# Existence of Gibbs measures relative to Brownian motion

Volker Betz

*Zentrum Mathematik, Technische Universität München*
*Gabelsbergerstr. 49, 80290 München, Germany*
betz@mathematik.tu-muenchen.de

**Abstract**

We prove existence of infinite volume Gibbs measures relative to Brownian motion. We require the pair potential $W$ to fulfill a uniform integrability condition, but otherwise our restrictions on the potentials are relatively weak. In particular, our results are applicable to the massless Nelson model. We also prove an upper bound for path fluctuations under the infinite volume Gibbs measures.

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## 1 Introduction

Let us define a probability measure on $C([-T,T],\mathbb{R}^d)$ by

$$d\mu_T^{y,z}(x) = \frac{1}{Z_T(y,z)} e^{-\int_{-T}^{T} V(x_s)\,ds - \int_{-T}^{T} ds \int_{-T}^{T} dt\, W(x_t,x_s,|t-s|)}\, d\mathcal{W}_{[-T,T]}^{y,z}(x). \tag{1}$$

on $C([-T,T],\mathbb{R}^d)$. Here, $T>0$, $y,z\in\mathbb{R}^d$, $\mathcal{W}_{[-T,T]}^{y,z}$ is pinned Brownian motion starting in $y$ at time $-T$ and ending in $z$ at time $T$, $Z_T(y,z)$ normalizes $\mu_T^{y,z}$ to a probability measure, and $V:\mathbb{R}^d\to\mathbb{R}$ and $W:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ are measurable functions with some additional properties to be specified later. One choice of $V$ and $W$ that fits into the framework of the present paper is

$$d=3, \qquad V(x) = -1/|x| \quad \text{and} \quad W(x,y,t) = -1/(|x-y|^2+t^2+1). \tag{2}$$

A natural problem in the context of (1) is the existence and uniqueness, i.e. independence of the 'boundary conditions' $y,z\in\mathbb{R}^d$, of a limiting probability measure $\mu_\infty$ on $C(\mathbb{R},\mathbb{R}^d)$ as $T\to\infty$ in (1). $\mu_\infty$ will be called (infinite volume) Gibbs measure relative to Brownian motion; this terminology already suggests a close relationship with statistical mechanics. We will outline this connection as well as a link to the theory of large deviations toward the end end of this introduction, but first let us study (1) in its own right.

An easy special case of (1) is obtained by choosing $W = 0$. Then, via the Feynman-Kac formula, $\mu_T^{x,y}$ is related to the Schrödinger operator $H_0 = -\frac{1}{2}\Delta + V$. If

$$H_0 \text{ has a ground state } \psi_0 \in L^2(\mathbb{R}^d), \tag{3}$$

then the infinite volume Gibbs measure exists and is given by the stationary solution of the stochastic differential equation $dX_t = \frac{\nabla\psi_0}{\psi_0}(X_t) + dB_t$ (see [18] or equations (14) and (15)). We will take the point of view that the $W \neq 0$ case is a perturbation of the $W = 0$ case. The existence problem for $\mu_\infty$ can then be regarded as a generalization to the problem of finding stationary solutions for stochastic differential equations. An important difference of the two problems is that, unlike solutions to stochastic differential equations, the limiting measure will not be the measure of a Markov process if $W \neq 0$.

When looking for reasonable conditions on $V$ and $W$ that ensure existence of $\mu_\infty$, a natural requirement on $V$ is that it should lead to an infinite volume Gibbs measure at least in the case $W = 0$. (3) is a sufficient condition for this. As far as the 'perturbation' $W$ is concerned, we should require that its effect does not completely outweigh the effect of the $V$. In other words, $W$ has to be extensive, i.e.

$$\limsup_{T\to\infty} \frac{1}{T} \left| \int_{-T}^{T} ds \int_{-T}^{T} dt\, W(x_s, x_t, |s-t|) \right| < \infty \tag{4}$$

at least for a reasonable class of $x \in C(\mathbb{R}, \mathbb{R}^d)$. While (4) may not be sufficient for the existence of $\mu_\infty$ in general, additional conditions on $W$ should be more of a technical nature.

As for uniqueness, already the case $W = 0$ shows [1] that we can only expect $\mu_\infty$ to be unique among the measures supported on a subset of $C(\mathbb{R}, \mathbb{R}^d)$ which is characterized by a condition on the growth of paths at infinity. Once this restriction is made, according to the folklore a sufficient condition is that the interaction energy

$$I = \sup_{x\in C(\mathbb{R},\mathbb{R}^d)} \left| \int_{-\infty}^{0} ds \int_{0}^{\infty} dt\, W(x_t, x_s, |t-s|) \right| \tag{5}$$

between left and right half of the path is finite. Such a strong result is not available at present, but [16] and [13] have some results about uniqueness, and [16] gives an example where uniqueness fails when (5) is not fulfilled. In the present work, we have nothing to say about uniqueness, focussing on existence instead.

Several authors have by now studied the existence problem. All of them assume (4) in some form, but also need additional restrictions on $V$ and $W$. In [16], the first mathematical account on the subject, correlation inequalities are used, and consequently the potentials $V$ and $W$ have to fulfill certain convexity assumptions. In [13], a cluster expansion method is applied, requiring a small parameter (coupling constant) in front of $W$ as well as a $V$ that is growing faster than quadratically at infinity. Recently, [11] used an integration by parts formula. His restrictions on $W$ are weak, but strong assumptions on the asymptotic behaviour of $V$ are needed. In particular, $V$ has to grow at least quadratically at infinity.

In this work we establish a new method for proving existence of $\mu_\infty$, relying on a stopping time estimate. The main advantage over the existing approaches is that our restrictions on $V$ are almost as weak as (3). All cases from [16, 13, 11] are covered, and in addition we allow for $V$'s which do not grow at infinity. For the pair potential $W$, the main assumption essentially is that (4) holds uniformly in $x \in C(\mathbb{R}, \mathbb{R}^d)$. In addition, we need a 'pathwise shift condition' that is somewhat implicit but easy to verify for many concrete examples of $W$. If we assume that $V$ fits in the framework of [16], [13] or [11], then on the one hand the (uniform) integrability conditions on $W$ that we impose are stronger than those needed there. On the other hand, we neither need the convexity assumed in [16], nor the small parameter of [13], nor the differentiability needed in [11]. An important feature that our work shares with all of the above is that the interaction energy (5) between the left and the right half-line is not assumed to be finite.

As mentioned before, there exist connections or (1) with statistical mechanics as well as with the theory of large deviations. The latter connection is seen most clearly when we replace the exponent in (1) by

$$-\int_{-T}^{T} V(x_s)\,ds - \frac{1}{2T}\int_{-T}^{T} ds \int_{-T}^{T} dt\, \tilde{W}(x_s, x_t) \tag{6}$$

with some nice function $\tilde{W}$. (6) is then a functional of the local time, and thus is a special case of the theory of Donsker and Varadhan [8]. So in a sense, these systems are extremely well understood. It turns out that the limiting process for interactions like (6) is a Markov process. This is not the case for the actual system (1), which shows that although (6) and (1) may look similar, they yield very different limiting objects. In the language of statistical mechanics, (6) is a mean field interaction, while (1) is a local interaction.

To link (1) with statistical mechanics, more precisely with the theory of lattice spin systems, we discretize (1) by replacing Brownian motion with a random walk with state space $\mathbb{R}^d$ and Gaussian step size distribution. We then obtain a finite volume Gibbs measure on a one-dimensional system of $\mathbb{R}^d$-valued spins, with single site potential $V$, quadratic nearest neighbour interaction and long range pair interaction $W$. The reference measure is the product of $d$-dimensional Lebesgue-measures. An equivalent description of this spin system, a little bit closer to (1), is to incorporate the nearest-neighbour interaction into the reference measure, which then becomes the measure of a random walk pinned at $-T$ and $T$.

Although we will not do it here, our method can be easily adapted to the lattice context, where it yields a new way of proving existence of Gibbs measures for one-dimensional systems of unbounded spins. For such systems, extremely powerful methods are already available: there is the superstability estimate by D. Ruelle [17], applied in [12], which has the big advantage of not being restricted to one-dimensional systems; there are the results of R. L. Dobrushin [7, 9], which are valid only for one-dimensional systems, but extremely general otherwise. However, superstability corresponds to rapidly growing single site potential, while one of Dobrushin's few restrictions is that the interaction energy between left and right half-space must be bounded. Thus our method covers some new situations in the discrete context also.

Finally, although it should have become clear that Gibbs measures are interesting objects also from a purely probabilistic point of view, the original motivation for studying them is a physical one. Nelson [15] first used measures with a structure similar to (1) with a $W$ of the type given in (2) to study the ultraviolet divergence in a model of a quantum particle coupled to a scalar bosonic field, nowadays known as Nelson's model. In [20], Gibbs measures are used to estimate the effective mass of the polaron. Recently [14, 2] study various aspects of the ground state of Nelson's model by using Gibbs measures.

## 2 Finite volume Gibbs measures

We start by specifying conditions on the potentials $V$ and $W$ appearing in (1). A measurable function $V : \mathbb{R}^d \to \mathbb{R}$ is said to be in the Kato class [19], $V \in \mathcal{K}(\mathbb{R}^d)$, if

$$\sup_{x\in\mathbb{R}} \int_{\{|x-y|\leq 1\}} |V(y)|\, dy < \infty \qquad \text{in case } d = 1,$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{\{|x-y|\leq r\}} g(x-y)|V(y)|\, dy = 0 \qquad \text{in case } d \geq 2.$$

Here,

$$g(x) = \begin{cases} -\ln|x| & \text{if } d = 2 \\ |x|^{2-d} & \text{if } d \geq 3. \end{cases}$$

$V$ is locally in the Kato class, $V \in \mathcal{K}_{\rm loc}(\mathbb{R}^d)$, if $V 1_K \in \mathcal{K}(\mathbb{R}^d)$ for each compact set $K \subset \mathbb{R}^d$. $V$ is Kato-decomposable [4] if

$$V = V^+ - V^- \quad \text{with} \quad V^- \in \mathcal{K}(\mathbb{R}^d), V^+ \in \mathcal{K}_{\rm loc}(\mathbb{R}^d),$$

where $V^+$ is the positive part and $V^-$ is the negative part of $V$.

Our conditions on $V$ are:

(V1): $V : \mathbb{R}^d \to \mathbb{R}$ is Kato-decomposable.

(V2): The Schrödinger operator

$$H_0 = -\frac{1}{2}\Delta + V$$

(where $\Delta$ denotes the Laplace operator) acting in $L^2(\mathbb{R}^d)$ fulfills $\inf \operatorname{spec}(H_0) = 0$. Moreover, $H_0$ has a unique, strictly positive ground state $\psi_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, i.e. 0 is an eigenvalue of multiplicity one with corresponding eigenfunction $\psi_0$.

Condition (V1) guarantees that the factor $\exp(-\int_{-T}^{T} V(x_s)\, ds)$ appearing in (1) is integrable with respect to Brownian motion [19]. The existence of a ground state in (V2) ensures the existence of an infinite volume Gibbs measure in case $W = 0$, while $\inf \operatorname{spec}(H_0) = 0$ is included for convenience and can be achieved by simply adding a constant to $V$ and changing the normalizing constant in (1) accordingly. Finally, $\psi_0 \in L^1$ will be needed in the proof of Theorem 3.2, but is only a mild restriction, since in most cases of interest $\psi_0(x)$ decays exponentially for large $x$ [6].

Examples for potentials $V$ that fulfill $(V1)$ and $(V2)$ are continuous functions bounded below and growing at infinity, as well as functions bounded above but with the negative part having Coulomb type singularities.

Schrödinger operators with Kato-decomposable potentials have many nice properties [19]. In this paper we will need the fact that the kernel $K_t(x, y)$ of $e^{-tH_0}$ uniformly bounded and bounded away from zero on compact sets, and that $y \mapsto K_t(x, y)$ is integrable uniformly in $x$.

Turning to conditions on $W$, let us write

$$\mathcal{H}_\Lambda(x) = -\iint_\Lambda W(x_t, x_s, |t-s|)\, ds\, dt \qquad (x \in C(\mathbb{R}, \mathbb{R}^d)) \tag{7}$$

with $\Lambda \subset \mathbb{R}^2$. In case $\Lambda = [-T, T]^2$, we simply write $\mathcal{H}_T(x)$. $C^{(0)}(\mathbb{R}, \mathbb{R}^d)$ will denote functions which are continuous with the possible exception of the point 0 but have left and right hand side limits there. For $\tau > 0$ consider the map

$$\theta_\tau^{(0)} : C(\mathbb{R}, \mathbb{R}^d) \to C^{(0)}(\mathbb{R}, \mathbb{R}^d), \quad (\theta_\tau^{(0)} x)_t = \begin{cases} x_{t+\tau} & \text{if } t \geq 0, \\ x_{t-\tau} & \text{if } t < 0. \end{cases} \tag{8}$$

Finally, put

$$\alpha = \liminf_{|x| \to \infty} V(x) \leq \infty. \tag{9}$$

From the way this constant will enter into our proofs it will be clear that really the quantity

$$\liminf_{|x| \to \infty} V(x) - \inf \operatorname{spec}(H_0)$$

is the important one, a fact that is obscured by our choice $\inf \operatorname{spec}(H_0) = 0$ in (V2).

Our conditions on $W$ are

(W1): There exists $C_\infty < \infty$ such that

$$\int_{-\infty}^{\infty} |W(x_0, x_s, |s|)|\, ds < C_\infty \quad \text{and} \quad \int_{-\infty}^{\infty} |W(x_s, x_0, |s|)|\, ds < C_\infty, \tag{10}$$

uniformly in $x \in C(\mathbb{R}, \mathbb{R}^d)$.

(W2): There exist $D \geq 0$ and $0 \leq C < \alpha$ such that

$$\mathcal{H}_T(x) \leq \mathcal{H}_T(\theta_\tau^{(0)} x) + C\tau + D \tag{11}$$

for all $T, \tau > 0$ and all $x \in C(\mathbb{R}, \mathbb{R}^d)$.

An immediate consequence of (W1) is

$$\left|\mathcal{H}_{\mathbb{R}\times[-S,S]}(x)\right| \leq 2C_\infty S, \quad \text{and} \quad \left|\mathcal{H}_{[-S,S]\times\mathbb{R}}(x)\right| \leq 2C_\infty S. \tag{12}$$

(12) will be used frequently below.

(W2) looks a little mysterious at first, but the proof of Theorem 3.2 will show how it comes about naturally. To see when (W2) is fulfilled, note that by (12),

$$\begin{aligned} -\int_0^T ds \int_0^T dt\, W(x_t, x_s, |t-s|) &\leq 4C_\infty\tau - \int_\tau^{T+\tau} ds \int_\tau^{T+\tau} dt\, W(x_t, x_s, |t-s|) = \\ &= 4C_\infty\tau - \int_0^T ds \int_0^T dt W(x_{t+\tau}, x_{s+\tau}, |t-s|), \end{aligned}$$

and similarly for the region $[-T,0]^2$. Thus, if we suppose

$$I = \sup_{x\in C(\mathbb{R},\mathbb{R}^d)} \int_{-\infty}^0 ds \int_0^\infty dt\, |W(x_t, x_s, |t-s|)| < \infty,$$

then $8C_\infty < \alpha$ is a sufficient condition for (W2). In case $I = \infty$, it is not hard to see that if there exist $L, M > 0$ with

$$\int_{-T}^0 ds \int_0^T dt (W(x_s, x_t, |s-t|) - W(x_s, x_t, |s-t| + 2\tau)) \leq L\tau + M \tag{13}$$

uniformly in $x \in C(\mathbb{R},\mathbb{R}^d)$ and $T > 0$, then $12C_\infty + L < \alpha$ is a sufficient condition for (W2). (13) can be checked directly for many choices of $W$, and is in particular true if $t \mapsto W(x,y,t)$ is increasing for $t > 0$ and each fixed $x, y \in \mathbb{R}^d$. This covers the physically important case

$$W(x, y, |t|) = -\frac{1}{(|x-y|^2 + |t|^2 + 1)}$$

of the massless Nelson model [5, 13]. On the other hand, for

$$W(x, y, |t|) = \begin{cases} -\frac{1}{|t|^2+1} & \text{if } |x-y| \leq 2t \\ 0 & \text{otherwise} \end{cases}$$

($x, y \in \mathbb{R}$) together with the path $x_t = t$, we find that $\int_{-T}^0 ds \int_0^T dt\, W(x_s, x_t, |t-s|)$ diverges as $T \to \infty$, but e.g. $\int_{-T}^0 ds \int_0^T dt\, W(x_{s-1}, x_{t+1}, |t-s|) = 0$. Thus (W2) need not hold in general.

We now construct finite volume Gibbs measures. We will take a point of view that differs slightly from the one taken in equation 1 by incorporating the single site potential $V$ into the reference measure. This leads to a $P(\phi)_1$-process [18]. To make the paper reasonably self-contained, we include a short description of this process.

The $P(\phi)_1$-process corresponding to the potential $V$ is the stationary solution of the stochastic differential equation

$$dX_t = \frac{\nabla\psi_0}{\psi_0}(X_t)\, dt + dB_t, \tag{14}$$

where $B_t$ denotes Brownian motion in $\mathbb{R}^d$. Remember that $\psi_0$ is the ground state of $H_0$. The measure on $C(\mathbb{R}, \mathbb{R}^d)$ corresponding to this process will be denoted by $\mu_0$ and identified with the process. $\mu_0$ is a stationary strong Markov process with generator $\tilde{H}_0 = \psi_0^{-1} H_0 \psi_0$, where $\psi_0$ and $\psi_0^{-1}$ denote operators of multiplication.

The tool that links (14) and (1) is the Feynman-Kac formula. It says that for a bounded interval $I = [0, T] \subset \mathbb{R}$ and a $\mu_0$-integrable, $\mathcal{F}_I$-measurable function $f$,

$$\int f(x)\, d\mu_0(x) = \int \psi_0(x_0) e^{-\int_0^T V(x_s)\, ds} f(x) \psi_0(x_T)\, d\mathcal{W}(x). \tag{15}$$

Here, $\mathcal{W}$ denotes the infinite mass Wiener measure, and $\mathcal{F}_I$ is the $\sigma$-field over $C(\mathbb{R}, \mathbb{R}^d)$ generated by the point evaluations with points inside $I$. A corresponding notation for $\sigma$-fields will be used throughout the paper.

By (15), the invariant measure of $\mu_0$ has the Lebesgue-density $\psi_0^2$. Moreover, a refined version of the Feynman-Kac formula [18] shows that the transition density of $\mu_0$ is given in terms of the kernel $K_t(x, y)$ of $e^{-tH}$ by

$$E_{\mu_0}(f(x_t)|\mathcal{F}_{\{0\}})(y) = \frac{1}{\psi_0(y)} \int K_t(y, z) \psi_0(z) f(z)\, dz \qquad (y \in \mathbb{R}^d). \tag{16}$$

We perturb the process $\mu_0$ by the pair potential $W$, i.e. for $T > 0$ we define the probability measure $\mu_T$ on $C(\mathbb{R}, \mathbb{R}^d)$ by

$$d\mu_T(x) = \frac{1}{Z_T} e^{-\mathcal{H}_T(x)}\, d\mu_0(x), \tag{17}$$

where

$$Z_T = \int e^{-\mathcal{H}_T(x)}\, d\mu_0(x)$$

is the normalizing constant. Comparing (17) and (1), we see that instead of pinning the path to $y, z \in \mathbb{R}$ at time $-T$ resp. $T$ (“sharp boundary condition”), we now allow it to fluctuate according to $\mu_0$ outside $[-T, T]$, resulting in a “smeared-out boundary condition”. This is technically easier to handle and, as we will see in the course of the paper, good enough to prove existence of an infinite volume Gibbs measure.

Let us now check that $\mu_T$ is a finite volume Gibbs measure with respect to the potential $W$ and the reference measure $\mu_0$. For $S > 0$ write $\mathcal{T}_S$ instead of $\mathcal{F}_{[-S,S]^c}$, and for $\bar{x} \in C(\mathbb{R}, \mathbb{R}^d)$ denote by $\mu_0^{S,\bar{x}}$ the version of the regular conditional expectation $\mu_0(.|\mathcal{T}_S)$ that is given by

$$d\mu_0^{S,\bar{x}}(x) = \frac{1}{Z^S(\bar{x})} \exp\left(-\int_{-S}^{S} V(x_s)\, ds\right) d\left(\mathcal{W}^{\bar{x}}_{[-S,S]} \otimes \delta^{\bar{x}}_{[-S,S]^c}\right)(x). \tag{18}$$

Here, $\delta^{\bar{x}}_{[-S,S]^c}$ is the point measure on $C([-S, S]^c, \mathbb{R}^d)$ concentrated in $\bar{x}|_{[-S,S]^c}$, $\mathcal{W}^{\bar{x}}_S$ is pinned Brownian motion starting at time $-S$ in $\bar{x}(-S)$ and ending at time $S$ in $\bar{x}(S)$, and $Z^S(\bar{x})$ is the normalizing constant. Moreover, for $S < T$ define

$$\Lambda(S, T) = ([-T, T] \times [-S, S]) \cup ([-S, S] \times [-T, T]) \subset \mathbb{R}^2, \quad \text{and} \tag{19}$$

$$d\mu_T^{S,\bar{x}}(x) = \frac{1}{Z_T^S(\bar{x})} \exp(\mathcal{H}_{\Lambda(S,T)}(x))\, d\mu_0^{S,\bar{x}}(x). \tag{20}$$

In (20), $Z_T^S(\bar{x}) = E_{\mu_0^{S,\bar{x}}}(e^{\mathcal{H}_{\Lambda(S,T)}})$ is again the normalizing constant.

**Lemma 2.1** *For each $S < T$, $\bar{x} \mapsto \mu_T^{S,\bar{x}}$ is a version of the regular conditional expectation $\mu_T(.|\mathcal{T}_S)$. In other words, $\mu_T$ is a (finite volume) Gibbs measure with reference measure $\mu_0$ and potential $W$.*

*Proof:* Let $f, g \in L^\infty(C(\mathbb{R}, \mathbb{R}^d))$, and suppose $g$ is $\mathcal{T}$-measurable. Then

$$\begin{aligned}
& Z_T \int g(\bar{x}) E_{\mu_T^{S,\bar{x}}}(f)\, d\mu_T(\bar{x}) = \\
= \;& E_{\mu_0}\left( E_{\mu_0}\Big( g \frac{1}{E_{\mu_0}(e^{\mathcal{H}_{\Lambda(S,T)}}|\mathcal{T}_S)} E_{\mu_0}(f e^{\mathcal{H}_{\Lambda(S,T)}}|\mathcal{T}_S) e^{\mathcal{H}_{[-T,T]^2}} \Big| \mathcal{T}_S \Big) \right) = \\
= \;& E_{\mu_0}\left( g \frac{E_{\mu_0}(f e^{\mathcal{H}_{\Lambda(S,T)}}|\mathcal{T}_S)}{E_{\mu_0}(e^{\mathcal{H}_{\Lambda(S,T)}}|\mathcal{T}_S)} e^{\mathcal{H}_{[-T,T]^2 \setminus \Lambda(S,T)}} E_{\mu_0}\Big( e^{\mathcal{H}_{\Lambda(S,T)}} \Big| \mathcal{T}_S \Big) \right) = \\
= \;& Z_T E_{\mu_T}(fg).
\end{aligned}$$

Dividing by $Z_T$ finishes the proof. □

# 3 Infinite volume Gibbs measures

We say that a sequence $\nu_n$ of measures on $C(\mathbb{R}, \mathbb{R}^d)$ converges locally weakly to a measure $\nu$ if for each bounded interval $I \subset \mathbb{R}$, the restrictions of $\nu_n$ to $\mathcal{F}_I$ converge weakly to the restriction of $\nu$ to $\mathcal{F}_I$. It is easy to see that, when $C(\mathbb{R}, \mathbb{R}^d)$ equipped with the topology of uniform convergence on compact sets, local weak convergence is equivalent to weak convergence.

An infinite volume cluster point of a family $(\nu_T)_{T>0}$ of probability measures is a cluster point of any sequence $(\nu_{t_n})_{n \in \mathbb{N}}$, where $t_n \to \infty$ as $n \to \infty$.

We will show that the family $(\mu_T)$ is relatively compact in the topology of local weak convergence. From this the existence of an infinite volume cluster point follows immediately. To prove relative compactness, we use a well-known theorem due to Prohorov. Recall that a family $(\nu_T)$ of probability measures on $\Omega = C([a,b], \mathbb{R}^d)$ is called tight if

(T1): For all $\eta > 0$ there exists $R > 0$ such that

$$\nu_T(\{x \in \Omega : |x_a| > R\}) < \eta \qquad \text{uniformly in } T$$

.

(T2): For all $\eta > 0$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\nu_T(\{x \in \Omega : w_\delta([a,b]) > \varepsilon\}) < \eta \qquad \text{uniformly in } T,$$

where

$$w_\delta([a,b]) = \sup\{|x_s - x_t| : s, t \in [a,b], |s-t| < \delta.\}$$

Prohorov's theorem states that a tight family of measures is relatively compact in the weak topology [3].

Usually, (T2) is rather harder to show than (T1). In our special case, however, (T2) follows without too much work from (T1). Without loss in generality, we may (and will) restrict our attention to $[a, b] = [-S, S]$.

**Lemma 3.1** *Define $(\mu_T)_{T>0}$ as in (17), and assume (V1), (V2) and (W1). If $(\mu_T)_{T>0}$ fulfills (T1) then it fulfills (T2) as well.*

*Proof:* Fix $\eta > 0$ and $\varepsilon > 0$. Now by (T1) and the time reversibility of $\mu_T$ for all $T$ it is possible to choose $R$ such that

$$E_{\mu_T}(|x_{-S}| > R) < \eta/4 \quad \text{and} \quad E_{\mu_T}(|x_S| > R) < \eta/4 \quad \text{uniformly in } T.$$

Putting

$$B = \{|x_{-S}| \leq R \text{ and } |x_S| \leq R\} \subset C(\mathbb{R}, \mathbb{R}^d),$$

and

$$f_\delta(x) = 1_{\{w_\delta([-S,S]) > \varepsilon\}}(x) \quad (x \in C(\mathbb{R}, \mathbb{R}^d)),$$

we clearly have $\mu_T(B) > 1 - \eta/2$ uniformly in $T$ and $|f_\delta| < 1$, and thus

$$E_{\mu_T}(f_\delta) \leq \eta/2 + E_{\mu_T}(f_\delta 1_B) = \eta/2 + E_{\mu_T}\left(E_{\mu_T}\left(f_\delta 1_B | \mathcal{T}_S\right)\right). \tag{21}$$

Using Lemma 2.1, we find

$$\begin{aligned} E_{\mu_T}(f_\delta 1_B | \mathcal{T}_S)(\bar{x}) &= \frac{1}{Z_T^S(\bar{x})} \int e^{\mathcal{H}_{\Lambda(S,T)}(x)} f_\delta(x) 1_B(x)\, d\mu_0^{S,\bar{x}}(x) \leq \\ &\leq e^{8C_\infty S} \int f_\delta(x) 1_B(x)\, d\mu_0^{S,\bar{x}}(x) \\ &= e^{8C_\infty S} \int f_\delta(x)\, d\mu_0^{S,\bar{x}}(x) 1_B(\bar{x}). \end{aligned} \tag{22}$$

The inequality above follows from (12) and the definition of $Z_T^S(\bar{x})$. Now it is easy to see that the restriction of the family $\{\mu_0^{S,\bar{x}} : \bar{x} \in B\}$ to $\mathcal{F}_{[-S,S]}$ is tight. In fact, this follows from the compactness of $\{x, y \in \mathbb{R}^d : |x| \leq R, |y| \leq R\}$. Thus we can find $\delta > 0$ such that

$$\sup_{\bar{x} \in C(\mathbb{R}, \mathbb{R}^d)} \int f_\delta(x)\, d\mu_0^{S,\bar{x}}(x) 1_B(\bar{x}) < e^{-8C_\infty S} \frac{\eta}{2}.$$

Using this in (22) and plugging the resulting expression into (21), we arrive at

$$\mu_T(f_\delta) \leq \eta/2 + (\eta/2)\mu_T(1_B) \leq \eta,$$

which is what we had to show. $\square$

**Theorem 3.2** *Assume (V1),(V2),(W1) and (W2). Then $(\mu_T)_{T>0}$ fulfills (T1).*

*Proof:* Since by (12) and the stationarity of $\mu_0$ we have

$$e^{-2|t|C_\infty} \int f(x_t)\, d\mu_T(x) \leq \int f(x_0)\, d\mu_T(x) \leq e^{2|t|C_\infty} \int f(x_t)\, d\mu_T(x)$$

for all $t \in \mathbb{R}, T > 0$ and $f \in L^\infty(\mathbb{R}^d)$, it will be sufficient to prove the claim for $t = 0$. We do so in several steps.
**Step 1:** Let $E_{\mu_0}(f|x_0 = y)$ denote expectation with respect to the measure $\mu_0$ conditional on $x_0 = y$. Since $x \mapsto x_0$ has distribution $\psi_0^2 dx$, we have

$$\mu_T(|x_0| > R) = \frac{1}{Z_T} \int_{|y|>R} \psi_0^2(y) E_{\mu_0}\Big(e^{\mathcal{H}_T}\Big| x_0 = y\Big)\, dy. \tag{23}$$

In the next few steps, we will show that there exists $K > 0$ and $r > 0$ such that for all $T > 0$ and all $y \in \mathbb{R}^d$,

$$E_{\mu_0}\Big(e^{\mathcal{H}_T}\Big| x_0 = y\Big) \leq \frac{K}{\psi_0(y)} \inf_{|z|\leq r} E_{\mu_0}\Big(e^{\mathcal{H}_T}\Big| x_0 = z\Big). \tag{24}$$

Once we will have established (24), we can plug it into (23). Since moreover

$$\frac{1}{Z_T} \inf_{|z|\leq r} E_{\mu_0}\Big(e^{\mathcal{H}_T}\Big| x_0 = z\Big) \leq \sup_{|z|\leq r} \frac{1}{\psi_0^2(z)} \mu_T(|x_0| \leq r) \leq \tilde{K}$$

by an expression analogous to (23), we get

$$\mu_T(|x_0| > R) \leq K\tilde{K} \int_{|y|>R} \psi_0(y)\, dy. \tag{25}$$

The hypothesis $\psi_0 \in L^1$ from (V2) will then conclude the proof.
**Step 2:** In order to prove (24), we change the probability space we work on. Remember that $C^{(0)}$ was defined before equation (8), and consider

$$J : C^{(0)}(\mathbb{R}, \mathbb{R}^d) \to C([0,\infty[, \mathbb{R}^{2d}), \quad (x_t)_{t\in\mathbb{R}} \mapsto (x_t, x_{-t})_{t\geq 0}. \tag{26}$$

$(Jx)_0 \in \mathbb{R}^{2d}$ is defined via the left and right hand side limits of $x_t$ as $t \to 0$, and $J$ is a bijection after making some choice for the value of $x \in C^{(0)}(\mathbb{R}, \mathbb{R}^d)$ at the point 0. We will write $\mathbf{x} = (x', x'')$ for the elements of $C([0,\infty[, \mathbb{R}^{2d})$.
The image of $\mu_0(.|x_0 = z)$ under $J$ can be described explicitly. For $\mathbf{z} \in \mathbb{R}^{2d}$ denote by $\tilde{\mu}_0^{\mathbf{z}}$ the measure of the $\mathbb{R}^{2d}$-valued $P(\phi)_1$-process with potential $\tilde{V}(x, y) = V(x) + V(y)$, starting in $\mathbf{z}$. Explicitly, if we write $\tilde{\mathcal{F}}_T$ for the $\sigma$-field over $C([0,\infty[, \mathbb{R}^{2d})$ generated by point evaluations at points within $[0, T]$, then for every $\tilde{\mathcal{F}}_T$-measurable, bounded function $f$ we have

$$\int f(\mathbf{x})\, d\tilde{\mu}_0^{\mathbf{z}}(\mathbf{x}) = \frac{1}{\psi_0(z')\psi_0(z'')} \int e^{-\int_0^T (V(x'_s)+V(x''_s))\, ds} f(\mathbf{x})\psi_0(x'_T)\psi_0(x''_T)\, d\mathcal{W}^{\mathbf{z}}(\mathbf{x}). \tag{27}$$

Here, $\mathcal{W}^{\mathbf{z}}$ denotes $2d$-dimensional Wiener measure conditional on $\{\mathbf{x}_0 = \mathbf{z} = (z', z'')\}$, i.e. Brownian motion starting in $\mathbf{z}$. The Markov property and time reversibility of Brownian motion together with (15) imply that for each $z \in \mathbb{R}^d$, $\tilde{\mu}_0^{(z,z)}$ is the image of $\mu_0(.|x_0 = z)$ under $J$, i.e.

$$E_{\mu_0}(f \circ J|x_0 = z) = E_{\tilde{\mu}_0}^{(z,z)}(f).$$

Here, $E_{\tilde{\mu}_0}^{(z,z)}$ denotes expectation with respect to $\tilde{\mu}_0^{(z,z)}$.

Now it is easy to check that

$$\begin{aligned} \tilde{\mathcal{H}}_T(\mathbf{x}) \;\equiv\; & \mathcal{H}_T \circ J^{-1}(\mathbf{x}) = -\int_0^T ds \int_0^T dt\, \Big( W(x'_t, x'_s, |s-t|) + W(x''_t, x''_s, |s-t|) + \\ & +W(x'_t, x''_s, |s+t|) + W(x''_t, x'_s, |s+t|) \Big), \end{aligned} \tag{28}$$

and therefore

$$E_{\mu_0}\Big( e^{\mathcal{H}_T} \Big| x_0 = z \Big) = E_{\tilde{\mu}_0}^{(z,z)}(e^{\tilde{\mathcal{H}}_T}). \tag{29}$$

Thus we reduced our problem to investigating the expectation of $e^{\tilde{\mathcal{H}}_T}$ with respect to the strong Markov process $\tilde{\mu}_0^{\mathbf{z}}$ as a function of the starting point $\mathbf{z}$.

**Step 3:** First note that in the representation established in Step 2, hypothesis (11) takes the form

$$\tilde{\mathcal{H}}_T(\mathbf{x}) \le \tilde{\mathcal{H}}_T \circ \theta_\tau(\mathbf{x}) + C\tau + D \quad \text{for all } \mathbf{x} \in C([0,\infty[, \mathbb{R}^{2d}), T, \tau > 0. \tag{30}$$

Here $\theta_\tau = J\theta_\tau^{(0)} J^{-1}$ is the usual time shift that maps $(\mathbf{x}_t)_{t\ge 0}$ to $(\mathbf{x}_{t+\tau})_{t\ge 0}$. Our strategy is to use (30) together with the strong Markov property of $\tilde{\mu}_0$. For $r > 0$ let

$$\tau_r(\mathbf{x}) = \inf\{t \ge 0 : |\mathbf{x}_t| \le r\}$$

be the hitting time of the centered ball with radius $r$, and let $\mathcal{F}_{\tau_r}$ be the corresponding $\sigma$-field, i.e.

$$\mathcal{F}_{\tau_r} = \{A \in \tilde{\mathcal{F}} : A \cap \{\tau_r \le t\} \in \tilde{\mathcal{F}}_t \text{ for all } t \ge 0\}.$$

Then for each $\mathbf{x} \in \mathbb{R}^{2d}$,

$$\begin{aligned} E^{\mathbf{x}}(e^{\tilde{\mathcal{H}}_T}) &= E^{\mathbf{x}}(E^{\mathbf{x}}(e^{\tilde{\mathcal{H}}_T}|\mathcal{F}_{\tau_r})) \le E^{\mathbf{x}}(E^{\mathbf{x}}(e^{\tilde{\mathcal{H}}_T\circ\theta_{\tau_r}} e^{C\tau_r + D}|\mathcal{F}_{\tau_r})) = \\ &= E^{\mathbf{x}}(e^{C\tau_r + D} E^{\mathbf{x}}(e^{\tilde{\mathcal{H}}_T\circ\theta_{\tau_r}}|\mathcal{F}_{\tau_r})) = E^{\mathbf{x}}(e^{C\tau_r + D} E^{\mathbf{x}_{\tau_r}}(e^{\tilde{\mathcal{H}}_T})) \le \\ &\le \sup_{|\mathbf{y}|\le r} E^{\mathbf{y}}(e^{\tilde{\mathcal{H}}_T}) E^{\mathbf{x}}(e^{C\tau_r + D}). \end{aligned} \tag{31}$$

All expectations above and henceforth are with respect to $\tilde{\mu}_0$. It remains to get a good estimate on the second factor on the right hand side of (31) and to estimate the supremum in the first factor against an infimum. This will be done in Steps 4 and 5.

**Step 4:** Here we show that there exists $r > 0$ and $\gamma > 0$ such that for all $\mathbf{x} \in \mathbb{R}^{2d}$ we have

$$E^{\mathbf{x}}(e^{C\tau_r}) \le 1 + \frac{C\,\|\psi_0\|_\infty}{\gamma} \left( \frac{1}{\psi_0(x')} + \frac{1}{\psi_0(x'')} \right). \tag{32}$$

To do so, we pick $\gamma$ with $0 < \gamma < \alpha - C$ and $r$ so large that $V(x) > C + \gamma$ for all $x \in \mathbb{R}^d$ with $|x| > r/\sqrt{2}$. Obviously,

$$\{\mathbf{x} \in \mathbb{R}^{2d} : |\mathbf{x}| > r\} \subset \{\mathbf{x} \in \mathbb{R}^{2d} : |x'| > r/\sqrt{2}\} \cup \{\mathbf{x} \in \mathbb{R}^{2d} : |x''| > r/\sqrt{2}\},$$

and with (27) it follows that

$$\begin{aligned}
&\psi_0(z')\psi_0(z'')\mu_1^{\mathbf{z}}(\tau_r > t) = \\
&= \int e^{-\int_0^t (V(x_s')+V(x_s''))\,ds} 1_{\{|\mathbf{x}_s|>r \ \forall s \le t\}} \psi_0(x_t')\psi_0(x_t'')\,d\mathcal{W}^{\mathbf{z}}(\mathbf{x}) \le \\
&\le \int e^{-\int_0^t V(x_s')\,ds} e^{-\int_0^t V(x_s'')\,ds} \left(1_{\{|(x_s')|>r/\sqrt{2} \ \forall s \le t\}} + 1_{\{|(x_s'')|>r/\sqrt{2} \ \forall s \le t\}}\right) \times \\
&\quad \times \psi_0(x_t')\psi_0(x_t'')\,d\mathcal{W}^{z'}(x')\,d\mathcal{W}^{z''}(x'') = \\
&= \psi_0(z'') \int e^{-\int_0^t V(x_s')\,ds} 1_{\{|(x_s')|>r/\sqrt{2} \ \forall s \le t\}} \psi_0(x_t')\,d\mathcal{W}^{z'}(x') + \\
&\quad + \psi_0(z') \int e^{-\int_0^t V(x_s'')\,ds} 1_{\{|(x_s'')|>r/\sqrt{2} \ \forall s \le t\}} \psi_0(x_t'')\,d\mathcal{W}^{z''}(x'') \le \\
&\le (\psi_0(z') + \psi_0(z''))\,\|\psi_0\|_\infty\, e^{-(C+\gamma)t}.
\end{aligned}$$

The second equality above is due the eigenvalue equation $e^{-tH_0}\psi_0 = \psi_0$ and the Feynman-Kac formula. It follows that

$$\tilde{\mu}_0^{\mathbf{z}}(\tau_r > t) \le \left(\frac{1}{\psi_0(z')} + \frac{1}{\psi_0(z'')}\right) \|\psi_0\|_\infty\, e^{-(C+\gamma)t},$$

and using the equality

$$E^{\mathbf{z}}(e^{C\tau_r}) = 1 + \int_0^\infty Ce^{Ct} E^{\mathbf{z}}(\tau_r > t)\,dt$$

we arrive at (32).

**Step 5:** Let $r > 0$ be as in Step 4. We will show that there exists $M > 0$ such that

$$\sup_{|\mathbf{y}| \le r} E^{\mathbf{y}}(e^{\tilde{\mathcal{H}}_T}) \le M \inf_{|\mathbf{y}| \le r} E^{\mathbf{y}}(e^{\tilde{\mathcal{H}}_T}) \tag{33}$$

uniformly in $T > 0$. Denote by $P_t(\mathbf{x}, \mathbf{y})$ the transition density from $\mathbf{x}$ to $\mathbf{y}$ in time $t$ of the process $\tilde{\mu}_0$. By (27) and (16) we have

$$P_t(\mathbf{x}, \mathbf{y}) = \frac{\psi_0(y')\psi_0(y'')}{\psi_0(x')\psi_0(x'')} K_t(x', y') K_t(x'', y''). \tag{34}$$

$\psi_0$ and $K_t$ are both uniformly bounded and bounded away from zero on compact sets, thus for each $R > 0$ the quantity

$$S_t(R, r) = \sup \left\{ \frac{P_t(\mathbf{x}, \mathbf{z})}{P_t(\mathbf{y}, \mathbf{z})} : \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{2d}, |\mathbf{x}| \le r, |\mathbf{y}| \le r, |\mathbf{z}| \le R \right\}$$

is finite. Defining $\tilde{\mathcal{H}}^1_T$ like in (28) but with the integrals starting at 1 rather than at 0, we see from (12) that

$$\tilde{\mathcal{H}}_T(\mathbf{x}) - 4C_\infty \le \tilde{\mathcal{H}}^1_T(\mathbf{x}) \le \tilde{\mathcal{H}}_T(\mathbf{x}) + 4C_\infty$$

for all $\mathbf{x}$ and all $T$. Putting $B = \{|\mathbf{x}_1| < R\}$, for each $\mathbf{y}$ with $|\mathbf{y}| < r$ we have

$$E^{\mathbf{y}}(e^{\tilde{\mathcal{H}}_T}) \le e^{4C_\infty} E^{\mathbf{y}}(1_B e^{\tilde{\mathcal{H}}^1_T}) + e^{C+D} E^{\mathbf{y}}(1_{B^c} e^{\tilde{\mathcal{H}}_T \circ \theta_1}). \tag{35}$$

Defining $\bar{\mathcal{H}}_T$ as in (28) but with $|s+t+2|$ appearing instead of $|s+t|$ everywhere, in the first term on the right hand side of (35) we find

$$\begin{aligned} E^{\mathbf{y}}(1_B e^{\tilde{\mathcal{H}}^1_T}) &= \int_{|\mathbf{z}|<R} P_1(\mathbf{y},\mathbf{z}) E^{\mathbf{z}}(e^{\bar{\mathcal{H}}_{T-1}})\, d\mathbf{z} \le \\ &\le S_1(R,r) \int_{|\mathbf{z}|\le R} P_1(\mathbf{x},\mathbf{z}) E^{\mathbf{z}}(e^{\bar{\mathcal{H}}_{T-1}})\, d\mathbf{z} = \\ &= S_1(R,r) E^{\mathbf{x}}(1_B e^{\tilde{\mathcal{H}}^1_T}) \le S_1(R,r) e^{4C_\infty} E^{\mathbf{x}}(e^{\tilde{\mathcal{H}}_T}) \end{aligned} \tag{36}$$

for each $\mathbf{x}$ with $|\mathbf{x}| \le r$. Turning to the second term on the right hand side of (35), equations (31) and (32) give

$$\begin{aligned} &E^{\mathbf{y}}(1_{B^c} e^{\tilde{\mathcal{H}}_T \circ \theta_1}) = \int_{|\mathbf{z}|>R} P_1(\mathbf{y},\mathbf{z}) E^{\mathbf{z}}(e^{\tilde{\mathcal{H}}_T})\, d\mathbf{z} \le \\ &\le \sup_{|\mathbf{x}|\le r} E^{\mathbf{x}}(e^{\tilde{\mathcal{H}}_T}) \int_{|\mathbf{z}|>R} P_1(\mathbf{y},\mathbf{z}) E^{\mathbf{z}}(e^{C\tau_r + D})\, d\mathbf{z} \le \\ &\le \sup_{|\mathbf{x}|\le r} E^{\mathbf{x}}(e^{\tilde{\mathcal{H}}_T}) e^D \int_{|\mathbf{z}|>R} P_1(\mathbf{y},\mathbf{z}) \left(1 + \frac{C\,\|\psi_0\|_\infty}{\gamma}\left(\frac{1}{\psi_0(z')} + \frac{1}{\psi_0(z'')}\right)\right) d\mathbf{z}. \end{aligned} \tag{37}$$

By (34) and the eigenvalue equation, we have

$$\begin{aligned} &\int P_1(\mathbf{y},\mathbf{z}) \left(\frac{1}{\psi_0(z')} + \frac{1}{\psi_0(z'')}\right) d\mathbf{z} = \\ &= \frac{1}{\psi_0(y')} \int K_1(y'',z)\, dz + \frac{1}{\psi_0(y'')} \int K_1(y',z)\, dz. \end{aligned} \tag{38}$$

By (V1), the above integrals are bounded in $y'$ and $y''$, respectively [19], and thus the right hand side of (38) is uniformly bounded on $\{\mathbf{y} : |\mathbf{y}| < r\}$. This implies that there exists $\bar{R} > 0$ and $\delta < 1$ such that

$$\int_{|\mathbf{z}|>\bar{R}} P_1(\mathbf{y},\mathbf{z}) \left(1 + \frac{C\,\|\psi_0\|_\infty}{\gamma}\left(\frac{1}{\psi_0(z')} + \frac{1}{\psi_0(z'')}\right)\right) d\mathbf{z} \le e^{-(C+2D)}\delta$$

uniformly on $\{\mathbf{y} : |\mathbf{y}| < r\}$. Plugging this result together with (36) into (35), we arrive at

$$E^{\mathbf{y}}(e^{\tilde{\mathcal{H}}_T}) \le S_1(\bar{R},r) e^{8C_\infty} E^{\mathbf{x}}(e^{\tilde{\mathcal{H}}_T}) + \delta \sup_{|\mathbf{z}|\le r} E^{\mathbf{z}}(e^{\tilde{\mathcal{H}}_T}), \tag{39}$$

which is valid for all $\mathbf{x}, \mathbf{y}$ with $|\mathbf{x}|, |\mathbf{y}| \leq r$. By taking the supremum over $\mathbf{y}$ and the infimum over $\mathbf{x}$ in (39) and rearranging, we find

$$\sup_{|\mathbf{y}|\leq r} E^{\mathbf{y}}(e^{\tilde{\mathcal{H}}_T}) \leq \frac{S_1(\bar{R}, r)e^{8C_\infty}}{1-\delta} \inf_{|\mathbf{y}|\leq r} E^{\mathbf{y}}(e^{\tilde{\mathcal{H}}_T}),$$

which concludes Step 5 and the proof. □

The two previous statements show relative compactness of the restrictions $\{\mu_T|_{\mathcal{F}_{[-S,S]}} : T > 0\}$ for any $S > 0$. From here, it is only a small step to relative compactness in the topology of local weak convergence.

**Theorem 3.3** *Assume (V1),(V2),(W1) and (W2). Then $\{\mu_T : T \geq 0\}$ is relatively compact in the topology of local weak convergence. Consequently, the family has an infinite volume cluster point.*

*Proof:* Take $S > 0$ and fix any sequence $(T_n) \subset \mathbb{R}^+$. By Lemma 3.1, Theorem 3.2 and the tightness argument, for each fixed $S > 0$ there exists a subsequence $(t_n)$ of $(T_n)$ such that $(\mu_{t_n}|_{\mathcal{F}_{[-S,S]}})$ converges weakly to some probability measure $\mu_\infty$ on $C([-S,S], \mathbb{R}^d)$ . In case $L = \limsup_{n\to\infty} T_n < \infty$ we are done by choosing $S > L$. In case $L = \infty$, we observe that convergence of $(\mu_{t_n}|_{\mathcal{F}_{[-R,R]}})_{n\in\mathbb{N}}$ implies convergence of $(\mu_{t_n}|_{\mathcal{F}_{[-S,S]}})_{n\in\mathbb{N}}$ if $R > S$, and thus a diagonal sequence argument does the job. This second case also provides us with an infinite volume cluster point. □

Let us denote by $\mu$ any cluster point of the family $(\mu_T)_{T>0}$ obtained by Theorem 3.3. Due to the good control on the stationary density we obtain in Theorem 3.2, we have the following estimate on the growth of paths under $\mu$.

**Lemma 3.4** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be monotone increasing with $f(x) \to \infty$ as $x \to \infty$, and suppose that*

$$\sum_{n=1}^{\infty} \int_{|y|>f(n)} \psi_0(y)\, dy < \infty \tag{40}$$

*Then for $\mu$-almost every path $x \in C(\mathbb{R}, \mathbb{R}^d)$, we have*

$$\limsup_{|t|\to\infty} \frac{|x_t|}{f(|t|)} \leq 1.$$

*Proof:* By path continuity and time reversibility, it is obviously enough to prove that for each $k \in \mathbb{N}$,

$$\mu\left(\limsup_{n\to\infty} \frac{|x_{(n/k)}|}{f(n/k)} > 1\right) = 0.$$

Since the above event is equal to $\{x : |x_{n/k}| > f(n/k) \text{ infinitely often}\}$, the first Borel-Cantelli lemma will yield the result once we have checked that

$$\sum_{n=1}^{\infty} \mu(|x_{n/k}| > f(n/k)) < \infty. \tag{41}$$

By the stationarity of $\mu$ and equation (25), there exists a constant $M$ such that

$$\mu(|x_{n/k}| > f(n/k)) = \mu(|x_0| > f(n/k)) \leq M \int_{|y|>f(n/k)} \psi_0(y)\, dy$$

for $n$ large enough. Since

$$\sum_{n=1}^{\infty} \int_{|y|>f(n/k)} \psi_0(y)\, dy \leq k \sum_{n=1}^{\infty} \int_{|y|>f(n)} \psi_0(y)\, dy,$$

(40) implies (41). □

In many cases, estimates on the decay of $\psi_0$ can be obtained via $V$. In [6] it is shown that for $s \geq 0$ the estimate $\liminf_{|x|\to\infty} V(x)/|x|^{2s} > 0$ implies the existence of constants $A > 0, \beta > 0$ such that

$$\psi_0(y) \leq A \exp(-\beta |y|^{s+1})$$

for all $y \in \mathbb{R}^d$. In this case, Lemma 3.4 implies

$$\limsup_{|t|\to\infty} \frac{|x_t|}{(\gamma \ln |t|)^{s+1}} = 0$$

for each $\gamma > 1/\beta$ and $\mu$-almost all $x \in C(\mathbb{R}, \mathbb{R}^d)$. This result has been obtained (for $s > 1$) in [13] via the cluster expansion.

We conclude this paper by showing the infinite volume analogue of Lemma 2.1. We refer to Section 2 for notation and additionally introduce

$$\begin{aligned} \Lambda(S) &= (\mathbb{R} \times [-S, S]) \cup ([-S, S] \times \mathbb{R}), &(42)\\ d\mu^{S,\bar{x}} &= \frac{1}{Z^S(\bar{x})} \exp(\mathcal{H}_{\Lambda(S)}(x))\, d\mu_0^{S,\bar{x}}(x). &(43) \end{aligned}$$

Note that the normalizing constant $Z^S(\bar{x})$ is finite for each $\bar{x} \in C(\mathbb{R}, \mathbb{R}^d)$ due to (12).

**Proposition 3.5** *For each $S > 0$ and each infinite volume cluster point $\mu$ of $(\mu_T)$, $\bar{x} \mapsto \mu_T^{S,\bar{x}}$ is a version of the regular conditional probability $\mu(.|\mathcal{T}_S)$. In other words, $\mu$ is a Gibbs measure for the reference measure $\mu_0$ and the potential $W$.*

*Proof:* By Lemma 2.1, we have for $f, g \in L^\infty(C(\mathbb{R}, \mathbb{R}^d))$ with $\mathcal{T}_S$-measurable $g$ that

$$\int g(\bar{x}) E_{\mu_T^{S,\bar{x}}}(f)\, d\mu_T(\bar{x}) = E_{\mu_T}(fg). \tag{44}$$

We have to show that (44) remains true when we replace $\mu_T$ by $\mu$ and $\mu_T^{S,\bar{x}}$ by $\mu^{S,\bar{x}}$. By a monotone class argument, we may in assume that $f$ and $g$ are $\mathcal{F}_{[-R,R]}$-measurable for some $R > S$. Taking a sequence $(t_n)$ such that $\mu_{t_n}$ converges to $\mu$, we immediately see that the right hand side of (44) converges to $E_\mu(fg)$. As for the left hand side, (12) guarantees that $\mathcal{H}_{\Lambda(S,T)}(q)$ converges to $\mathcal{H}_{\Lambda(S)}(q)$ uniformly in $q \in C(\mathbb{R}, \mathbb{R}^d)$ as $T \to \infty$, and thus the left hand side converges to $\int g(\bar{x}) E_{\mu^{S,\bar{x}}}(f)\, d\mu(\bar{x})$. □

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