THE CRITICAL TEMPERATURE OF DILUTE BOSE GASES: A TENTATIVE EXACT APPROACH USING SPATIAL PERMUTATIONS

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We attempt to calculate the change of critical temperature of the dilute Bose gas, that is due to interactions. The system of N bosons in the three-dimensional cubic box Λ is described by the Schrödinger operator

$$H = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \leq i < j \leq N} U(x_i - x_j),$$

acting in $L^2_{\rm sym}(\Lambda^N)$. We always suppose that the interaction potential U is repulsive and finite range; we let a denote its scattering length.

Our approach starts with the Feynman-Kac representation, where quantum particles are represented by Brownian bridges, and the restriction to the symmetric subspace is implemented by explicitly summing over permutations $\pi \in \mathcal{S}_N$. A good reference is [4]. We a re led to a model of "spatial random permutations", whose state space is $\Lambda^N \times \mathcal{S}_N$, i.e. it consists of all pairs $(\boldsymbol{x}, \pi)$, with $\boldsymbol{x} = (x_1, \dots, x_N)$ representing N positions in Λ, and π a permutation of N elements. One considers the Gibbs weight $\mathrm{e}^{-H(\boldsymbol{x},\pi)}$ with "Hamiltonian"

$$H_1(\boldsymbol{x}, \pi) = \frac{1}{4\beta} \sum_{i=1}^{N} |x_i - x_{\pi(i)}|^2 + \sum_{1 \leq i < j \leq N} V_{ij}(\boldsymbol{x}, \pi).$$

The first term is due to Brownian bridges and it forces the spatial permutation to have only small jumps. The second term gives the interaction between the jumps $x_i \mapsto x_{\pi(i)}$ and $x_j \mapsto x_{\pi(j)}$. Its explicit expression is a bit complicated, namely

$$\begin{aligned} V_{ij}(\boldsymbol{x}, \pi) &= \int \bigl[1 - \mathrm{e}^{-\frac14 \int_0^{4\beta} U(\omega(s)) \mathrm{d}s}\bigr] \mathrm{d}\widehat{W}^{4\beta}_{x_i - x_j, x_{\pi(i)} - x_{\pi(j)}}(\omega) \\ &= K\bigl(x_i - x_j, x_{\pi(i)} - x_{\pi(j)}\bigr). \end{aligned}$$

Here, $\widehat{W}$ denotes the normalized Wiener measure for Brownian bridges, and K is the integral kernel of the operator $\mathrm{e}^{2\beta\Delta} - \mathrm{e}^{\beta(2\Delta - U)}$. Notice that $V_{ij}(\boldsymbol{x}, \pi)$ depends only on $x_i, x_{\pi(i)}, x_j, x_{\pi(j)}$. Our system is not exactly equivalent to the Bose gas, but it should describe it exactly in the dilute regime $a\rho^{1/3} \ll 1$ [8].

The usual definition of the Bose-Einstein condensation involves the off-diagonal correlations $\langle a^*(x)a(y)\rangle$. But in this approach we consider an alternate criterion pioneered by Feynman [3] and Sütő [6, 7], and that should be equivalent when interactions are weak. Namely, there is Bose-Einstein condensation if and only if there are infinite permutation cycles.

At this stage we perform certain expansions and simplifications, which we hope to retain the main features of the model — the critical temperature should be identical to first order in $a\rho^{1/3}$. But these steps remain to be justified in a mathematically rigorous fashion. We then obtain the simpler Hamiltonian

$$H_2(\boldsymbol{x},\pi) = \frac{4\pi\beta a N^2}{|\Lambda|} + 8\pi\beta\rho_{\rm c}^{(0)} a N + \frac{1}{4\beta}\sum_{i=1}^N |x_i - x_{\pi(i)}|^2 + \sum_{\ell\geq 1}\alpha'_\ell r_\ell(\pi),$$

where $r_\ell(\pi)$ gives the number of cycles of length ℓ in the permutation π. The weights α'_ℓ are given by

$$\alpha'_\ell = \frac{2\ell a}{(4\pi\beta)^{1/2}}\Big[\tfrac12 \sum_{j=1}^{\ell-1}\Big(\frac{\ell}{j(\ell-j)}\Big)^{3/2} - \zeta(\tfrac32)\Big].$$

One can check that the weights are negative, and that they converge to $-(6-\gamma_{1/2})(4\pi\beta)^{-1/2}a$ as $\ell\to\infty$. The critical density of this model is conjectured to be

$$\rho_{\rm c}^{(a)} = (4\pi\beta)^{-3/2}\sum_{\ell\geq 1}\frac{{\rm e}^{-\alpha'_\ell}}{\ell^{3/2}}.$$

This was rigorously proved in the case where the weights α'_ℓ go to 0 faster than $1/\log\ell$ as $\ell\to\infty$ [2]. We assume here that the formula remains true with the present weights. Let $\rho_1^{(a)}$ be the critical density for the occurrence of infinite cycles in this model. To first order, the change in the critical temperature is given by

$$\frac{\rho_{\rm c}^{(a)}-\rho_{\rm c}^{(0)}}{\rho_{\rm c}^{(0)}} \approx -\frac{1}{\zeta(\frac32)}\sum_{\ell\geq1}\frac{\alpha'_\ell}{\ell^{3/2}} = \frac{2\sqrt{\pi}a\beta^{-1/2}}{\zeta(\frac32)}.$$

In the physics literature, people have rather consider the change in the critical temperature. The result above translates into

$$\frac{T_{\rm c}^{(a)}-T_{\rm c}^{(0)}}{T_{\rm c}^{(0)}} \approx -\frac{8\pi a\rho^{1/3}}{3\zeta(\frac32)^{4/3}} \approx -2.33\,a\rho^{1/3}.$$

We have found that the change in the critical temperature is linear in a and that it is negative. The latter point is totally unexpected, as it goes against the findings of the physics community. Indeed, many papers have been devoted to this question, and physicists have recently reached the consensus that $\frac{T_{\rm c}^{(a)}-T_{\rm c}^{(0)}}{T_{\rm c}^{(0)}} \approx 1.3\,a\rho^{1/3}$. See [5] for a review of the physics literature, and also for a rigorous upper bound on the critical temperature.

The discrepancy between our result and the physics consensus is puzzling. We trust the physics literature, especially when over a dozen articles point towards the same conclusion. On the other hand, our approximations seem reasonable and they should lead to the correct critical temperature, to first order in the scattering length. This conundrum will hopefully be solved in the future.

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