^{S+1,\bar x}(x).\end{aligned}$$

Moreover, by the special choice of $B$, an expression similar to (2.25) and the fact that $K_t(\xi,\eta)$ (using the notation as in (2.25)) and $\psi_0(\xi)$ are both bounded and bounded away from zero on the compact set $\{(\xi,\eta)\in\mathbb{R}^{2d}:|\xi|<R,|\eta|<R\}$, we get

$$\sup_{\bar x\in C(\mathbb{R},\mathbb{R}^d)}\int 1_{A\cap B}(x)\,d\mu_0^{S+1,\bar x}(x)\le M\mu_0(A)$$

for some $M>0$. Inserted in (2.39), this gives

$$\mu_T(A)\le\varepsilon/2+Me^{8C_\infty(S+1)}\mu_0(A)\qquad\text{for each } A\in\mathcal{F}_S,$$

and now local uniform domination follows by choosing $\delta<\varepsilon e^{-8C_\infty(S+1)}/2M$. □

It remains to prove (U), which turns out to require most of the work.

**Theorem 2.3.7** *Assume (V1), (V2), (V3), (W1) and (W2). Then $(\mu_T)_{T>0}$ fulfills (U).*

*Proof:* Since by (2.36) and the stationarity of $\mu_0$ we have

$$e^{-8|t|C_\infty}\int f(x_t)\,d\mu_T(x) \le \int f(x_0)\,d\mu_T(x) \le e^{8|t|C_\infty}\int f(x_t)\,d\mu_T(x)$$

for all $t \in \mathbb{R}, T > 0$ and $f \in L^\infty(\mathbb{R}^d)$, it will be sufficient to prove the claim for $t = 0$. We do so in several steps.

**Step 1:** Let $\mathbb{E}_{\mu_0}(f|x_0 = y)$ denote expectation with respect to the measure $\mu_0$ conditional on $x_0 = y$. Since $x \mapsto x_0$ has distribution $\psi_0^2 dx$, we have

$$\mu_T(|x_0| > R) = \frac{1}{Z_T}\int_{|y|>R} \psi_0^2(y)\mathbb{E}_{\mu_0}\Big(e^{-\mathcal{H}_T}\Big|x_0 = y\Big)dy. \tag{2.40}$$

In the next few steps, we will show that there exists $K > 0$ and $r > 0$ such that for all $T > 0$ and all $y \in \mathbb{R}^d$,

$$\mathbb{E}_{\mu_0}\Big(e^{-\mathcal{H}_T}\Big|x_0 = y\Big) \le \frac{K}{\psi_0(y)}\inf_{|z|\le r}\mathbb{E}_{\mu_0}\Big(e^{-\mathcal{H}_T}\Big|x_0 = z\Big). \tag{2.41}$$

Once we will have established (2.41), we can plug it into (2.40). Since moreover

$$\frac{1}{Z_T}\inf_{|z|\le r}\mathbb{E}_{\mu_0}\Big(e^{-\mathcal{H}_T}\Big|x_0 = z\Big) \le \left(\int_{|z|<r}\psi_0^2(z)\,dz\right)^{-1}\mu_T(|x_0| \le r) \le \tilde{K}$$

by an expression analogous to (2.40), we get

$$\mu_T(|x_0| > R) \le K\tilde{K}\int_{|y|>R}\psi_0(y)\,dy. \tag{2.42}$$

The hypothesis $\psi_0 \in L^1$ from (V3) will then conclude the proof.

**Step 2:** In order to prove (2.41), we change the probability space we work on. Consider

$$J : C^{(0)}(\mathbb{R},\mathbb{R}^d) \to C([0,\infty[,\mathbb{R}^{2d}), \quad (x_t)_{t\in\mathbb{R}} \mapsto (x_t, x_{-t})_{t\ge 0}, \tag{2.43}$$

with $C^{(0)}$ from Definition 2.3.1. $(Jx)_0 \in \mathbb{R}^{2d}$ is defined via the left and right hand side limits of $x_t$ as $t \to 0$, and $J$ is a bijection after making some choice for the value of $x \in C^{(0)}(\mathbb{R},\mathbb{R}^d)$ at the point 0. We will write $\mathbf{x} = (x', x'')$ for the elements of $C([0,\infty[,\mathbb{R}^{2d})$.

The image of $\mu_0(.|x_0 = z)$ under $J$ can be described explicitly. For $\mathbf{z} \in \mathbb{R}^{2d}$ denote by $\tilde{\mu}_0^{\mathbf{z}}$ the measure of the $\mathbb{R}^{2d}$-valued $P(\phi)_1$-process with potential $\tilde{V}(x,y) = V(x) + V(y)$, starting in $\mathbf{z}$. Explicitly, if we write $\tilde{\mathcal{F}}_T$ for the $\sigma$-field over $C([0,\infty[,\mathbb{R}^{2d})$

generated by point evaluations at points within $[0,T]$, then for every $\tilde{\mathcal{F}}_T$-measurable, bounded function $f$ we have

$$\int f(\mathbf{x})\, d\tilde{\mu}_0^{\mathbf{z}}(\mathbf{x}) = \frac{1}{\psi_0(z')\psi_0(z'')} \int e^{-\int_0^T (V(x'_s)+V(x''_s))\, ds} f(\mathbf{x})\psi_0(x'_T)\psi_0(x''_T)\, d\mathcal{W}^{\mathbf{z}}(\mathbf{x}). \tag{2.44}$$

Here, $\mathcal{W}^{\mathbf{z}}$ denotes $2d$-dimensional Wiener measure conditional on $\{\mathbf{x}_0 = \mathbf{z} = (z', z'')\}$, i.e. Brownian motion starting in $\mathbf{z}$. The Markov property and time reversibility of Brownian motion together with the Feynman-Kac formula imply that for each $z \in \mathbb{R}^d$, $\tilde{\mu}_0^{(z,z)}$ is the image of $\mu_0(.|x_0 = z)$ under $J$, i.e.

$$\mathbb{E}_{\mu_0}(f \circ J | x_0 = z) = \mathbb{E}_{\tilde{\mu}_0}^{(z,z)}(f).$$

Here, $\mathbb{E}_{\tilde{\mu}_0}^{(z,z)}$ denotes expectation with respect to $\tilde{\mu}_0^{(z,z)}$.

Now it is easy to check that

$$\begin{aligned}\tilde{\mathcal{H}}_T(\mathbf{x}) \;\equiv\; \mathcal{H}_T \circ J^{-1}(\mathbf{x}) = \int_0^T ds \int_0^T dt \Big( & W(x'_t, x'_s, |s-t|) + W(x''_t, x''_s, |s-t|) + \\ & + W(x'_t, x''_s, |s+t|) + W(x''_t, x'_s, |s+t|)\Big),\end{aligned} \tag{2.45}$$

and therefore

$$\mathbb{E}_{\mu_0}\Big(e^{-\mathcal{H}_T}\Big| x_0 = z\Big) = \mathbb{E}_{\tilde{\mu}_0}^{(z,z)}(e^{-\tilde{\mathcal{H}}_T}). \tag{2.46}$$

Thus we reduced our problem to investigating the expectation of $e^{-\tilde{\mathcal{H}}_T}$ with respect to the strong Markov process $\tilde{\mu}_0^{\mathbf{z}}$ as a function of the starting point $\mathbf{z}$.

**Step 3:** First note that in the representation established in Step 2, hypothesis (2.35) takes the form

$$-\tilde{\mathcal{H}}_T(\mathbf{x}) \le -\tilde{\mathcal{H}}_T \circ \theta_\tau(\mathbf{x}) + C\tau + D \quad \text{for all } \mathbf{x} \in C([0,\infty[, \mathbb{R}^{2d}), T, \tau > 0. \tag{2.47}$$

Here $\theta_\tau = J\theta_\tau^{(0)}J^{-1}$ is the usual time shift that maps $(\mathbf{x}_t)_{t\ge 0}$ to $(\mathbf{x}_{t+\tau})_{t\ge 0}$. Our strategy is to use (2.47) together with the strong Markov property of $\tilde{\mu}_0$. For $r > 0$ let

$$\tau_r(\mathbf{x}) = \inf\{t \ge 0 : |\mathbf{x}_t| \le r\}$$

be the hitting time of the centered ball with radius $r$, and let $\mathcal{F}_{\tau_r}$ be the corresponding $\sigma$-field, i.e.

$$\mathcal{F}_{\tau_r} = \{A \in \tilde{\mathcal{F}} : A \cap \{\tau_r \le t\} \in \tilde{\mathcal{F}}_t \text{ for all } t \ge 0\}.$$

In the next step we will see that $\tau_r$ is finite almost surely, and thus for each $\mathbf{x} \in \mathbb{R}^{2d}$,

$$\begin{aligned}\mathbb{E}^{\mathbf{x}}(e^{-\tilde{\mathcal{H}}_T}) &= \mathbb{E}^{\mathbf{x}}(\mathbb{E}^{\mathbf{x}}(e^{-\tilde{\mathcal{H}}_T}|\mathcal{F}_{\tau_r})) \le \mathbb{E}^{\mathbf{x}}(\mathbb{E}^{\mathbf{x}}(e^{-\tilde{\mathcal{H}}_T\circ\theta_{\tau_r}} e^{C\tau_r+D}|\mathcal{F}_{\tau_r})) = \\ &= \mathbb{E}^{\mathbf{x}}(e^{C\tau_r+D}\mathbb{E}^{\mathbf{x}}(e^{-\tilde{\mathcal{H}}_T\circ\theta_{\tau_r}}|\mathcal{F}_{\tau_r})) = \mathbb{E}^{\mathbf{x}}(e^{C\tau_r+D}\mathbb{E}^{\mathbf{x}_{\tau_r}}(e^{-\tilde{\mathcal{H}}_T})) \le \\ &\le \sup_{|\mathbf{y}|\le r} \mathbb{E}^{\mathbf{y}}(e^{-\tilde{\mathcal{H}}_T})\mathbb{E}^{\mathbf{x}}(e^{C\tau_r+D}).\end{aligned} \tag{2.48}$$

All expectations above and henceforth are with respect to $\tilde{\mu}_0$. It remains to get a good estimate on the second factor on the right hand side of (2.48) and to estimate the supremum in the first factor against an infimum. This will be done in Steps 4 and 5.

**Step 4:** Here we show that there exists $r > 0$ and $\gamma > 0$ such that for all $\mathbf{x} \in \mathbb{R}^{2d}$ we have

$$\mathbb{E}^{\mathbf{x}}(e^{C\tau_r}) \leq 1 + \frac{C\,\|\psi_0\|_\infty}{\gamma}\left(\frac{1}{\psi_0(x')} + \frac{1}{\psi_0(x'')}\right). \tag{2.49}$$

To do so, we pick $\gamma$ with $0 < \gamma < \alpha - C$ and $r$ so large that $V(x) > C + \gamma$ for all $x \in \mathbb{R}^d$ with $|x| > r/\sqrt{2}$. This is possible by assumption (W2). Obviously,

$$\{\mathbf{x} \in \mathbb{R}^{2d} : |\mathbf{x}| > r\} \subset \{\mathbf{x} \in \mathbb{R}^{2d} : |x'| > r/\sqrt{2}\} \cup \{\mathbf{x} \in \mathbb{R}^{2d} : |x''| > r/\sqrt{2}\},$$

and with (2.44) it follows that

$$\begin{aligned}
&\psi_0(z')\psi_0(z'')\tilde{\mu}_0^{\mathbf{z}}(\tau_r > t) = \\
&= \int e^{-\int_0^t (V(x'_s)+V(x''_s))\,ds} 1_{\{|\mathbf{x}_s|>r\ \forall s\leq t\}} \psi_0(x'_t)\psi_0(x''_t)\,d\mathcal{W}^{\mathbf{z}}(\mathbf{x}) \leq \\
&\leq \int e^{-\int_0^t V(x'_s)\,ds} e^{-\int_0^t V(x''_s)\,ds} \left(1_{\{|(x'_s)|>r/\sqrt{2}\ \forall s\leq t\}} + 1_{\{|(x''_s)|>r/\sqrt{2}\ \forall s\leq t\}}\right) \times \\
&\quad \times \psi_0(x'_t)\psi_0(x''_t)\,d\mathcal{W}^{z'}(x')\,d\mathcal{W}^{z''}(x'') = \\
&= \psi_0(z'') \int e^{-\int_0^t V(x'_s)\,ds} 1_{\{|(x'_s)|>r/\sqrt{2}\ \forall s\leq t\}} \psi_0(x'_t)\,d\mathcal{W}^{z'}(x') + \\
&\quad + \psi_0(z') \int e^{-\int_0^t V(x''_s)\,ds} 1_{\{|(x''_s)|>r/\sqrt{2}\ \forall s\leq t\}} \psi_0(x''_t)\,d\mathcal{W}^{z''}(x'') \leq \\
&\leq (\psi_0(z') + \psi_0(z''))\,\|\psi_0\|_\infty\, e^{-(C+\gamma)t}.
\end{aligned}$$

The second equality above is due the eigenvalue equation $e^{-tH_0}\psi_0 = \psi_0$ and the Feynman-Kac formula. It follows that

$$\tilde{\mu}_0^{\mathbf{z}}(\tau_r > t) \leq \left(\frac{1}{\psi_0(z')} + \frac{1}{\psi_0(z'')}\right)\|\psi_0\|_\infty\, e^{-(C+\gamma)t},$$

and using the equality

$$\mathbb{E}^{\mathbf{z}}(e^{C\tau_r}) = 1 + \int_0^\infty Ce^{Ct}\mathbb{E}^{\mathbf{z}}(\tau_r > t)\,dt$$

we arrive at (2.49).

**Step 5:** Let $r > 0$ be as in Step 4. We will show that there exists $M > 0$ such that

$$\sup_{|\mathbf{y}|\leq r} \mathbb{E}^{\mathbf{y}}(e^{-\tilde{\mathcal{H}}_T}) \leq M \inf_{|\mathbf{y}|\leq r} \mathbb{E}^{\mathbf{y}}(e^{-\tilde{\mathcal{H}}_T}) \tag{2.50}$$

uniformly in $T > 0$. Denote by $P_t(\mathbf{x}, \mathbf{y})$ the transition density from $\mathbf{x}$ to $\mathbf{y}$ in time $t$ of the process $\tilde{\mu}_0$, and by $K_t(x, y)$ the integral kernel of $e^{-tH}$. The Feynman-Kac formula implies

$$E_{\mu_0}(f(x_t)|\mathcal{F}_{\{0\}})(y) = \frac{1}{\psi_0(y)} \int K_t(y, z)\psi_0(z)f(z)\, dz \qquad (y \in \mathbb{R}^d), \tag{2.51}$$

and together with (2.44) this shows

$$P_t(\mathbf{x}, \mathbf{y}) = \frac{\psi_0(y')\psi_0(y'')}{\psi_0(x')\psi_0(x'')} K_t(x', y')K_t(x'', y''). \tag{2.52}$$

$\psi_0$ and $K_t$ are both uniformly bounded and bounded away from zero on compact sets (cf. (S2) in Appendix A.2), thus for each $R > 0$ the quantity

$$S_t(R, r) = \sup \left\{ \frac{P_t(\mathbf{x}, \mathbf{z})}{P_t(\mathbf{y}, \mathbf{z})} : \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{2d}, |\mathbf{x}| \le r, |\mathbf{y}| \le r, |\mathbf{z}| \le R \right\}$$

is finite. Defining $\tilde{\mathcal{H}}^1_T$ like in (2.45) but with the integrals starting at 1 rather than at 0, we see from (2.36) that

$$-\tilde{\mathcal{H}}_T(\mathbf{x}) - 4C_\infty \le -\tilde{\mathcal{H}}^1_T(\mathbf{x}) \le -\tilde{\mathcal{H}}_T(\mathbf{x}) + 4C_\infty$$

for all $\mathbf{x}$ and all $T$. Putting $B = \{|\mathbf{x}_1| < R\}$, for each $\mathbf{y}$ with $|\mathbf{y}| < r$ we have

$$\mathbb{E}^{\mathbf{y}}(e^{-\tilde{\mathcal{H}}_T}) \le e^{4C_\infty}\mathbb{E}^{\mathbf{y}}(1_B e^{-\tilde{\mathcal{H}}^1_T}) + e^{C+D}\mathbb{E}^{\mathbf{y}}(1_{B^c} e^{-\tilde{\mathcal{H}}_T \circ \theta_1}). \tag{2.53}$$

Defining $\bar{\mathcal{H}}_T$ as in (2.45) but with $|s+t+2|$ appearing instead of $|s+t|$ everywhere, in the first term on the right hand side of (2.53) we find

$$\begin{aligned}
\mathbb{E}^{\mathbf{y}}(1_B e^{-\tilde{\mathcal{H}}^1_T}) &= \int_{|\mathbf{z}|<R} P_1(\mathbf{y}, \mathbf{z})\mathbb{E}^{\mathbf{z}}(e^{-\bar{\mathcal{H}}_{T-1}})\, d\mathbf{z} \le \\
&\le S_1(R, r) \int_{|\mathbf{z}|\le R} P_1(\mathbf{x}, \mathbf{z})\mathbb{E}^{\mathbf{z}}(e^{-\bar{\mathcal{H}}_{T-1}})\, d\mathbf{z} = \\
&= S_1(R, r)\mathbb{E}^{\mathbf{x}}(1_B e^{-\tilde{\mathcal{H}}^1_T}) \le S_1(R, r)e^{4C_\infty}\mathbb{E}^{\mathbf{x}}(e^{-\tilde{\mathcal{H}}_T})
\end{aligned} \tag{2.54}$$

for each $\mathbf{x}$ with $|\mathbf{x}| \le r$. Turning to the second term on the right hand side of (2.53), equations (2.48) and (2.49) give

$$\begin{aligned}
&\mathbb{E}^{\mathbf{y}}(1_{B^c} e^{-\tilde{\mathcal{H}}_T \circ \theta_1}) = \int_{|\mathbf{z}|>R} P_1(\mathbf{y}, \mathbf{z})\mathbb{E}^{\mathbf{z}}(e^{-\tilde{\mathcal{H}}_T})\, d\mathbf{z} \le \\
&\le \sup_{|\mathbf{x}|\le r} \mathbb{E}^{\mathbf{x}}(e^{-\tilde{\mathcal{H}}_T}) \int_{|\mathbf{z}|>R} P_1(\mathbf{y}, \mathbf{z})\mathbb{E}^{\mathbf{z}}(e^{C\tau_r + D})\, d\mathbf{z} \le \\
&\le \sup_{|\mathbf{x}|\le r} \mathbb{E}^{\mathbf{x}}(e^{-\tilde{\mathcal{H}}_T})e^D \int_{|\mathbf{z}|>R} P_1(\mathbf{y}, \mathbf{z}) \left(1 + \frac{C\,\|\psi_0\|_\infty}{\gamma} \left(\frac{1}{\psi_0(z')} + \frac{1}{\psi_0(z'')}\right)\right) d\mathbf{z}.
\end{aligned} \tag{2.55}$$

By (2.52) and the eigenvalue equation, we have

$$\int P_1(\mathbf{y},\mathbf{z})\left(\frac{1}{\psi_0(z')}+\frac{1}{\psi_0(z'')}\right)d\mathbf{z} = \\ = \frac{1}{\psi_0(y')}\int K_1(y'',z)\,dz+\frac{1}{\psi_0(y'')}\int K_1(y',z)\,dz. \tag{2.56}$$

By (S3) from Appendix A.2, the integrals appearing above are bounded as functions of $y'$ and $y''$, respectively, and thus the right hand side of (2.56) is uniformly bounded on $\{\mathbf{y}:|\mathbf{y}|<r\}$. This implies that there exists $\bar{R}>0$ and $\delta<1$ such that

$$\int_{|\mathbf{z}|>\bar{R}} P_1(\mathbf{y},\mathbf{z})\left(1+\frac{C\,\|\psi_0\|_\infty}{\gamma}\left(\frac{1}{\psi_0(z')}+\frac{1}{\psi_0(z'')}\right)\right)d\mathbf{z}\le e^{-(C+2D)}\delta$$

uniformly on $\{\mathbf{y}:|\mathbf{y}|<r\}$. Plugging this result together with (2.54) into (2.53), we arrive at

$$\mathbb{E}^{\mathbf{y}}(e^{-\tilde{\mathcal{H}}_T})\le S_1(\bar{R},r)e^{8C_\infty}\mathbb{E}^{\mathbf{x}}(e^{-\tilde{\mathcal{H}}_T})+\delta\sup_{|\mathbf{z}|\le r}\mathbb{E}^{\mathbf{z}}(e^{-\tilde{\mathcal{H}}_T}), \tag{2.57}$$

which is valid for all $\mathbf{x},\mathbf{y}$ with $|\mathbf{x}|,|\mathbf{y}|\le r$. By taking the supremum over $\mathbf{y}$ and the infimum over $\mathbf{x}$ in (2.57) and rearranging, we find

$$\sup_{|\mathbf{y}|\le r}\mathbb{E}^{\mathbf{y}}(e^{-\tilde{\mathcal{H}}_T})\le\frac{S_1(\bar{R},r)e^{8C_\infty}}{1-\delta}\inf_{|\mathbf{y}|\le r}\mathbb{E}^{\mathbf{y}}(e^{-\tilde{\mathcal{H}}_T}),$$

which concludes Step 5 and the proof. □

Putting things together, we easily arrive at

**Theorem 2.3.8** *Assume (V1), (V2), (V3), (W1) and (W2). Then for the potential $W$ and the reference measure $\mu_0$ an infinite volume Gibbs measure exists.*

*Proof:* By Theorem 2.3.7 and Proposition 2.3.6, e.g. the sequence $(\mu_n)_{n\in\mathbb{N}}$ has a cluster point $\mu$. By Proposition 2.1.12, $\mu$ is an infinite volume Gibbs measure. □

Let us denote by $\mu$ any Gibbs measure obtained via Theorem 2.3.8. Due to the good control on the stationary density we obtain in Theorem 2.3.7, we have the following estimate on the growth of paths under $\mu$.

**Lemma 2.3.9** *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be monotone increasing with $f(x)\to\infty$ as $x\to\infty$, and suppose that*

$$\sum_{n=1}^{\infty}\int_{|y|>f(n)}\psi_0(y)\,dy<\infty \tag{2.58}$$

*Then for $\mu$-almost every path $x\in C(\mathbb{R},\mathbb{R}^d)$, we have*

$$\limsup_{n\to\infty}\frac{|x_n|}{f(n)}\le 1.$$

*Proof:* The first Borel-Cantelli lemma will yield the result once we have checked that

$$\sum_{n=1}^{\infty} \mu(|x_n| > f(n)) < \infty. \tag{2.59}$$

By the stationarity of $\mu$ and equation (2.42), there exists a constant $M$ such that

$$\mu(|x_n| > f(n)) = \mu(|x_0| > f(n)) \leq M \int_{|y|>f(n)} \psi_0(y)\, dy$$

for $n$ large enough, and the claim follows by assumption (2.58). □

From Theorem 2.2.10 we infer that for $s \geq 0$ the estimate

$$\liminf_{|x|\to\infty} V(x)/|x|^{2s} > 0$$

implies the existence of constants $A > 0, \beta > 0$ such that

$$\psi_0(y) \leq A \exp(-\beta |y|^{s+1})$$

for all $y \in \mathbb{R}^d$. In this case, Lemma 2.3.9 implies

$$\limsup_{n\to\infty} \frac{|x_n|}{(\gamma \ln(n))^{1/(s+1)}} = 0 \tag{2.60}$$

for each $\gamma > 1/\beta$ and $\mu$-almost all $x \in C(\mathbb{R}, \mathbb{R}^d)$. Using the cluster expansion and restricting to $s > 1$, [29] prove (2.60) with $n \in \mathbb{N}$ replaced by $t \in \mathbb{R}$, which is slightly stronger.

Although we will not do it here, our method from Theorem 2.3.7 can be easily adapted to the lattice context, where it yields a new way of proving existence of Gibbs measures for one-dimensional systems of unbounded spins as given in Example 2.1.1. For such systems, extremely powerful methods are already available: there is the superstability estimate by D. Ruelle [43], applied in [28], which has the big advantage of not being restricted to one-dimensional systems; there are the results of R. L. Dobrushin [11, 12], which are valid only for one-dimensional systems, but extremely general otherwise. However, superstability corresponds to rapidly growing single site potential, while one of Dobrushin's few restrictions is that the interaction energy between left and right half-space must be bounded. Thus our method covers some new situations in the discrete context also.

# Chapter 3

# Ground state expectations for Nelson's model

## 3.1 The existence question

The main results of this chapter will appear in Sections 3.2 and 3.3, where we use Gibbs measures to evaluate ground state expectations of the Nelson Hamiltonian. Before we do so, however, we should discuss existence of the ground state. This discussion is done in the present section, together with an overview about known results concerning the connections of Gibbs measures with the question of existence of a ground state.

For convenience, we repeat some notions from Chapters 1 and 2.

### 3.1.1 The setup:

a) $H_{\mathcal{F}}$ is the Hamiltonian of Nelson's model in Fock space. We have

$$H_{\mathcal{F}} = H_{\mathrm{p}} \otimes 1 + 1 \otimes H_{\mathrm{f}} + H_{\mathrm{I}},$$

where $H_{\mathrm{p}}$ is defined in (1.31), $H_{\mathrm{f}}$ in (1.29) and $H_{\mathrm{I}}$ in (1.30). Recall that we also assume (1.26) - (1.28) throughout.

b) The Nelson Hamiltonian in function space is given by

$$H = \Theta^{-1} H_{\mathcal{F}} \Theta,$$

where $\Theta$ is the combination of the ground state transform and the Wiener-Itô-Segal isomorphism, cf. Proposition 1.5.4 and the text below. In the language of function space, (1.28) asserts that we have to have $\varrho \in K$ and $|B|\varrho \in K$.

c) $H$ acts in $L^2(\mathsf{P}_0)$, where $\mathsf{P}_0 = \mathsf{N}_0 \otimes \mathsf{G}$, $\mathsf{N}_0$ is the stationary measure of the $P(\phi)_1$-process corresponding to $H_{\mathrm{p}}$, and $\mathsf{G}$ is the stationary measure of the infinite dimensional Ornstein-Uhlenbeck process given in Definition 1.4.8.

d) The finite volume Gibbs measures $\mathcal{N}_T$ and $\mathcal{P}_T$ are given in Remark 2.1.9. Their infinite volume limits are denoted by $\mathcal{N}$ and $\mathcal{P}$, cf. Corollary 2.2.5. The stationary measures of $\mathcal{N}$ and $\mathcal{P}$ are denoted by $\mathsf{N}$ and $\mathsf{P}$.

e) Two special choices of the parameters $\omega$ and $\varrho$ are particularly important and will be singled out here. Taking

$$\omega(k) = |k| \qquad \text{and} \qquad \hat{\varrho}(k) = e1_{\{r<|k|<R\}}$$

with $0 < r \ll 1$ and $R \gg 1$, we arrive at the **massless Nelson model** with (standard) infrared cutoff $r$, ultraviolet cutoff $R$ and coupling strength $e$. Taking

$$\omega(k) = \sqrt{|k|^2 + m^2} \qquad \text{and} \qquad \hat{\varrho}(k) = e1_{\{|k|<R\}}$$

with $m > 0$ and $R \gg 1$, we get the **massive Nelson model** with mass $m$, ultraviolet cutoff $R$ and coupling strength $e$.

Proving existence of a ground state $\Psi \in L^2(\mathsf{P}_0)$ of the operator $H$ (or, equivalently, existence of a ground state $\Psi_{\mathcal{F}}$ of the operator $H_{\mathcal{F}}$) is a difficult task in case $\operatorname{essinf}_{k\in\mathbb{R}^d}\omega(k) = 0$. The reason is that in this case, $H$ has no spectral gap, i.e. the infimum of the essential spectrum of $H$ is equal to the infimum of its whole spectrum. The best result available in our context is

**Theorem 3.1.2** *[48] Let $H$ be the Nelson Hamiltonian. In addition to (1.26) - (1.28) assume*

*(i):* $I = \int \frac{|\hat{\varrho}|^2}{2\omega^3}\, dk < \infty$. *This condition will be referred to as the* **infrared condition**.

*(ii):* $\Sigma - E_{\mathrm{p}} > \int \frac{|\hat{\varrho}|^2 k^2}{\omega(2\omega + k^2)}\, dk$,
*where $\Sigma$ is the infimum of the essential spectrum of $H_{\mathrm{p}}$, and $E_{\mathrm{p}} = \inf \operatorname{spec} H_{\mathrm{p}}$.*

*Then $H$ has a ground state $\Psi \in L^2(\mathsf{P}_0)$.*

In [17] more general particle-field couplings are allowed. When specialized to our setting, the assumptions in [17] correspond to $\Sigma = \infty$.

(ii) is essentially a restriction on the coupling strength and is needed for currently available proofs. Physically, one would assume that it is possible to do without (ii). Indication in this direction has been given by Griesemer et. al. [20], who study a slightly different model. Note however that if $\lim_{|q|\to\infty} V(q) = \infty$, then $\Sigma = \infty$, and (ii) follows from (1.26) - (1.28).

Condition (i) has already been discussed in 1.5.15, where an explanation for the name 'infrared condition' was given, and also in Remark 2.3.3 c) in the context

of Gibbs measures. In the language of function space, (i) reads $|B|^2\varrho \in K$. The quantity $I$ also appears explicitly in many of the estimates of Section 3.3. It has been shown in [30] that, given some technical assumptions, the infrared condition is also necessary for the existence of a ground state $\Psi \in L^2(\mathsf{P}_0)$. Interestingly, if (i) fails, we have a scenario where the local limit $\mathcal{P}$ of the sequence

$$d\mathcal{P}_T(q,\phi) = \frac{1}{Z_T}\exp\left(-\int_{-T}^{T}\phi_s(\varrho_{q(s)})\,ds\right)\,d\mathcal{P}_0(q,\phi) \tag{3.1}$$

from (2.16) exists, but is not locally absolutely continuous to $\mathcal{P}_0$. In fact, $\mathcal{P}|_{\mathcal{F}_{\{0\}}}$ is then singular with respect to $\mathsf{P}_0 = \mathcal{P}_0|_{\mathcal{F}_{\{0\}}}$. According to our definition, $\mathcal{P}$ is thus not a Gibbs measure relative to the reference measure $\mathcal{P}_0$. We will spend the rest of this section with a slightly informal discussion of this phenomenon.

In order to show that $\mathcal{P}_T$ converges locally, fix $\bar{q} \in C(\mathbb{R},\mathbb{R}^d)$ in (3.1) and write $\mathcal{P}_T^{\bar{q}}$ for the measure $\mathcal{P}_T$ conditional on $\{q = \bar{q}\}$. $\mathcal{P}_T^{\bar{q}}$ is a Gaussian measure with mean

$$\int \phi_t(f)\,d\mathcal{P}_T^{\bar{q}}(\phi) = M_T^{t,\bar{q}}(f) = -\int_{-T}^{T} ds \int dk\,\frac{\overline{\hat{f}(k)}\hat{\varrho}(k)e^{-ik\bar{q}_s}}{2\omega(k)}e^{-\omega(k)|t-s|}$$

($f \in K$, $t \in \mathbb{R}$) and covariance equal to that of $\mathcal{G}$. Moreover,

$$\int F(q,\phi)\,d\mathcal{P}_T(q,\phi) = \int\left(\int F(q,\phi)\,d\mathcal{P}_T^q(\phi)\right)\,d\mathcal{N}_T(q), \tag{3.2}$$

where $\mathcal{N}_T$ is from (2.17). In words, $\mathcal{P}_T$ is a mixture of Gaussian measures that differ from $\mathcal{G}$ only by their mean value.

By the results of Section 2.3, $\mathcal{N}_T$ converges locally to some limiting measure $\mathcal{N}$ in many cases where (i) from Theorem 3.1.2 fails. By the convergence theory for Gaussian measures ([8], Section 3.8), it is now sufficient to show that the set

$$\mathbf{M} = \{M_T^{t,\bar{q}} : T \in \mathbb{R}, t \in [-S,S], \bar{q} \in C(\mathbb{R},\mathbb{R}^d)\} \subset \Phi \tag{3.3}$$

is relatively compact in $\Phi$, cf. [30] for details. If we do assume the infrared condition for a moment, from

$$\begin{aligned}
&\int_{-T}^{T} ds \int dk \left|\frac{\overline{\hat{f}(k)}\hat{\varrho}(k)e^{-ik\bar{q}_s}}{2\omega(k)}e^{-\omega(k)|t-s|}\right| \\
&\quad\leq \int_{-\infty}^{\infty} ds \int \frac{|\hat{f}||\hat{\varrho}|e^{-\omega|t-s|}}{2\omega}\,dk = \int \frac{|\hat{f}||\hat{\varrho}|}{\omega^2}\,dk \leq \\
&\quad\leq \|f\|_K\left(\int \frac{2|\hat{\varrho}|^2}{\omega^3}\,dk\right)^{1/2} = 2\,\|f\|_K\sqrt{I} < \infty,
\end{aligned} \tag{3.4}$$

it follows that $\mathbf{M}$ is bounded in $K$, and since the embedding $K \to \Phi$ is compact, $\mathbf{M}$ is relatively compact in $\Phi$. Of course, we are more interested in the case where the infrared condition is not fulfilled. Then we need to enlarge $\Phi$ slightly compared to the setup from Chapter 1. While there we required $A^{-1}$ and $|B|^{-1}A^{-1}$ to be Hilbert-Schmidt operators, we now also assume that $|B|A^{-1}$ is a compact operator in $K$. By our conditions on $\omega$, $D(|B|) \cap D(|B|^{-1})$ is dense in $K$, and thus we can construct $A$ easily with the method given after Proposition 1.4.11. [30] gives a concrete example of such an operator. A calculation analogous to (3.4) shows that the set $\{|B|^{-1}\phi : \phi \in \mathbf{M}\}$ is bounded in $K$, and thus $\{\|A|B|^{-1}\phi\|_\Phi : \phi \in \mathbf{M}\}$ is bounded. Our additional assumption now gives relative compactness of $\mathbf{M}$ in $\Phi$. We summarize:

**Proposition 3.1.3** *Under the assumptions made above, $\mathcal{P}_T$ converges locally to a probability measure $\mathcal{P}$. For $F \in L^1(\mathcal{P})$,*

$$\int F(q,\phi)d\mathcal{P}(q,\phi) = \int \left( \int F(q,\phi)d\mathcal{P}^q(\phi) \right) d\mathcal{N}(q), \tag{3.5}$$

*where $\mathcal{P}^q$ is the Gaussian measure on $\Phi$ with mean*

$$\int \phi_t(f)\, d\mathcal{P}^q(\phi) = M^{t,q}(f) = -\int_{-\infty}^{\infty} ds \int dk\, \frac{\overline{\hat{f}(k)}\hat{\varrho}(k)e^{ikq_s}}{2\omega(k)} e^{-\omega(k)|t-s|}$$

*and covariance equal to that of $\mathcal{G}$.*

In the case of the massless Nelson model in space dimension $d = 3$, $I = \infty$ implies that $\mathcal{P}|_{\mathcal{F}_{\{0\}}}$ is singular with respect to $\mathsf{P}_0$. The reason for this lies in formula (3.5): For fixed $q \in C(\mathbb{R}, \mathbb{R}^d)$ with $|q_s|^2 \leq C|s|$, one uses the inequality

$$\cos(kq_s) \geq 1 - \frac{1}{2}|q_s|^2|k|^2 \geq 1 - \frac{C}{2}|s||k|^2$$

in a calculation similar to (3.4) to show that the mean of $\mathcal{P}^q|_{\mathcal{F}_{\{0\}}}$ is not an element of $K$. The Cameron-Martin criterion (cf. [8], Theorem 2.4.5) then implies that $\mathcal{P}^q|_{\mathcal{F}_{\{0\}}}$ and $\mathsf{P}_0$ are mutually singular. A similar argument shows that for each $q, \tilde{q}$ with the above growth restriction, $\mathcal{P}^q|_{\mathcal{F}_{\{0\}}}$ and $\mathcal{P}^{\tilde{q}}|_{\mathcal{F}_{\{0\}}}$ are mutually absolutely continuous, implying the existence of a set $N \in \mathcal{F}_0$ with $\mathcal{P}_0(N) = 1$ but $\mathcal{P}^q(N) = 0$ for all $q$ growing not faster than a square root. Since by results like Lemma 2.3.9, the latter set of paths $q$ carries full $\mathsf{N}$-measure, Fubini's theorem and (3.5) imply $\mathcal{P}(N) = 0$, proving the claim.

$\mathcal{P}$ is, however, a Gibbs measure with respect to a different reference measure $\mathcal{P}_1$ (and a different interaction). $\mathcal{P}_1$ is obtained from $\mathcal{P}_0$ by shifting the mean of the Gaussian part of $\mathcal{P}_0$ so that $\mathcal{P}_1$ is mutually absolutely continuous with $\mathcal{P}^q|_{\mathcal{F}_{\{0\}}}$

for each $q$ not growing too fast. We refer to [31] for details. It is also shown there that the Nelson Hamiltonian has a ground state in $L^2(\mathsf{P}_1)$ in this case. The proof uses specific features of the massless Nelson model and assumes a rapidly growing potential in $H_{\mathrm{p}}$. These results give rise to an interesting

**3.1.4 Open question:** Suppose that a Hamiltonian $H = H_0 + \mathcal{V}$ gives rise to a family of finite volume Gibbs measures via a Feynman-Kac formula, analogously to $\mathcal{P}_T$. Suppose further that the local limit of the finite volume Gibbs measures exists and is locally absolutely continuous with respect to the reference process with generator $H_0$. (In other words, assume that an infinite volume Gibbs measure for the potential $\mathcal{V}$ and the reference process with generator $H_0$ exists.)
Is this a sufficient condition for the existence of a ground state of $H$ in the $L^2$-space over the stationary measure of the reference process?

We would guess that the answer is yes, but unfortunately were unable to prove it. Thus for the rest of this work, we will assume that the infrared condition is fulfilled.

## 3.2 Ground state expectations as Gibbs averages

We now establish an explicit formula for writing expectations $\langle \Psi, L\Psi \rangle_{L^2(\mathsf{P}_0)}$ of an operator $L$ as Gibbs averages with respect to $\mathcal{N}$. The results of this and the next section come from [4].

We have the following

**3.2.1 Standing assumptions:**

(i): $H$ has a ground state $\Psi$ in $L^2(\mathsf{P}_0)$.

(ii): The infrared condition is fulfilled, i.e.

$$I = 2 \sup_{q \in C(\mathbb{R}, \mathbb{R}^d)} \int_{-\infty}^{0} ds \int_{0}^{\infty} dt \, |W(q_s - q_t, s - t)| = \int \frac{|\hat{\varrho}|^2}{2\omega^3} \, dk < \infty. \qquad (3.6)$$

In order to state our main theorem, we introduce some special elements of $K$. For each $T \in [0, \infty], q \in C(\mathbb{R}, \mathbb{R}^d)$ we define

$$\begin{aligned} f^+_{T,q} &= -\int_0^T e^{-\frac{1}{2}|s||B|^{-2}} \varrho_{q(s)} \, ds, \\ f^-_{T,q} &= -\int_{-T}^0 e^{-\frac{1}{2}|s||B|^{-2}} \varrho_{q(s)} \, ds, \end{aligned}$$

and write $f_q^{\pm}(k)$ for $f_{\infty,q}^{\pm}(k)$. More explicitly, we have

$$\begin{aligned}
\hat{f}_{T,q}^{+}(k) &= -\int_0^T \hat{\varrho}(k) e^{-ikq_s} e^{-\omega(k)|s|}\, ds, \\
\hat{f}_{T,q}^{-}(k) &= -\int_{-T}^0 \hat{\varrho}(k) e^{-ikq_s} e^{-\omega(k)|s|}\, ds, \\
\left\langle f_q^{-}, f_q^{+} \right\rangle_K &= -2\int_{-\infty}^0 ds \int_0^\infty dt\, W(q_t - q_s, t-s), \qquad (3.7)
\end{aligned}$$

and

$$\left\| f_q^{\pm} \right\|_K^2 \leq \int \frac{|\hat{\varrho}(k)|^2}{2\omega(k)^3}\, dk = I < \infty. \qquad (3.8)$$

Thus indeed $f_q^{\pm} \in K$.

**Theorem 3.2.2** *Let $L$ be a bounded operator on $L^2(\mathsf{G})$. Then*

$$\begin{aligned}
\langle \Psi, (1 \otimes L)\Psi \rangle_{L^2(\mathsf{P}_0)} &= \int \left\langle :\exp(\phi(f_q^{-})):, L\ :\exp(\phi(f_q^{+})): \right\rangle_{L^2(\mathsf{G})} \times \\
&\quad \times \exp\left( 2\int_{-\infty}^0 ds \int_0^\infty dt\, W(q_t - q_s, t-s) \right) d\mathcal{N}(q). \qquad (3.9)
\end{aligned}$$

*Proof:* Put

$$\Psi_T := \frac{1}{\|e^{-TH}1\|} e^{-TH} 1.$$

Then by the Feynman-Kac-Nelson formula, we have in $L^2$-sense

$$\Psi_T(\bar{q}, \bar{\phi}) = \frac{1}{\sqrt{Z_T}} \int \exp\left( -\int_0^T \phi_s(\varrho_{q(s)})\, ds \right) d\mathcal{P}_0^{\bar{q},\bar{\phi}}(q, \phi), \qquad (3.10)$$

where $\mathcal{P}_0^{\bar{q},\bar{\phi}} = \mathcal{N}_0^{\bar{q}} \otimes \mathcal{G}^{\bar{\phi}}$ denotes the measure $\mathcal{P} = \mathcal{N}_0 \otimes \mathcal{G}$ conditional on $\{q_0 = \bar{q}, \phi_0 = \bar{\phi}\}$. $\mathcal{G}^{\bar{\phi}}$ is a Gaussian measure with mean

$$M^{\bar{\phi},t}(f) \equiv \int \phi_t(f)\, d\mathcal{G}^{\bar{\phi}}(\phi) = \bar{\phi}(e^{-\frac{1}{2}|t||B|^{-2}} f) \qquad (t \in \mathbb{R}, f \in K) \qquad (3.11)$$

and covariance

$$\int \phi_t(f)\phi_s(g)\, d\mathcal{G}^{\bar{\phi}}(\phi) - M^{\bar{\phi},t}(f) M^{\bar{\phi},s}(g) = \int \frac{\hat{f}\overline{\hat{g}}}{2\omega} \left( e^{-\omega|t-s|} - e^{-\omega(|t|+|s|)} \right) dk; \quad (3.12)$$

(3.11) and (3.12) follow from Lemma 1.4.9, cf. also the proof of Proposition 1.4.10. Now the integration with respect to $\mathcal{G}^{\bar{\phi}}$ in (3.10) can be carried out with the result

$$\begin{aligned}
&\Psi_T(\bar{q}, \bar{\phi}) = \frac{1}{\sqrt{Z_T}} \int \exp(\bar{\phi}(f_{T,q}^{+})) \times \qquad (3.13) \\
&\times \exp\left( \frac{1}{2} \int_0^T ds \int_0^T dt \int dk \frac{|\hat{\varrho}(k)|^2}{2\omega(k)} \cos(k(q_s - q_t)) \left( e^{-\omega(k)|t-s|} - e^{-\omega(k)(t+s)} \right) \right) d\mathcal{N}_0^{\bar{q}}.
\end{aligned}$$

By Proposition 1.5.9 a) we have,

$$\exp(\bar\phi(f^+_{T,q})) = {:}\exp(\bar\phi(f^+_{T,q})){:}\ \exp\left(\frac{1}{2}\int_0^T ds\int_0^T dt\int dk\,\frac{|\hat\varrho(k)|^2}{2\omega(k)}\cos(k(q_s-q_t))e^{-\omega(k)(t+s)}\right),$$

and hence

$$\Psi_T(\bar q,\bar\phi) = \frac{1}{\sqrt{Z_T}}\int {:}\exp(\bar\phi(f^+_{T,q})){:}\ \exp\left(-\int_0^T ds\int_0^T dt\,W(q_s-q_t,s-t)\right)\,d\mathcal{N}_0^{\bar q}. \tag{3.14}$$

By the time reversibility of $\mathcal{N}_0^{\bar q}$, also

$$\Psi_T(\bar q,\bar\phi) = \frac{1}{\sqrt{Z_T}}\int {:}\exp(\bar\phi(f^-_{T,q})){:}\ \exp\left(-\int_{-T}^0 ds\int_{-T}^0 dt\,W(q_s-q_t,s-t)\right)\,d\mathcal{N}_0^{\bar q} \tag{3.15}$$

holds. Now we write (3.15) for the left entry and (3.14) for the right entry of the scalar product $\langle\Psi_T,(1\otimes L)\Psi_T\rangle$ and use the fact that for $\mathcal{F}_{[0,\infty[}$-measurable $f,g\in L^1(\mathcal{N}_0)$,

$$\int\left(\int f\,d\mathcal{N}_0^{\bar q}\int g\,d\mathcal{N}_0^{\bar q}\right)d\mathsf{N}_0(\bar q) = \int f(q^+)g(q^-)\,d\mathcal{N}_0(q)$$

(with $q^+_s = q_s$ and $q^-_s = q_{-s}$ for $s\geq 0$), to write $\langle\Psi_T,(1\otimes L)\Psi_T\rangle$ as an integral with respect to $\mathcal{N}_0$. Then we add and subtract the term $2\int_{-T}^0 ds\int_0^T dtW(q_s-q_t,s-t)$ in the exponent and incorporate the term with the minus sign into the measure $\mathcal{N}_T$, cf. (2.17). The result reads

$$\begin{aligned}\langle\Psi_T,(1\otimes L)\Psi_T\rangle_{L^2(\mathsf{P}_0)} \;=\; &\int\big\langle\,{:}\exp(\phi(f^-_{T,q})){:}\,,L\ {:}\exp(\phi(f^+_{T,q})){:}\,\big\rangle_{L^2(\mathsf{G})}\times \\ &\times\exp\left(2\int_{-T}^0 ds\int_0^T dt\,W(q_t-q_s,t-s)\right)d\mathcal{N}_T(q).\end{aligned} \tag{3.16}$$

This is the finite $T$ version of (3.9). It remains to justify the passing to the limit $T\to\infty$. On the left hand side of (3.16), this is immediate since $\Psi_T\to\Psi$ in $L^2(\mathsf{N}_0\otimes\mathsf{G})$ and $L$ is continuous. On the right hand side, we already know that $\mathcal{N}_T\to\mathcal{N}$ in the topology of local convergence, and thus it only remains to show that the integrand converges uniformly in $q\in C(\mathbb{R},\mathbb{R}^d)$. For the second factor of the integrand this is a consequence of (3.6). As for the first factor, we find that for $\omega(k)\neq 0$,

$$|\hat f^\pm_{T,q}(k)|\leq\frac{|\hat\varrho(k)|}{\omega(k)}\qquad\text{uniformly in } T \text{ and } q,$$

and

$$\hat f^\pm_{T,q}(k)\overset{T\to\infty}{\longrightarrow}\hat f^\pm_q(k)\qquad\text{uniformly in } q.$$

Thus $:\exp(\phi(f^+_{T,q})):\to :\exp(\phi(f^+_q)):$ in $L^2(\mathsf{G})$ and uniformly in $q$ by dominated convergence. Since the same argument applies to $f^-_{T,q}$ and $L$ is continuous, the claim follows. □

Most operators of physical interest are not bounded. Therefore we need to extend formula (3.9) to unbounded operators.

**Proposition 3.2.3** *Let $L$ be a self-adjoint operator in $L^2(\mathsf{G})$ with*

$$\int \big\| L\ :\exp(\phi(f^\pm_q)):\big\|^2_{L^2(\mathsf{G})}\, d\mathcal{N}(q) < \infty. \tag{3.17}$$

*Then $\Psi \in D(1\otimes L)$, and (3.9) holds.*

*Proof:* Let $E$ be the projection valued measure corresponding to $L$, and let $L_N = \int_{-N}^N \lambda\, dE(\lambda)$ for $N\in\mathbb{N}$. Then $L_N$ is a bounded operator, hence (3.9) holds for $L_N$. Using (3.6) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\|(1\otimes L_N)\Psi\|^2_{L^2(\mathsf{P}_0)}\\
&= \int \big\langle :\exp(\phi(f^-_q)):, L_N^2\ :\exp(\phi(f^+_q)):\big\rangle_{L^2(\mathsf{G})}\, e^{2\int_{-\infty}^0 ds\int_0^\infty dt\, W(q_t-q_s,t-s)} d\mathcal{N}(q)\\
&\le e^I \int \big\| L_N\ :\exp(\phi(f^-_q)):\big\|_{L^2(\mathsf{G})} \big\| L_N\ :\exp(\phi(f^+_q)):\big\|_{L^2(\mathsf{G})}\, d\mathcal{N}(q)\\
&\le e^I \int \big\| L\ :\exp(\phi(f^-_q)):\big\|_{L^2(\mathsf{G})} \big\| L\ :\exp(\phi(f^+_q)):\big\|_{L^2(\mathsf{G})}\, d\mathcal{N}(q),
\end{aligned}$$

which is finite according to (3.17). This shows that $\Psi\in D(1\otimes L)$ and $(1\otimes L_N)\Psi\to(1\otimes L)\Psi$ as $N\to\infty$. On the other hand, it follows from (3.17) that

$$:\exp(\phi(f^\pm_q)):\in D(L) \text{ for } \mathcal{N}\text{-almost all } q.$$

From this we conclude

$$\big\langle :\exp(\phi(f^-_q)):, L_N\ :\exp(\phi(f^+_q)):\big\rangle_{L^2(\mathsf{G})} \to \big\langle :\exp(\phi(f^-_q)):, L\ :\exp(\phi(f^+_q)):\big\rangle_{L^2(\mathsf{G})}$$

for $\mathcal{N}$ almost all $q$ as $N\to\infty$. Since by Proposition 1.5.9 b) we have

$$\begin{aligned}
&\big\langle :\exp(\phi(f^-_q)):, L_N\ :\exp(\phi(f^+_q)):\big\rangle_{L^2(\mathsf{G})}\\
&\quad\le \big\| :\exp(\phi(f^-_q)):\big\|_{L^2(\mathsf{G})} \big\| L_N\ :\exp(\phi(f^+_q)):\big\|_{L^2(\mathsf{G})}\\
&\quad\le e^{I/2} \big\| L\ :\exp(\phi(f^+_q)):\big\|_{L^2(\mathsf{G})}
\end{aligned}$$

for all $q$, and the right hand side of the above is $\mathcal{N}$-integrable, the dominated convergence theorem implies

$$\begin{aligned}
&\int \big\langle :\exp(\phi(f^-_q)):, L_N\ :\exp(\phi(f^+_q)):\big\rangle_{L^2(\mathsf{G})}\, e^{2\int_{-\infty}^0 ds\int_0^\infty dt\, W(q_t-q_s,t-s)} d\mathcal{N}(q) \to\\
&\quad\to \int \big\langle :\exp(\phi(f^-_q)):, L\ :\exp(\phi(f^+_q)):\big\rangle_{L^2(\mathsf{G})}\, e^{2\int_{-\infty}^0 ds\int_0^\infty dt\, W(q_t-q_s,t-s)} d\mathcal{N}(q)
\end{aligned}$$

as $N \to \infty$. This finishes the proof. □

We now present one minor extension and two important special cases of (3.9).

**Corollary 3.2.4** *Let $g \in L^\infty(\mathbb{R}^d)$, and suppose $L$ satisfies the assumptions of Proposition 3.2.3. Then*

$$\begin{aligned}\langle \Psi, (g \otimes L)\Psi \rangle_{L^2(\mathsf{P}_0)} &= \int \big\langle :\exp(\phi(f_q^-)):, L\ :\exp(\phi(f_q^+)): \big\rangle_{L^2(\mathsf{G})} \times \\ &\quad \times g(q_0) \exp\left(2\int_{-\infty}^0 ds \int_0^\infty dt\, W(q_t - q_s, t-s)\right) d\mathcal{N}(q).\end{aligned}$$

*Here $g$ is again used to denote the operator of multiplication with $g$.*

Note that if $L$ is chosen to be the identity operator, then we arrive at $\langle \Psi, g\Psi \rangle_{L^2(\mathsf{P}_0)} = \int g(q_0)\, d\mathcal{N}$, a formula that also follows from Corollary 2.2.5.

**Corollary 3.2.5** *For $\beta > 0$ and $g \in K$, put*

$$M(\beta) = \left\langle \Psi, e^{\beta\phi(g)}\Psi \right\rangle_{L^2(\mathsf{P}_0)}.$$

*$M$ is the moment generating function for the random variable $\phi \mapsto \phi_0(g)$ under $\mathcal{P}$, and*

$$\begin{aligned} M(\beta) &= \int e^{\beta\phi_0(g)}\, d\mathcal{P}(q,\phi) \qquad (3.18)\\ &= \int \exp\left(\frac{\beta^2}{2}\int \frac{|\hat{g}|^2}{2\omega}\, dk - \beta\int_{-\infty}^\infty ds \int dk\, \frac{\hat{\varrho}(k)\overline{\hat{g}(k)}e^{-ikq_s}}{2\omega(k)} e^{-\omega(k)|s|}\right) d\mathcal{N}.\end{aligned}$$

*By (3.4), $M(\beta)$ is finite for all $\beta$, and hence*

$$\langle \Psi, \phi(g)^n \Psi \rangle_{L^2(\mathsf{P}_0)} = \frac{d^n}{d\beta^n} M(\beta)|_{\beta=0} \quad \text{for all } n \in \mathbb{N}. \qquad (3.19)$$

Note that, although (3.18) can in principle be deduced from Proposition 3.2.3, it can be obtained more easily by using (3.5), i.e. by fixing $q \in C(\mathbb{R}, \mathbb{R}^d)$ and integrating the function $\phi \mapsto e^{\phi_0(g)}$ with respect to the conditional Gaussian measure.

The next statement deals with second quantization and differential second quantization of operators. Recall that $\theta : L^2(\mathsf{G}) \to \mathcal{F}$ denoted the Wiener-Itô-Segal isomorphism, cf. 1.5.4.

**Corollary 3.2.6** *Let $L$ be a bounded self-adjoint operator on $L^2(\mathbb{R}^d)$, and write $\tilde{\Gamma}(L) = \theta^{-1}\Gamma(L)\theta$ and $d\tilde{\Gamma}(L) = \theta^{-1}d\Gamma(L)\theta$. Then $\Psi \in D(\tilde{\Gamma}(L))$, $\Psi \in D(d\tilde{\Gamma}(L))$ and*

$$\begin{aligned}\left\langle \Psi, \tilde{\Gamma}(L)\Psi \right\rangle_{L^2(\mathsf{P}_0)} &= \int \exp\left(\left\langle f_q^-, B^{-1}LBf_q^+ \right\rangle_K\right) e^{2\int_{-\infty}^0 ds \int_0^\infty dt\, W(q_s - q_t, s-t)}\, d\mathcal{N}(q),\\ \left\langle \Psi, d\tilde{\Gamma}(L)\Psi \right\rangle_{L^2(\mathsf{P}_0)} &= \int \left\langle f_q^-, B^{-1}LBf_q^+ \right\rangle_K d\mathcal{N}(q). \qquad (3.20)\end{aligned}$$

*Proof:* First note that by (3.8),

$$\left\|B^{-1}LBf_q^{\pm}\right\|_K = \left\|LBf_q^{\pm}\right\|_{L^2(\mathbb{R}^d)} \leq \|L\| \left\|f_q^{\pm}\right\|_K$$

is uniformly bounded in $q$. On the other hand,

$$\left\|\tilde{\Gamma}(L)\ :\exp(\phi(f_q^{\pm})):\right\|^2_{L^2(\mathsf{G})} = \exp\left(\left\|B^{-1}LBf_q^{+}\right\|^2_K\right)$$

follows directly from Proposition 1.5.9 b), c). Furthermore,

$$\left\|d\tilde{\Gamma}(L)\ :\exp(\phi(f_q^{\pm})):\right\|^2_{L^2(\mathsf{G})} = \left(\left\|B^{-1}LBf_q^{\pm}\right\|^2_K + \left\langle f_q^{\pm}, B^{-1}LBf_q^{\pm}\right\rangle^2_K\right) e^{\left\|f_q^{\pm}\right\|^2_K}$$

can be obtained from the definitions of differential second quantization (1.25), Wick exponentials (Definition 1.5.8) and of $d\tilde{\Gamma}(L)$ above. Since

$$\left\|B^{-1}LBf\right\|_K = \|LBf\|_{L^2(\mathbb{R}^d)} \leq \|L\| \|f\|_K$$

for all $f \in K$, (3.17) is fulfilled, and Proposition 3.2.3 now gives $\Psi \in D(\tilde{\Gamma}(L))$ and $\Psi \in D(d\tilde{\Gamma}(L))$. Now that this is established, formulas (3.20) follow directly from (3.9) and Proposition 1.5.9 b) - d) and (3.7). □

## 3.3 Bounds on ground state expectations

We will now apply the results of the previous section in order to investigate some qualitative properties of the ground state of $H$. Alternatively (and equivalently), we will instead refer to the ground state $\Psi_{\mathcal{F}} = \Theta\Psi$ of $H_{\mathcal{F}}$ whenever this is convenient. Recall that $I < \infty$, the infrared constant, was defined in (3.5).

**Example 3.3.1 Boson number distribution**
Let $P_n$ be the projection onto the $n$-th Fock space component (or $n$-boson sector). Then $\tilde{P}_n = \theta P_n \theta^{-1}$ is the projection onto the closure of the subspace spanned by $\{:\phi(f)^n:, f \in K\} \subset L^2(\mathsf{G})$. By the definition of Wick exponentials, we have

$$\left\langle :\exp(\phi(f_q^-)):, \tilde{P}_n\ :\exp(\phi(f_q^+)):\right\rangle_{L^2(\mathsf{G})} = \frac{1}{n!}\left\langle f_q^+, f_q^-\right\rangle^n_K,$$

and with (3.7) and Theorem 3.2.2 we find

$$\begin{aligned} p_n \ \equiv\ \left\langle \Psi, 1 \otimes \tilde{P}_n \Psi\right\rangle_{L^2(\mathsf{P}_0)} &= \int \frac{1}{n!}\left(-2\int_{-\infty}^{0} ds \int_0^{\infty} dt\, W(q_s - q_t, s-t)\right)^n \times \\ &\times \exp\left(2\int_{-\infty}^{0} ds \int_0^{\infty} dt\, W(q_s - q_t, s-t)\right) d\mathcal{N}. \end{aligned}$$

$p_n$ is the probability of finding $n$ bosons in the ground state of $H_{\mathcal{F}}$. Obviously,

$$p_n \leq \frac{I^n}{n!} e^I. \tag{3.21}$$

Denoting by $\mathbf{N} = d\Gamma(1)$ the number operator, the superexponential bound (3.21) implies

$$\left\langle \Psi_{\mathcal{F}}, 1 \otimes e^{\alpha \mathbf{N}} \Psi_{\mathcal{F}} \right\rangle_{L^2(\mathbb{R}^d)\times\mathcal{F}} < \infty \quad \text{for each } \alpha > 0$$

and is useful in the context of scattering theory [24].
Let us now assume in addition that $W(q,t) < 0$ for all $q$ and all $t$. This is true e.g. for the massive Nelson model, cf. 3.1.1 e). In this case there exists $D \leq I$ with

$$\frac{D^n}{n!} e^{-I} \leq p_n \leq \frac{I^n}{n!}. \tag{3.22}$$

The right hand side of (3.22) is again obvious, and the left hand side follows from

$$\begin{aligned} p_n &\geq \frac{1}{n!} e^{-I} \int \left( -2 \int_{-\infty}^{0} ds \int_0^\infty dt\, W(q_s - q_t, s-t) \right)^n d\mathcal{N} \\ &\geq \frac{1}{n!} e^{-I} \left( - \int 2 \int_{-\infty}^{0} ds \int_0^\infty dt\, W(q_s - q_t, s-t)\, d\mathcal{N} \right)^n . \end{aligned}$$

$D$ is then the expectation of the double integral above.

In the next two examples we will look at the mean value and variance of the random variable $\phi \mapsto \phi_0(g)$ under $\mathcal{P}$ for $g \in K$, using the results of Corollary 3.2.5.

**Example 3.3.2 Average field strength**
For $n = 1$, (3.19) yields

$$\begin{aligned} \langle \Psi, \phi(g) \Psi \rangle_{L^2(\mathsf{P}_0)} &= - \int dk \int_{-\infty}^{\infty} ds\, \frac{\hat{\varrho}(k) \overline{\hat{g}(k)}}{2\omega(k)} e^{-\omega(k)|s|} \left( \int e^{-ikq_s}\, d\mathcal{N}(q) \right) \\ &= - \int dk \int dq \psi_0^2(q) \lambda^2(q) \frac{\hat{\varrho}(k) \overline{\hat{g}(k)} e^{-ikq}}{\omega(k)^2}, \end{aligned} \tag{3.23}$$

where $\lambda^2(q) = \int \Psi^2(\phi, q)\, d\mathsf{G}(\phi)$ is the stationary density of $\mathcal{N}$ with respect to $\mathsf{N}_0$, and $\psi_0^2$ is the density of $\mathsf{N}_0$ with respect to Lebesgue measure (and the square of the ground state of $H_{\mathrm{p}}$).
Writing $\chi = \psi_0^2 \lambda^2$ for the position density of the particle, and taking $g$ to be a delta function in momentum space and in position space, respectively, we find

$$\langle \Psi, \phi(k) \Psi \rangle_{L^2(\mathsf{P}_0)} = - \frac{\hat{\varrho}(k) \hat{\chi}(k)}{(2\pi)^{d/2} \omega^2(k)} \qquad (k \in \mathbb{R}^d),$$

and

$$\langle \Psi, \phi(q)\Psi \rangle_{L^2(\mathsf{P}_0)} = (\chi * V_\omega * \varrho)(q) \qquad (q \in \mathbb{R}^d), \tag{3.24}$$

respectively. Here $V_\omega$ denotes the Fourier transform of $-1/\omega^2$ and is the Coulomb potential for massless bosons, i.e. for $\omega(k) = |k|$. (3.24) is the classical field generated by a particle with position distribution $\chi(q)\,dq$. Note that equality (3.23) follows also from the equations of motion and the stationarity of $\Psi$.

**Example 3.3.3 Field fluctuations**
For $n = 2$, (3.19) becomes

$$\left\langle \Psi, \phi(g)^2\Psi \right\rangle_{L^2(\mathsf{P}_0)} = \int \frac{|\hat{g}(k)|^2}{2\omega(k)}\,dk + \int \left( \int_{-\infty}^{\infty} ds \int dk \frac{\hat{\varrho}(k)\overline{\hat{g}(k)}e^{ikq_s}}{2\omega(k)} e^{-\omega(k)|s|} \right)^2 d\mathcal{N}.$$

By using the previous result and the Cauchy-Schwarz inequality, we find that

$$\left\langle \Psi, \phi(g)^2\Psi \right\rangle_{L^2(\mathsf{P}_0)} - \langle \Psi, \phi(g)\Psi \rangle^2_{L^2(\mathsf{P}_0)} \geq \int \frac{|\hat{g}|^2}{2\omega}\,dk = \int \phi(g)^2\,d\mathsf{G}(\phi).$$

The latter term represents the fluctuations of the free field. We thus see that fluctuations increase by coupling the field to the particle.

We now treat special cases of Corollary 3.2.6.

**Example 3.3.4 Average number of bosons at given momentum**
For real-valued $g \in L^\infty$ consider

$$\int a_k^* a_k g(k)\,dk = d\Gamma(g),$$

cf. Definition 1.3.2. By Corollary 3.2.6, we have $\Psi \in D(d\tilde{\Gamma}(g))$. With $g$ chosen to be the indicator function of some set $M \subset \mathbb{R}^d$, $\left\langle \Psi, d\tilde{\Gamma}(g)\Psi \right\rangle_{L^2(\mathsf{P}_0)}$ is the expected number of bosons with momentum within $M$. By (3.20) and the equality $(B^{-1}gBf)\hat{} = g\hat{f}$,

$$\left\langle \Psi, d\tilde{\Gamma}(g)\Psi \right\rangle_{L^2(\mathsf{P}_0)} = \int dk \frac{|\hat{\varrho}(k)|^2}{2\omega(k)} g(k) \int_{-\infty}^{0} ds \int_0^{\infty} dt\, e^{-\omega(k)(t-s)} \int \cos(k(q_t - q_s))\,d\mathcal{N} \tag{3.25}$$

On the one hand, from $\cos(kx) \leq 1$ we get

$$\left\langle \Psi, d\tilde{\Gamma}(g)\Psi \right\rangle_{L^2(\mathsf{P}_0)} \leq \int \frac{|\hat{\varrho}(k)|^2}{2\omega(k)^3} g(k)\,dk. \tag{3.26}$$

(3.26) is proven in [6] using the pullthrough formula.
On the other hand, from $1-(k^2x^2)/2 \le \cos(kx)$ we get

$$\begin{aligned}\int \cos(k(q_t-q_s))\,d\mathcal{N} &\ge 1-\frac{k^2}{2}\left(\int (q_t^2+q_s^2-2q_tq_s)\,d\mathcal{N}\right)\\ &\ge 1-k^2\int q^2\psi_0^2(q)\lambda^2(q)\,dq.\end{aligned}$$

The last inequality above follows from

$$\int q_sq_t\,d\mathcal{N} = \left\langle \Psi q, e^{-|t-s|(H-E_0)}\Psi q\right\rangle_{L^2(\mathsf{P}_0)} = \left\| e^{-(|t-s|/2)(H-E_0)}q\Psi\right\|^2_{L^2(\mathsf{P}_0)} \ge 0.$$

Writing $C=\int q^2\psi_0^2(q)\lambda^2(q)\,dq$, we have for $g\ge 0$ that

$$\left\langle \Psi, d\tilde{\Gamma}(g)\Psi\right\rangle_{L^2(\mathsf{P}_0)} \ge \int \frac{|\hat{\varrho}(k)|^2}{2\omega(k)^3}(1-Ck^2)g(k)\,dk.$$

The above results can be compactly (and somewhat formally) written as

$$\frac{|\hat{\varrho}(k)|^2}{2\omega(k)^3}(1-Ck^2) \le \langle \Psi_{\mathcal{F}}, 1\otimes a_k^*a_k\Psi_{\mathcal{F}}\rangle_{L^2(\mathbb{R}^d)\times\mathcal{F}} \le \frac{|\hat{\varrho}(k)|^2}{2\omega(k)^3}. \tag{3.27}$$

Here, $a_k^*a_k$ denotes the formal expression $d\Gamma(\delta(\cdot-k))$. The quantity in the middle of (3.27) is the Lebesgue density at momentum $k$ of the expected number of bosons. In particular, for the 3-dimensional massless Nelson model (cf. 3.1.1 e)) one can see from the lower bound how the infrared divergence occurs. Letting $r\to 0$ there, the expected number of bosons in the ground state with momenta in a neighborhood of 0 diverges.

**Example 3.3.5 Average number of bosons at given position**
Consider now the operator $L=g(-i\nabla)$ in $L^2(\mathbb{R}^d)$, with real-valued $g\in L^\infty$. If $g$ is the indicator of some bounded set $M\subset\mathbb{R}^d$, $\langle \Psi_{\mathcal{F}}, d\Gamma(L)\Psi_{\mathcal{F}}\rangle_{L^2(\mathbb{R}^d)\times\mathcal{F}}$ measures the expected number of bosons with position within $M$. It is common to write

$$d\Gamma(g(-i\nabla)) = \int a_q^*a_qg(q)\,dq.$$

In order to apply Corollary 3.2.6, we calculate

$$\left\langle f_q^-, B^{-1}LBf_q^+\right\rangle_K = \left\langle Bf_q^-, LBf_q^+\right\rangle_{L^2(\mathbb{R}^d)} = \left\langle (Bf_q^-)^\vee, g(Bf_q^+)^\vee\right\rangle_{L^2(\mathbb{R}^d)},$$

where $f^\vee$ denotes the inverse Fourier transform of $f$. We have

$$(Bf_q^+)^\vee(x) = \frac{1}{(2\pi)^{d/2}}\int_0^\infty ds\int \frac{\hat{\varrho}(k)}{\sqrt{2\omega(k)}}e^{ik(q_s+x)}e^{-\omega(k)|s|}\,dk,$$

and consequently

$$\begin{aligned}\left\langle f_q^-, B^{-1}LBf_q^+\right\rangle_K &= \frac{1}{(2\pi)^d}\int_{-\infty}^0 ds\int_0^\infty dt\int dk\int dk'\,\frac{\overline{\hat\varrho(k)}\hat\varrho(k')}{\sqrt{2\omega(k)}\sqrt{2\omega(k')}}\times\\ &\quad\times e^{i(k'q_t-kq_s)}e^{-\omega(k)|t|-\omega(k')|s|}\int dx\,g(x)e^{ix(k-k')}\leq\\ &\leq \frac{1}{(2\pi)^d}\left(\int\frac{|\hat\varrho|}{(2\omega)^{3/2}}\,dk\right)^2\|g\|_{L^1(\mathbb{R}^d)} = (*).\end{aligned}$$

In case of the massless Nelson model (with coupling strength equal to one for convenience), we have

$$(*) \leq \frac{1}{(2\pi)^d}\int\frac{|\hat\varrho|^2}{2\omega}\,dk\int\frac{|\hat\varrho|^2}{(2\omega)^2}\,dk\,\|g\|_{L^1(\mathbb{R}^d)} = \frac{1}{(2\pi)^d}\,\|\varrho\|_K^2\,\||B|\varrho\|_K^2\,\|g\|_{L^1(\mathbb{R}^d)}\,.$$

We thus see that the number of Bosons with position in a set $M\subset\mathbb{R}^d$ is bounded by a multiple of the volume of $M$. Moreover, it is interesting to note that this bound is insensitive to formally removing the infrared cutoff. On the other hand, the total number of bosons in the ground state is obtained by taking $g=1$ in this or the previous example, and we see from (3.27) that this quantity diverges when the infrared cutoff is formally removed.

### Example 3.3.6 Localization of the particle

As the final example we show exponential decay of the Lebesgue-density of the stationary measure of $\mathcal{N}$. We will need the following property of $H$:

**Proposition 3.3.7** *(Diamagnetic inequality) For $f,g\in L^2(\mathsf{P}_0)$ we have*

$$\left\langle f, e^{-tH}g\right\rangle_{L^2(\mathsf{P}_0)} \leq e^{tV_{\text{eff}}}\left\langle \|f\|_{L^2(\mathsf{G})}\,, e^{-t\tilde H_{\text{p}}}\,\|g\|_{L^2(\mathsf{G})}\right\rangle_{L^2(\mathsf{N}_0)},$$

*where*

$$V_{\text{eff}} = \frac{1}{2}\int\frac{|\hat\varrho(k)|^2}{\omega^2(k)}\,dk = 2\,\||B|\varrho\|_K^2 < \infty,$$

*and $\tilde H_{\text{p}} = (1/\psi_0)H_{\text{p}}\psi_0$.*

A proof can be found in [23].

The second ingredient we need is essentially the result due to Carmona [10] that we already used in Chapter 2. Recall that the Carmona class of potentials was defined in 2.2.9. From the proofs of Lemma 3.1 and Propositions 3.1 and 3.2 of [10] one can extract the following

**Lemma 3.3.8** *Let $V = V_1 - V_2$ be of the Carmona class, and use $\mathcal{W}^{\bar{q}}$ to denote the measure of Brownian motion on $\mathbb{R}^d$ starting in $\bar{q}$.*

*a) Suppose there exist $\gamma > 0, m > 0$ such that*

$$V_1(q) \geq \gamma \, |q|^{2m}$$

*outside a compact set. Put $t(q) = \max\{|q|^{1-m}, 1\}$. Then for each $E > 0$ there exist $D > 0$ and $\delta > 0$ such that*

$$\forall \bar{q} \in \mathbb{R}^d : \quad e^{t(\bar{q})E} \int e^{-\int_0^{t(\bar{q})} V(q_s)\, ds} \, d\mathcal{W}^{\bar{q}}(q) \; \leq \; D \exp(-\delta \, |\bar{q}|^{m+1}).$$

*b) Put $\alpha := \liminf_{|q|\to\infty} V(q)$, $t(q) := \beta \, |q|$ with $\beta > 0$. Then for each $E \in \mathbb{R}$ with $E < \alpha$, there exist $D > 0, \delta > 0$ and $\beta > 0$ such that*

$$\forall \bar{q} \in \mathbb{R}^d : \quad e^{t(\bar{q})E} \int e^{-\int_0^{t(\bar{q})} V(q_s)\, ds} \, d\mathcal{W}^{\bar{q}}(q) \; \leq \; D \exp(-\delta \, |\bar{q}|).$$

Recall that $\mathsf{N}$ denotes the stationary measure of $\mathcal{N}$, $\psi_0\lambda$ equals the square root of the Lebesgue density of $\mathsf{N}$ (cf. Example 3.3.2) and $E_0$ is the ground state energy of $H$. Our result now reads:

**Theorem 3.3.9** *For any $V$ fulfilling (V1) and (V2) from 2.2.1, we have $\psi_0\lambda \in L^\infty(\mathbb{R}^d)$. If, in addition, $V = V_1 - V_2$ is of the Carmona class, then there exists a version of $\psi_0\lambda$ (denoted by $q \mapsto \psi_0(q)\lambda(q)$) for which the following statements hold:*

*a) If $V$ satisfies the assumptions of Proposition 3.3.8 a), then there exist $D, \delta > 0$ with*

$$\forall q \in \mathbb{R}^d : \quad \psi_0(q)\lambda(q) \leq D \exp(-\delta \, |q|^{m+1}). \tag{3.28}$$

*b) Put $\alpha := \liminf_{|q|\to\infty} V_1(q)$. If $\alpha - (E_0 + V_{\text{eff}}) > 0$, then there exist $D > 0, \delta > 0$ such that*

$$\forall q \in \mathbb{R}^d : \quad \psi_0(q)\lambda(q) \leq D \exp(-\delta \, |q|). \tag{3.29}$$

*Proof:* We first show that $\psi_0\lambda \in L^\infty(\mathbb{R}^d)$. Since $H\Psi = E_0\Psi$, for $h \in L^\infty(\mathbb{R}^d), h \geq 0$, the diamagnetic inequality implies

$$\begin{aligned} \int h(q)\psi_0^2(q)\lambda^2(q)\, dq &= \langle h\Psi, \Psi \rangle_{L^2(\mathsf{P}_0)} = e^{tE_0} \left\langle h\Psi, e^{-tH}\Psi \right\rangle_{L^2(\mathsf{P}_0)} \\ &\leq e^{t(V_{\text{eff}}+E_0)} \left\langle h\lambda, e^{-t\tilde{H}_{\text{P}}}\lambda \right\rangle_{L^2(\mathsf{N}_0)} \\ &= e^{t(V_{\text{eff}}+E_0)} \int h(q)\psi_0(q)\lambda(q)(e^{-tH_{\text{P}}}\lambda\psi_0)(q)\, dq. \end{aligned} \tag{3.30}$$

Since we required $V$ to be in the Kato class, $e^{-tH_{\mathrm{P}}}$ takes $L^2(dq)$ into $L^\infty(dq)$ [46]. Thus we can find $C \in \mathbb{R}$ with

$$\int h(q)\psi_0^2(q)\lambda^2(q)\,dq \leq C \int h(q)\psi(q)\lambda(q)\,dq, \tag{3.31}$$

which implies $\psi_0\lambda \in L^\infty$. Using this result in (3.30) and the Feynman-Kac formula to express the kernel of $e^{-tH_{\mathrm{P}}}$, we get

$$\begin{aligned}
&\int h(q)\psi_0^2(q)\lambda^2(q)\,dq \leq \\
&\leq \; e^{t(V_{\mathrm{eff}}+E_0)} \int d\bar{q}\, h(\bar{q})\psi_0(\bar{q})\lambda(\bar{q}) \int e^{-\int_0^t V(q_s)\,ds}\psi_0(q_t)\lambda(q_t)\,d\mathcal{W}^{\bar{q}}(q) \\
&\leq \; e^{t(V_{\mathrm{eff}}+E_0)} \left\|\psi_0\lambda\right\|_{L^\infty}^2 \int d\bar{q}\, h(\bar{q}) \int e^{-\int_0^t V(q_s)\,ds}\,d\mathcal{W}^{\bar{q}}(q).
\end{aligned} \tag{3.32}$$

The version of $\psi_0\lambda$ mentioned above can now be explicitly defined by

$$\psi_0^2(q)\lambda^2(q) = \limsup_{n\to\infty} \int h_{q,n}(x)\psi_0^2(x)\lambda^2(x)\,dx,$$

where $h_{q,n}$ is any fixed sequence of $L^1$-functions converging in $L^1$ to a delta peak at $q$. We now use this sequence in (3.32). Since $\int \exp(-\int_0^t V(q_s)\,ds)\,d\mathcal{W}^{\bar{q}}$ is continuous in $\bar{q}$ and finite for all $\bar{q}$, the right hand side of (3.32) converges and we have

$$\psi_0^2(\bar{q})\lambda^2(\bar{q}) \leq e^{t(V_{\mathrm{eff}}+E_0)} \left\|\psi_0\lambda\right\|_{L^\infty}^2 \int e^{-\int_0^t V(q_s)\,ds}\,d\mathcal{W}^{\bar{q}}(q).$$

This inequality is valid for each $t > 0$, and therefore in case $V$ is in the Carmona class, we can use Proposition 3.3.8 with $E$ replaced by $V_{\mathrm{eff}} + E_0$ to conclude the proof. □

A version of the preceding result already appears in [6]. There it is shown that $\psi_0(q)\lambda(q)\exp(\alpha q) \in L^1(dq)$ for some $\alpha > 0$, while the present results (when applicable) imply $\psi_0(q)\lambda(q)\exp(\alpha q) \in L^\infty(dq)$ in case of a decaying external potential $V$ and superexponential localization in case of growing potentials.

# Appendix

## A.1 Fock space over $\mathbb{C}^N$

In this appendix it is shown how the symmetric Fock space over $\mathbb{C}^N$ arises in a natural way as a method for decomposing $L^2(\mathbb{R}^N), N < \infty$ into orthogonal subspaces. For each $l \in \{1, \ldots, N\}$ we choose an orthonormal basis $(f_n^{(l)})_{n \geq 0}$ of $L^2(\mathbb{R})$. Then the subspace of $L^2(\mathbb{R}^N)$ spanned by functions of the form

$$(x_1, \ldots, x_n) \mapsto f_{n_1}^{(1)}(x_1) \cdots f_{n_N}^{(N)}(x_N)$$

is dense in $L^2(\mathbb{R}^N)$. Put

$$L_M = \{(x_1, \ldots, x_N) \mapsto f_{n_1}^{(1)}(x_1) \cdots f_{n_N}^{(N)}(x_N) : \sum_{i=1}^{N} n_i = M\}.$$

The orthogonality of the $f_j$ implies that $F \in L_M$ and $G \in L_{M'}$ are orthogonal if $M \neq M'$, and $F, G \in L_M$ with $F(x_1, \ldots, x_n) = f_{n_1}(x_1) \cdots f_{n_N}(x_n), G(x_1, \ldots, x_n) = f_{m_1}(x_1) \cdots f_{m_N}(x_n)$ are orthogonal if $(n_1, \ldots, n_N) \neq (m_1, \ldots, m_N)$. Thus we can decompose $L^2(\mathbb{R}^N)$ into the orthogonal subspaces spanned by the $L_M$, and the dimension of each $L_M$ is equal to the number of integer-valued, $N$-dimensional vectors with $\sum_{i=1}^{N} n_i = M$. The symmetric Fock space is a way of counting the number of such vectors.

The idea is that each integer-valued vector $\vec{n} = (n_1, \ldots n_N)$ with $\sum_{i=1}^{N} n_i = M$ can be represented as a sum of $M$ standard basis vectors of $\mathbb{C}^N$, i.e. $\vec{n} = \sum_{i=1}^{M} \vec{e}_{k_j}$, $k_j \in \{1, \ldots, N\}$. The $M$-tuple of eigenvectors in turn can be viewed as a function $\phi : \{1, \ldots, N\}^M \to \mathbb{R}$ with is equal to one at exactly one point $(k_1, \ldots, k_M) \in \{1, \ldots N\}^M$ and zero otherwise. Of course, this is nothing but the tensor product $\vec{e}_{k_1} \otimes \ldots \otimes \vec{e}_{k_M}$. However, this representation is not unique, since each permutation of the $\vec{e}_i$, equivalently each permutation of the variables in $\phi$, leads to the same $\vec{n}$. This is where the symmetry condition enters: we declare two function $\phi_1, \phi_2 : \{1, \ldots, N\}^M \to \mathbb{R}$ of the above given form to be equivalent if $\phi_1$ arises from $\phi_2$ by a permutation of variables, i.e. if there exists a permutation $\pi$ on $\{1, \ldots, M\}$ such that $\phi(k_1, \ldots, k_M) = \phi(k_{\pi(1)}, \ldots, k_{\pi(M)})$ for all $k_1, \ldots, k_M \in \{1, \ldots, N\}$. In

the language of tensor products, an equivalent procedure is to take the symmetric tensor product

$$\vec{e}_{k_1}\hat{\otimes}\cdots\hat{\otimes}\vec{e}_{k_M} = \frac{1}{|\Pi(M)|}\sum_{\pi\in\Pi(M)} \vec{e}_{k_{\pi(1)}}\otimes\ldots\otimes\vec{e}_{k_{\pi(M)}},$$

where $\Pi(M)$ is the set of permutations on $\{1,\ldots,M\}$. A moment of thought reveals that both procedures result in a one-to-one correspondence of the resulting objects (equivalence classes or symmetric tensors) with the elements of $L_M$. Now we take linear combinations to find

$$\begin{aligned}\mathrm{span}(L_M) &\cong \mathrm{span}\{\phi : \{1,\ldots,N\}^M \to \mathbb{C}, \phi \text{ symmetric }\} \cong \\ &\cong \mathrm{span}\{\vec{v}_1\hat{\otimes}\ldots\hat{\otimes}\vec{v}_M : \vec{v}\in\mathbb{C}^N\} \equiv \mathcal{F}_N^{(M)}. \end{aligned} \tag{A.1}$$

$\mathcal{F}_N^{(M)}$ is the $M$-th Fock space segment of the symmetric Fock space over $\mathbb{C}^N$. Since the $L_M$ are orthogonal and span $L^2(\mathbb{R}^N)$ we have

$$L^2(\mathbb{R}^N) \cong \bigoplus_{M=0}^{\infty} \mathcal{F}_N^{(M)} \equiv \mathcal{F}_N.$$

$\mathcal{F}_N$ is the symmetric Fock space over $\mathbb{C}^N$.

To make the connection (at least formally) with the Fock space introduced in Section 1.3, let us take a finite box $\Lambda\subset\mathbb{R}^d$ and divide it into $N$ pieces $\Lambda_1,\ldots,\Lambda_N$. We use the numbers $1,\ldots,N$ from (A.1) to label these pieces, which means that each symmetric function $\phi : \{1,\ldots,N\}^M\to\mathbb{C}$ is regarded as a symmetric function $f_N : (\mathbb{R}^d)^M\to\mathbb{C}$ that is constant on each of the $\Lambda_j$ and zero outside $\Lambda$. Now by taking the limit $|\Lambda|\to\infty$ and $N\to\infty$ in such a way that $\Lambda_j\to 0$ for each $j$, and by imposing some summability conditions on the functions $f_N$ as $N\to\infty$, it is at least formally plausible that the limiting object $\mathcal{F}^{(M)} = \mathcal{F}_\infty^{(M)}$ will be the space of all symmetric, e.g. square integrable functions $f : (\mathbb{R}^d)^M\to\mathbb{C}$. This leads to Definition 1.3.1.

## A.2 Stochastic processes from Schrödinger operators

Here we give some basics on $P(\phi)_1$-processes [45, 46]. These processes are connected to Schrödinger operators via the Feynman-Kac formula and the ground state transformation. We consider a Schrödinger operator $H = -\frac{1}{2}\Delta + V$ in $L^2(\mathbb{R}^d)$. $V : \mathbb{R}^d\to\mathbb{R}$ is a measurable function which acts as an operator of multiplication. We first give the conditions we impose on $V$.

**Definition A.2.1** *A measurable function $V : \mathbb{R}^d \to \mathbb{R}$ is said to be in the* **Kato class** *[46], $V \in \mathcal{K}(\mathbb{R}^d)$, if*

$$\sup_{x\in\mathbb{R}} \int_{\{|x-y|\le 1\}} |V(y)|\, dy < \infty \qquad \text{in case } d = 1,$$

*and*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{\{|x-y|\le r\}} g(x-y)|V(y)|\, dy = 0 \qquad \text{in case } d \ge 2.$$

*Here,*

$$g(x) = \begin{cases} -\ln|x| & \text{if } d = 2 \\ |x|^{2-d} & \text{if } d \ge 3. \end{cases}$$

*$V$ is locally in the Kato class, $V \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$, if $V 1_K \in \mathcal{K}(\mathbb{R}^d)$ for each compact set $K \subset \mathbb{R}^d$. $V$ is* **Kato decomposable** *[9] if*

$$V = V^+ - V^- \quad \text{with} \quad V^- \in \mathcal{K}(\mathbb{R}^d), V^+ \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d),$$

*where $V^+$ is the positive part and $V^-$ is the negative part of $V$.*

In the context of Brownian motion, the Kato class turns out to be a very natural class of functions. Namely, a function $V$ is in the Kato class if and only if [1]

$$\sup_{y\in\mathbb{R}^d} \int \left( \int_0^t |V(x_s)|\, ds \right) d\mathcal{W}^y(x) \xrightarrow{t\to 0} 0.$$

Here, $\mathcal{W}^y$ is Brownian motion starting in $y \in \mathbb{R}^d$.

In the following, we will always assume that $V$ is Kato-decomposable. In that case, $H$ is self-adjoint on $D(H_0)$ and bounded from below, and for each $t > 0$ the operator $e^{-tH}$ has an integral kernel given via the Feynman-Kac formula:

$$\left(e^{-tH} f\right)(y) = \int e^{-\int_0^t V(x_s)\, ds} f(x_t)\, d\mathcal{W}^y(x). \tag{A.2}$$

As the case $V = 0$ shows, $H$ need not have eigenvalues. However, we are only interested in such $H$ that do have eigenvalues, more precisely in those for which the infimum of the spectrum is an eigenvalue. If this eigenvalue exists, it is necessarily of multiplicity one, and the corresponding eigenfunction (called ground state) can be chosen to be strictly positive; this follows from a Perron-Frobenius argument, since by (A.2) $e^{-tH}$ has a strictly positive kernel. A sufficient condition on $V$ for the existence of a ground state is $\liminf_{|x|\to\infty} V(x) = \infty$, but ground states exist also for many other choices of $V$, like e.g. the Coulomb potential in three dimensions.

Let us summarize our assumptions on $V$:

$(A1)$ : $V$ is Kato-decomposable.
$(A2)$ : $H$ has a ground state $\psi_0 \in L^2(\mathbb{R}^d)$. We take $\|\psi_0\|_{L^2(\mathbb{R}^d)} = 1$.
$(A3)$ : $\inf \operatorname{spec}(H) = 0$.

(A3) is there only for convenience and can always be achieved by adding a constant to $V$.

We now use the ground state transformation to convert $H$ into the generator of a stochastic process. The ground state transformation is the unitary map

$$L^2(\psi_0^2\, dx) \to L^2(\mathbb{R}^d), \qquad f \mapsto \psi_0 f.$$

we will use $\psi_0$ to denote this transformation, and $1/\psi_0$ for the inverse. Using $H\psi_0 = 0$, we obtain for the operator $L = \frac{1}{\psi_0} H \psi_0$ the formula

$$(Lf)(x) = -\frac{1}{2}\Delta f(x) - \left\langle \frac{\nabla \psi_0}{\psi_0}(x), \nabla f(x) \right\rangle_{\mathbb{R}^d}$$

for all $f \in L^2(\psi_0^2\, dx)$ which are in the domain of $L$. $L$ is the generator of a stochastic process satisfying the stochastic differential equation

$$dX_t = \frac{\nabla \psi_0}{\psi_0}(X_t)\, dt + dB_t. \tag{A.3}$$

The stationary solution of (A.3) will be called $P(\phi)_1$-process. It is a strong Markov process. The path measure of this process will be denoted by $\mu$, and the transition probabilities are given by

$$\begin{aligned} E_\mu(f(X_t)|X_0 = y) &= (e^{-tL} f)(y) = \frac{1}{\psi_0(y)}(e^{-tH}(\psi_0 f))(y) = \\ &= \frac{1}{\psi_0(y)} \int e^{-\int_0^t V(x_s)\, ds} \psi_0(x_t) f(x_t)\, d\mathcal{W}^y(x). \end{aligned} \tag{A.4}$$

Since the stationary measure of $\mu$ is $\psi_0^2\, dx$, according to the Feynman-Kac formula the finite dimensional distributions of $\mu$ are given by

$$\begin{aligned} \int f_1(X_{t_1}) \cdots f_n(X_{t_n})\, d\mu(X) &= \left\langle \psi_0 f_1, e^{-(t_2-t_1)H} f_2 \ldots e^{-(t_n - t_{n-1})H} f_n \psi_0 \right\rangle_{L^2(\mathbb{R}^d)} = \\ &= \left\langle f_1, e^{-(t_2-t_1)L} f_2 \ldots e^{-(t_n - t_{n-1})L} f_n \right\rangle_{L^2(\psi_0^2\, dx)} \end{aligned} \tag{A.5}$$

for $t_1 < t_2 < \ldots < t_n$.

We also use this appendix to collect some analytic properties of the semigroup $e^{-tH}$ that will be needed in the course of this work. The reference for all of them is [46].

(S1): For each $1 \leq p \leq q \leq \infty$ and each $t > 0$, $e^{-tH}$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$.

(S2): For each $t > 0$, the integral kernel $K_t(\xi, \eta)$ of $e^{-tH}$ is uniformly bounded on $\mathbb{R}^{2d}$ and bounded away from zero on compact subsets of $\mathbb{R}^{2d}$. The same is true for the ground state $\psi_0$ on $\mathbb{R}^d$.

(S3): For each $t > 0$, $\eta \mapsto K_t(\xi, \eta)$ is Lebesgue-integrable uniformly in $\xi \in \mathbb{R}^d$.

## A.3 The finite dimensional Ornstein-Uhlenbeck process

We now apply the results from Appendix A.2 to obtain the $N$-dimensional Ornstein-Uhlenbeck process appearing in Section 1.2. In this case, all of the above objects are given by explicit formulas, which we state without further comments. The (non-renormalized) Schrödinger operator $H_{\mathrm{OU,nr}}$ is given by

$$(H_{\mathrm{OU,nr}} f)(x) = -\frac{1}{2}\Delta f(x) + \frac{1}{2}\langle x, Ax\rangle_{\mathbb{R}^N} f(x),$$

where $A$ is a symmetric matrix with strictly positive eigenvalues. The (normalized) ground state is

$$\psi_{\mathrm{OU}} = \left(\frac{\det\sqrt{A}}{(2\pi)^N}\right)^{1/2} \exp\left(-\frac{1}{2}\left\langle x, \sqrt{A}x\right\rangle_{\mathbb{R}^N}\right).$$

The renormalized Schrödinger operator, i.e. the one with $\inf \mathrm{spec} H = 0$, is

$$H_{\mathrm{OU}} = -\frac{1}{2}\Delta + \frac{1}{2}\langle x, Ax\rangle_{\mathbb{R}^N} - \frac{1}{2}\mathrm{tr}\sqrt{A}.$$

The stochastic generator $L_{\mathrm{OU}}$ is given by

$$(L_{\mathrm{OU}} f)(x) = -\frac{1}{2}\Delta f(x) + \left\langle \sqrt{A}x, \nabla f(x)\right\rangle_{\mathbb{R}^N},$$

and the stochastic differential equation is

$$dX_t = -\sqrt{A}X_t\, dt + dB_t. \tag{A.6}$$

It can be checked with the help of Itô's formula that the solution of (A.6) with initial condition $X_0 = y$ is given by

$$X_t = e^{-t\sqrt{A}}y + \int_0^t e^{-|t-s|\sqrt{A}}\, dB_s. \tag{A.7}$$

As a stochastic integral with deterministic integrand, $X_t$ is a Gaussian process, and thus is characterized by its mean value

$$E(X_t) = e^{-t\sqrt{A}}y$$

and (matrix-valued) covariance function

$$C(t,s) = (E((X_s)_i(X_t)_j) - E(X_s)_i E(X_t)_j)_{1\le i,j\le N} = \frac{1}{2}\sqrt{A}^{-1}\left(e^{-|t-s|\sqrt{A}} - e^{-(t+s)\sqrt{A}}\right)$$

for $t, s \ge 0$. The covariance of the stationary solution of (A.6) is obtained by sending $t, s \to \infty$ and keeping $t - s$ fixed, with the result

$$C_{\mathrm{stat}}(t,s) = \frac{1}{2}\sqrt{A}^{-1}e^{-|t-s|\sqrt{A}}.$$

The mean value of the stationary process is 0.

# Bibliography


[1] M. Aizenman, B. Simon: Brownian motion and Harnack's inequality for Schrödinger operators. Comm. Pure Appl. Math. **35**, 209-271 (1982).

[2] G. Benfatto, E. Presutti, M. Pulvirenti: DLR Measures for one-dimensional harmonic systems. Z. Wahrscheinlichkeitstheorie verw. Gebiete **41**, 305-312 (1978).

[3] V. Betz: Existence of Gibbs measures relative to Brownian motion. mp-arc 01-285, submitted for publication (2001).

[4] V. Betz, F. Hiroshima, J. Lőrinczi, R. Minlos, H. Spohn: Ground state properties of the Nelson Hamiltonian - A Gibbs measure-based approach. To appear in Rev. Math. Phys (2001).

[5] V. Betz, J. Lőrinczi: A Gibbsian description of $P(\phi)_1$-processes. mp-arc 99-223, submitted for publication (2000).

[6] V. Bach, J. Fröhlich, I. M. Sigal: Quantum electrodynamics of confined non-relativistic particles. Adv. Math. **137**, 299-395 (1998).

[7] P. Billingsley: Convergence of probability measures. New York, London, Sidney, Toronto: John Wiley & sons, 1968.

[8] V. Bogachev: Gaussian measures. AMS mathematical surveys and monographs 62, AMS Publications, 1998.

[9] K. Broderix, D. Hundertmark, H. Leschke: Continuity properties of Schrödinger semigroups with magnetic fields, *Rev. Math. Phys.* **12**, 181-255 (2000).

[10] R. Carmona: Pointwise bounds for Schrödinger eigenstates, *Commun. Math. Phys.* **62**, 97-106 (1978).

[11] R. L. Dobrushin: Analyticity of correlation functions in one-dimensional classical systems with slowly decreasing potentials. *Commun. Math. Phys.* **32**, 269 - 289 (1973).

[12] R. L. Dobrushin: Analyticity of correlation functions for one-dimensional classical systems with power-law decay of the potential. *Math. USSR Sbornik* **23**, 13-44 (1973).

[13] R. Durett: Probability: Theory and Examples. Wadsworth Publishing Company, 1996.

[14] R. P. Feynman, A. R. Hibbs: Quantum Mechanics and Path Integrals. McGraw Hill 1965.

[15] J. Fröhlich: On the infrared problem in a model of scalar electrons and massless scalar bosons. Ann. Inst. H. Poincaré **19**, 1-103 (1973).

[16] J. Fröhlich: Existence of dressed one electron states in a class of persistent models. Fortschr. Phys. **22**,159-198 (1974).

[17] C. Gérard: On the existence of ground states for massless Pauli-Fierz Hamiltonians. Ann. Henri Poincaré **1**, 443-459. (2000)

[18] H.-O. Georgii: Gibbs Measures and Phase Transitions. Berlin, New York: de Gruyter, 1988.

[19] J. Glimm, A. Jaffe: Quantum Physics: A Functional Integral Point of View. New York: Springer, 1981.

[20] M. Griesemer, E. Lieb, M. Loss: Ground states in non-relativistic quantum electrodynamics. Inv. Math., to appear. (2001)

[21] Y. Hariya: A new approach to construct Gibbs measures on $C(\mathbb{R}, \mathbb{R}^d)$. Preprint (2001).

[22] T. Hida, H-H. Kuo, J. Potthoff, L. Streit: White Noise. Dordrecht, Boston, London: Kluwer Academic Publishers, 1993.

[23] F. Hiroshima: Diamagnetic inequalities for systems of nonrelativistic particles with a quantized field. Rev. Math. Phys. **8**, 185-203 (1996).

[24] M. Hübner, H. Spohn: Radiative decay: nonperturbative approaches. Rev. Math. Phys. **7**, 363-387 (1995).

[25] Iwata, K, 1987. Reversible measures of a $P(\phi)_1$ time evolution, in Probabilistic Methods in Mathematical Physics, Proc. Taniguchi Symp., Katata-Kyoto, 1985, Academic Press, pp. 195-209.

[26] O. Kallenberg: Foundations of Modern Probability, New York, Berlin, Heidelberg: Springer 1997.

[27] C. Kipnis, S.R.S. Varadhan: Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Commun. Math. Phys.* **104**, 1-19 (1986).

[28] J.L. Lebowitz, E. Presutti: Statistical mechanics of systems of unbounded spins. *Commun. Math. Phys.* **50**, 195-218 (1976).

[29] J. Lőrinczi, R. Minlos: Gibbs measures for Brownian paths under the effect of an external and a small pair potential. J. Stat. Phys. **105**, 607-649 (2001).

[30] J. Lőrinczi, R. Minlos, H. Spohn: The infrared behavior in Nelson's model of a quantum particle coupled to a massless scalar field. Ann. Henri Poincaré **3**, 1-28 (2001).

[31] J. Lőrinczi, R. A. Minlos, H. Spohn: Infrared regular representation of the three dimensional massless Nelson model. Submitted for publication (2001).

[32] E. Nelson: Schrödinger particles interacting with a quantized scalar field, Proceedings of a conference on analysis in function space, Ed. W. T. Martin, I. Segal, MIT Press, Cambridge 1964, p. 87.

[33] E. Nelson: Interaction of nonrelativistic particles with a quantized scalar field, J. Math. Phys. **5**, 1990-1997 (1964).

[34] E. Nelson: Probability theory and euclidean quantum field theory. In: Constructive Quantum field theory, ed. G. Velo and A. Wightman. Berlin, Heidelberg, New York: Springer, 1973.

[35] E. Nelson: The free Markoff field, *Journ. Funct. Anal.* **12**, 211-227 (1973).

[36] E. Nelson: Construction of quantum fields from Markoff fields, *Journ. Funct. Anal.* **12**, 97-112 (1973).

[37] N. Obata: White noise calculus and Fock space. Berlin, Heidelberg: Springer, 1994.

[38] H. Osada, H. Spohn: Gibbs measures relative to Brownian motion. *Ann. Probab.* **27**, 1183-1207 (1999).

[39] J. Rosen, B. Simon: Fluctuations in $P(\phi)_1$ processes. *Ann. Probab.* **4**, No 2, 155-174 (1976).

[40] M. Reed, B. Simon: Methods of Modern Mathematical Physics, 1. Functional analysis. London: Academic Press, 1980.

[41] M. Reed, B. Simon: Methods of Modern Mathematical Physics, 2. Fourier Analysis, Self-Adjointness. London: Academic Press, 1975.

[42] M. Reed, B. Simon: Methods of Modern Mathematical Physics, 4. Analysis of Operators. London: Academic Press, 1978.

[43] D. Ruelle: Superstable interactions in classical statistical mechanics. *Commun. Math. Phys* **18**, 127-159 (1970).

[44] B. Simon: The $P(\phi)_2$ Euclidean (quantum) field theory. Princeton, New York: Princetom Univ. Press (1974).

[45] B. Simon: Functional Integration and Quantum Physics. New York, San Francisco, London: Academic Press, 1979.

[46] B. Simon: Schrödinger semigroups. *Bull. AMS* **7**, 447-526 (1982).

[47] H. Spohn: Effective mass of the polaron: a functional integral approach. *Ann. Phys.* **175** No. 2, 278-318 (1987).

[48] H. Spohn: Ground state of a quantum particle coupled to a scalar boson field. Lett. Math. Phys. **44**, 9-16 (1998).

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