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SPATIAL RANDOM PERMUTATIONS AND POISSON-DIRICHLET LAW OF CYCLE LENGTHS

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Abstract. We study spatial permutations with cycle weights that are bounded or slowly diverging. We show that a phase transition occurs at an explicit critical density. The long cycles are macroscopic and their cycle lengths satisfy a Poisson-Dirichlet law.

Keywords: Spatial random permutations, cycle weights, Poisson-Dirichlet distribution.

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1. Introduction

The structure for spatial permutations consists of a large box $\Lambda \subset \mathbb{R}^d$, a large number N of points in Λ, and permutations of those points such that all permutation jumps remain small. The relevant parameter is the density $\rho = |\Lambda|/N$. In many models there is a critical density $\rho_{\rm c}$ that corresponds to a transition from a phase with only finite cycles (when $\rho < \rho_{\rm c}$) to a phase where a nonzero fraction of points belong to infinite cycles (when $\rho > \rho_{\rm c}$). The goal of the present article is twofold. First, we prove that such a transition occurs in a class of models of spatial random permutations with cycle weights. Second, we show that the cycle structure of infinite cycles satisfies a Poisson-Dirichlet law.

The main motivation for our models comes from the interacting Bose gas of quantum statistical mechanics. The possible relevance of long permutation cycles to Bose-Einstein condensation was pointed out by Matsubara [17] and Feynman [10]. Sütő made important clarifications for the ideal Bose gas, showing in particular that long cycles are macroscopic [19, 20]. In the recent article [5], we derived a simplified model of spatial permutations where the original interactions between quantum particles are replaced by cycle weights. Those calculations involved five conjectures. The last conjecture dealt with the validity of the formula for the critical density and it is proved in the present article.

Models of spatial permutations are also attractive *per se*. They have both specific and general features. One general feature that is especially striking is the Poisson-Dirichlet law for the distribution of cycle lengths. The literature on the subject is huge, see e.g. [1, 14, 12] for a sample. The Poisson-Dirichlet distribution is expected to make an appearance in other models with spatial structure and permutations such as the random stirring model [13, 21]. This was proved recently by Schramm on the complete graph [18]; see also Berestycki [2] for several useful observations and clarifications.

The models considered here are "annealed" in the sense that spatial positions vary and they are integrated upon. Annealed models are both simpler and more relevant for the Bose gas. But the "quenched " models, where the positions are fixed and chosen according to a suitable point process, look very interesting in probability theory. One conjectures that long cycles satisfy the same Poisson-Dirichlet law as in the annealed model — the only difference being the critical density. This is supported by numerical evidence [11, 15]. An unrelated but very interesting problem is the complete description of Gibbs states, involving crossing fluxes that depend on the boundary conditions. Such a description was recently achieved by Biskup and Richthammer in the one-dimensional model [7].

2. Setting & results

The state space of the (annealed) model of spatial permutations with cycle weights is $\Omega_{\Lambda,N} = \Lambda^N \times \mathcal{S}_N$, where $\Lambda \subset \mathbb{R}^d$ is a cubic box of size L, and $\mathcal{S}_N$ is the symmetric group of permutations of N elements. We denote by $|\Lambda| = L^d$ the volume of Λ. We equip $\Omega_{\Lambda,N}$ with the product of the Borel σ-algebra on Λ^N and the discrete σ-algebra on $\mathcal{S}_N$. We introduce a "Hamiltonian" and its corresponding Gibbs state. Namely, the Hamiltonian is a function $H : \Omega_{\Lambda,N} \to \mathbb{R}$ that we suppose of the form

$$H(\boldsymbol{x}, \pi) = \sum_{i=1}^N \xi(x_i - x_{\pi(i)}) + \sum_{\ell \geqslant 1} \alpha_\ell r_\ell(\pi). \tag{2.1}$$

Here, $\boldsymbol{x} = (x_1, \dots, x_N) \in \Lambda^N$ and $\pi \in \mathcal{S}_N$. We always suppose that $\mathrm{e}^{-\xi}$ is continuous and spherically symmetric, that it has positive Fourier transform, and that it is normalized, $\int_{\mathbb{R}^d} \mathrm{e}^{-\xi(x)} \, \mathrm{d}x = 1$. Notice that ξ is allowed to take the value $+\infty$. The cycle weights $\alpha_1, \alpha_2, \dots$ are fixed parameters. Finally, $r_\ell(\pi)$ denotes the number of ℓ-cycles in the permutation π.

Boundary conditions are not expected to play a prominent rôle here, and we therefore choose those that make proofs simpler. These are the "periodized" boundary conditions, where we replace ξ by the function ξ_Λ, defined by

$$\mathrm{e}^{-\xi_\Lambda(x)} = \sum_{z \in \mathbb{Z}^d} \mathrm{e}^{-\xi(x - Lz)}. \tag{2.2}$$

When $\mathrm{e}^{-\xi}$ has bounded support with diameter smaller than L we recover the usual periodic boundary conditions. We let H_Λ be as H in (2.1), but with ξ_Λ instead of ξ. The Gibbs state is given by the probability measure

$$\mathrm{Prob}(\mathrm{d}\boldsymbol{x}, \pi) = \frac{1}{N! Y} \mathrm{e}^{-H_\Lambda(\boldsymbol{x}, \pi)} \, \mathrm{d}\boldsymbol{x} \tag{2.3}$$

on $\Omega_{\Lambda,N}$, where $\mathrm{d}\boldsymbol{x}$ is the Lebesgue measure on Λ^N and Y is a suitable normalization, namely

$$Y = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_n} \int_{\Lambda^N} \mathrm{e}^{-H_\Lambda(\boldsymbol{x}, \pi)} \, \mathrm{d}\boldsymbol{x}. \tag{2.4}$$

In typical realizations of the system, points are spread all over the space because of the Lebesgue measure that prevents accumulations. The lengths of permutation jumps $\|x_i - x_{\pi(i)}\|$ stay bounded uniformly in Λ because of the jump weights $\mathrm{e}^{-\xi(x_i - x_{\pi(i)})}$. The lengths of permutation cycles depend on the density of the system. For small density, points are far apart and jumps are unlikely, which typically results in small cycles. But as the density increases, points have more and more possibilities to hop, and a phase transition takes place where "infinite" cycles appear. The cycle weights modify the critical density and also the distribution of cycle lengths, see below. The model is illustrated in Fig. 1.

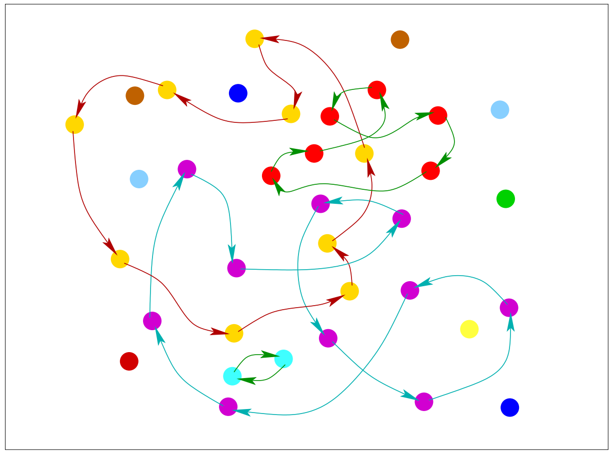

FIGURE 1. A typical realization of a spatial permutation. As $|\Lambda|, N \to \infty$, the jumps remain finite but the cycle length may diverge.

This model arises naturally from the Feynman-Kac representation of the dilute Bose gas. The jump function is then $\xi(x) = \frac{1}{4\beta}\|x\|^2$ (plus a normalization constant), with β the inverse temperature of the system. Notice that if the original quantum system has periodic boundary conditions, we get the periodized Gaussian function. Cycle weights were introduced in [3] as a crude way to account for the particle interactions. But the calculations of [5] suggest that the cycle weights can be chosen so that the model describes the Bose gas exactly in the dilute regime. We do not write here the precise formula for the weights, but we observe that they satisfy the asymptotic $\alpha_j = -c(1 - O(j^{-1/5}))$, so that α_j converges as $j \to \infty$ fast enough for our purpose.

We are solely interested in properties of permutations and we introduce random variables that are functions on $\mathcal{S}_N$ rather than $\Omega_{\Lambda,N}$. Let $\ell^{(1)}(\pi), \ell^{(2)}(\pi), \dots$ denote the cycle lengths in non-increasing order, repeated with multiplicities. We will prove that, above the critical density, the cycle lengths scale like N and they converge in distribution to Poisson-Dirichlet. The latter is conveniently defined using the *Griffiths-Engen-McCloskey distribution* GEM(θ), which is the distribution for

$$\big(X_1,\, (1 - X_1)X_2\,, (1 - X_1)(1 - X_2)X_3\,, \dots\big),$$

where $X_1, X_2, \dots$ are i.i.d. beta random variables with parameter $(1, \theta)$; that is, $\mathrm{Prob}(X_i > s) = (1-s)^\theta$ for $0 \leqslant s \leqslant 1$. The *Poisson-Dirichlet distribution* PD(θ) is the law obtained by rearranging those numbers in non-increasing order. See [1, 14] for more information and background. In the sequel, we say that a sequence of random variables $Y_n^{(1)}, Y_n^{(2)}, \dots$ converges in distribution to Poisson-Dirichlet as $n \to \infty$ if, for each fixed k, the joint distribution of $Y_n^{(1)}, \dots, Y_n^{(k)}$ converges weakly to the joint distribution of the first k random variables in Poisson-Dirichlet. This is denoted

$$(Y_n^{(1)}, Y_n^{(2)}, \dots) \Rightarrow \mathrm{PD}(\theta). \tag{2.5}$$

As already mentioned, we make the important assumption that the jump function has nonnegative Fourier transform. This allows to define the "dispersion relation" $\varepsilon(k)$, $k \in \mathbb{R}^d$, by the equation

$$\mathrm{e}^{-\varepsilon(k)} = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} kx} \, \mathrm{e}^{-\xi(x)} \, \mathrm{d}x. \tag{2.6}$$

Notice that $\varepsilon(k)$ is real and that $\varepsilon(0) = 0$ since $\int \mathrm{e}^{-\xi} = 1$. We suppose that $\varepsilon(k) \geqslant a\|k\|^\eta$ for small k, for some $a > 0$ and $\eta < d$. It is easy to see that $\varepsilon(k)$ is always greater than $a|k|^2$ so this assumption always holds in dimensions $d > 2$. Among possible jump functions other than Gaussians, let us mention $\mathrm{e}^{-\xi(x)} = \mathrm{const}\,(|x|+1)^{-\gamma}$ with $1 < \gamma < 2$ in $d = 1$, for which $\eta = \gamma - 1$. As for the cycle weights, we consider three cases:

(i) $\lim_{j\to\infty} \alpha_j = \alpha$ with $\alpha > 0$, and $\sum_j |\alpha_j - \alpha| < \infty$.
(ii) $\lim_{j\to\infty} \alpha_j = \alpha$ with $\alpha \leqslant 0$, and $\sum_j \frac{1}{j}|\alpha_j - \alpha| < \infty$.
(iii) $\alpha_j = \gamma \log j$ with $\gamma > 0$.

We now introduce the fraction ν of points in infinite cycles. It is obvious that finite systems can only host finite cycles, so the definition of ν must involve the thermodynamic limit. Given a finite number K, let ν_K denote the fraction of points in cycles of length larger than K. Precisely,

$$\nu_K = \liminf_{|\Lambda|, N\to\infty} E\Big(\frac{1}{N} \sum_{i:\ell^{(i)}>K} \ell^{(i)}\Big). \tag{2.7}$$

Here and in the sequel, the limit $|\Lambda|, N \to \infty$ means that both go to infinity while keeping the density $\rho = N/|\Lambda|$ fixed. This is the standard thermodynamic limit of statistical mechanics. We then define

$$\nu = \lim_{K\to\infty} \nu_K. \tag{2.8}$$

This limit exists since (ν_K) is decreasing and bounded. Let $\bar\nu_K$ denote the $\limsup$ of (2.7). We expect that $\bar\nu_K = \nu_K$ but we do not prove it. On the other hand, we will prove in Section 5 that $\bar\nu_K$ also converges to ν as $K \to \infty$.

Next we introduce the *critical density* by

$$\rho_{\mathrm{c}} = \sum_{j\geqslant 1} \mathrm{e}^{-\alpha_j} \int_{\mathbb{R}^d} \mathrm{e}^{-j\varepsilon(k)} \, \mathrm{d}k. \tag{2.9}$$

It follows from our assumptions that the critical density is finite. Indeed, the numbers $\mathrm{e}^{-\alpha_j}$ are bounded, so ρ_{c} is bounded by the integral of a geometric series, $\int \frac{1}{\mathrm{e}^{\varepsilon(k)}-1}$, which is finite.

We propose now two theorems that confirm that ρ_{c} is indeed the critical density of the model, at least in several interesting situations. The formula (2.9) is presumably valid beyond the cases treated in this article, but the precise extent of its validity is not clear. The first theorem states that macroscopic cycles occur precisely above the critical density, and that they obey the Poisson-Dirichlet law.

Theorem 2.1. *Assume that $\alpha_j \to \alpha$ as described above. Then*

(a) *the fraction of points in infinite cycles is given by*

$$\nu = \max\Big(0, 1 - \frac{\rho_{\mathrm{c}}}{\rho}\Big);$$

(b) *when $\nu > 0$, i.e. when $\rho > \rho_{\rm c}$, the cycle structure converges in distribution to Poisson-Dirichlet:*

$$\Big(\frac{\ell^{(1)}}{\nu N}, \frac{\ell^{(2)}}{\nu N}, \dots\Big) \Rightarrow \mathrm{PD}(\,\mathrm{e}^{-\alpha}\,).$$

Such a law was already observed in absence of spatial structure, and when the cycle weights are constant. This case is known as the Ewens distribution, see e.g. [9, 12, 1]. Results about weights that are asymptotically Ewens can be found in [16, 6]. Spatial permutations with small cycle weights, i.e. when the limit is $\alpha = 0$, were studied in [4].

The second theorem concerns cycle weights that diverge logarithmically — it is somehow the limit $\alpha \to \infty$ of Theorem 2.1. Cycle weights have a striking effect as a single giant cycle occurs above the critical density! This is in accordance with a similar observation for non-spatial permutations [6].

Theorem 2.2. *Assume that $\alpha_j = \gamma \log j$ with $\gamma > 0$. Then*

(a) *the fraction of points in infinite cycles is given by*

$$\nu = \max\Big(0, 1 - \frac{\rho_{\rm c}}{\rho}\Big);$$

(b) *when $\nu > 0$, i.e. when $\rho > \rho_{\rm c}$, there is a single giant cycle that contains almost all points in infinite cycles:*

$$\frac{\ell^{(1)}}{\nu N} \Rightarrow 1.$$

The rest of this article is devoted to the proof of the results above. We reformulate the problem in the Fourier space in Section 3, following Sütő [20]. The model involves a measure on occupation numbers of Fourier modes, and of random permutations of those numbers. In Section 4 we obtain information about occupation numbers using techniques of Buffet and Pulé [8], and using certain estimates of our recent joint work with Velenik [6]. Random permutations within each mode involve the cycle weights and are thus similar to those studied in [6]. Combining all those results allow us to prove Theorems 2.1 and 2.2 in Section 5.

3. Random permutations and Fourier modes

The goal of this section is to introduce an alternative model of random permutations that involves Fourier modes, and that has the same marginal distribution on cycle lengths. Let $\Lambda^* = \frac{1}{L}\mathbb{Z}^d$ be the space dual to Λ in the sense of Fourier theory.

3.1. **The marginal distribution of cycle lengths.** Recall that the cycle structure of a permutation $\pi \in \mathcal{S}_N$ is the sequence of cycle lengths $\boldsymbol{\ell} = (\ell^{(1)}, \ell^{(2)}, \dots, \ell^{(m)})$, with $\ell^{(i)} \geqslant \ell^{(i+1)}$ and $\ell^{(m)} \geqslant 1$; the number of cycles m depends on π, $1 \leqslant m \leqslant N$. Those numbers form a partition of $\{1, \dots, N\}$. Another way to write $\boldsymbol{\ell}$ is to introduce the occupation numbers $\boldsymbol{r} = (r_1, \dots, r_N)$, where $r_j = \#\{i : \ell^{(i)} = j\}$. We always have

$$\sum_{i=1}^m \ell^{(i)} = \sum_{j=1}^N j r_j = N. \tag{3.1}$$

One should not confuse the occupation numbers $\boldsymbol{r}$ with the occupation numbers $\boldsymbol{n} = (n_k)$ to be introduced later; they are not related in any direct way.

Proposition 3.1. *The marginal of the probability measure* (2.3) *on occupation numbers is*

$$\operatorname{Prob}(\boldsymbol{r}) = \frac{1}{Y} \prod_{j=1}^N \frac{1}{r_j!} \Big(\frac{\mathrm{e}^{-\alpha_j}}{j} \sum_{k \in \Lambda^*} \mathrm{e}^{-j\varepsilon(k)} \Big)^{r_j},$$

with Y the normalization of (2.4).

Proof. The marginal probability on permutations is

$$\begin{aligned} \operatorname{Prob}(\pi) &= \frac{1}{N!Y} \int_{\Lambda^N} \mathrm{e}^{-H_\Lambda(\boldsymbol{x},\pi)} \, \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{N!Y} \int_{\Lambda^N} \mathrm{e}^{-\sum_{i=1}^N \xi_\Lambda(x_i - x_{\pi(i)}) - \sum_{j \geqslant 1} \alpha_j r_j(\pi)} \, \mathrm{d}x_1 \dots \mathrm{d}x_N. \end{aligned} \tag{3.2}$$

We observe that integrals factorize according to permutation cycles. The contribution of a cycle of length j is (with $y_{j+1} \equiv y_1$)

$$\mathrm{e}^{-\alpha_j} \int_{\Lambda^j} \mathrm{e}^{-\sum_{i=1}^j \xi_\Lambda(y_i - y_{i+1})} \, \mathrm{d}y_1 \dots \mathrm{d}y_j = \mathrm{e}^{-\alpha_j} |\Lambda| \sum_{z \in \mathbb{Z}^d} \big(\mathrm{e}^{-\xi} \big)^{*j} (Lz). \tag{3.3}$$

The Fourier transform of $(\mathrm{e}^{-\xi})^{*j}$ is $\mathrm{e}^{-j\varepsilon(k)}$. The Poisson summation formula states that

$$\sum_{z \in \mathbb{Z}^d} f(Lz) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} \widehat{f}(k). \tag{3.4}$$

We then get

$$\operatorname{Prob}(\pi) = \frac{1}{N!Y} \prod_{j=1}^N \Big(\mathrm{e}^{-\alpha_j} \sum_{k \in \Lambda^*} \mathrm{e}^{-j\varepsilon(k)} \Big)^{r_j(\pi)}. \tag{3.5}$$

All permutations of a given equivalence class have the same probability, and there are

$$\frac{N!}{\prod_j j^{r_j} r_j!} \tag{3.6}$$

elements in the equivalence class defined by $\boldsymbol{r}$. We get the claim by multiplying the above probability by this number. ◻

3.2. **Decomposition of permutations according to Fourier modes.** We denote by $\boldsymbol{n} = (n_k)$ the occupation numbers indexed by $k \in \Lambda^*$, and by $\mathcal{N}_{\Lambda,N}$ the set of occupation numbers such that $\sum_{k \in \Lambda^*} n_k = N$. Next, we introduce permutations that are also indexed by Fourier modes, $\boldsymbol{\pi} = (\pi_k)$. Let $\mathcal{M}_{\Lambda,N}$ be the set of pairs $(\boldsymbol{n}, \boldsymbol{\pi})$ where $\boldsymbol{n} \in \mathcal{N}_{\Lambda,N}$ and $\boldsymbol{\pi} = (\pi_k)$ with $\pi_k \in \mathcal{S}_{n_k}$ for each $k \in \Lambda^*$. We introduce a probability measure on $\mathcal{N}_{\Lambda,N}$:

$$\operatorname{Prob}(\boldsymbol{n}) = \frac{1}{Y} \prod_{k \in \Lambda^*} \mathrm{e}^{-\varepsilon(k) n_k} h_{n_k} \tag{3.7}$$

with

$$h_n = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \mathrm{e}^{-\sum_{j \geqslant 1} \alpha_j r_j(\pi)}, \tag{3.8}$$

and $h_0 = 1$. We will check later that the normalization Y is the same as given in (2.4). Then we introduce the probability of a pair $(\boldsymbol{n}, \boldsymbol{\pi})$ by

$$\operatorname{Prob}(\boldsymbol{n}, \boldsymbol{\pi}) = \frac{1}{Y} \prod_{k \in \Lambda^*} \frac{1}{n_k!} \mathrm{e}^{-\varepsilon(k) n_k - \sum_{j \geqslant 1} \alpha_j r_j(\pi_k)}. \tag{3.9}$$

Notice that (3.7) is the marginal of (3.9) with respect to $\boldsymbol{\pi}$. The conditional probability $\operatorname{Prob}(\boldsymbol{\pi}|\boldsymbol{n})$, where $\pi_k \in \mathcal{S}_{n_k}$ for all k, is given by

$$\operatorname{Prob}(\boldsymbol{\pi}|\boldsymbol{n}) = \prod_{k\in\Lambda^*} \Bigl(\frac{1}{n_k! h_{n_k}} \, \mathrm{e}^{-\sum_{j \geqslant 1} \alpha_j r_j(\pi_k)} \Bigr). \tag{3.10}$$

That is, given $\boldsymbol{n}$, each π_k is independent and distributed as nonspatial random permutations with cycle weights (see Eq. (5.1) below). Given $\boldsymbol{\pi}$, let $r_j = \sum_k r_j(\pi_k)$.

Proposition 3.2. *The marginal of the probability measure* (3.9) *with respect to* $\boldsymbol{r}$ *is identical to the marginal of the probability measure* (2.3).

Proof. We check that the marginal of (3.9) gives the formula of Proposition 3.1. For this, let $\boldsymbol{r}$ be a collection of occupation numbers, and write $(r_{jk}) : \boldsymbol{r}$ for the set of all integers r_{jk} such that $\sum_k r_{jk} = r_j$ for all j. Then,

$$\begin{aligned} \operatorname{Prob}(\boldsymbol{r}) &= \frac{1}{Y} \sum_{(r_{jk}):\boldsymbol{r}} \sum_{\substack{(\boldsymbol{n},\boldsymbol{\pi}):\\ r_j(\pi_k)=r_{jk}}} \prod_{k\in\Lambda^*} \Bigl(\frac{1}{n_k!} \, \mathrm{e}^{-\varepsilon(k)n_k - \sum_j \alpha_j r_j(\pi_k)} \Bigr) \\ &= \frac{1}{Y} \sum_{(r_{jk}):\boldsymbol{r}} \prod_{k\in\Lambda^*} \Bigl(\frac{1}{\prod_j j^{r_{jk}} r_{jk}!} \, \mathrm{e}^{-\varepsilon(k) \sum_j j r_{jk} - \sum_j \alpha_j r_{jk}} \Bigr). \end{aligned} \tag{3.11}$$

We have summed over π_k that are compatible with r_{jk}, using the formula (3.6) for the number of elements. The bracket above factorizes according to j. Using

$$\prod_{k\in\Lambda^*} \frac{\mathrm{e}^{-\alpha_j r_{jk}}}{j^{r_{jk}}} = \Bigl(\frac{\mathrm{e}^{-\alpha_j}}{j} \Bigr)^{r_j}, \tag{3.12}$$

we get

$$\operatorname{Prob}(\boldsymbol{r}) = \frac{1}{Y} \prod_{j \geqslant 1} \Bigl[\Bigl(\frac{\mathrm{e}^{-\alpha_j}}{j} \Bigr)^{r_j} \sum_{(r_{jk}):r_j} \prod_{k\in\Lambda^*} \frac{1}{r_{jk}!} \, \mathrm{e}^{-j\varepsilon(k) r_{jk}} \Bigr]. \tag{3.13}$$

For each fixed j, the multinomial theorem gives

$$\sum_{(r_{jk}):r_j} \prod_{k\in\Lambda^*} \frac{\mathrm{e}^{-j\varepsilon(k) r_{jk}}}{r_{jk}!} = \frac{1}{r_j!} \Bigl(\sum_{k\in\Lambda^*} \mathrm{e}^{-j\varepsilon(k)} \Bigr)^{r_j}. \tag{3.14}$$

Then $\operatorname{Prob}(\boldsymbol{r})$ is indeed given by the formula of Proposition 3.1. This also proves that Y is the correct normalization that makes (3.7) and (3.9) probability measures. $\square$

4. Properties of occupation numbers

We study in this section the probability measure of occupation numbers of Fourier modes, $\operatorname{Prob}(\boldsymbol{n})$, that is defined in (3.7). We show that the typical $\boldsymbol{n}$ has the following properties:

- $\frac{n_0}{N} = \max(0, 1 - \frac{\rho_c}{\rho})$;
- $\frac{1}{N} \sum_{0<|k|<\delta} n_k$ is small when δ is small.
- For all $\delta > 0$, $\frac{1}{N} \sum_{|k| \geqslant \delta} n_k 1_{n_k > M}$ is small when M is large.

We recall two properties of the normalizations h_n defined in (3.8). First, their generating function is equal to

$$\sum_{n \geqslant 0} \mathrm{e}^{-\gamma n} \, h_n = \exp \sum_{j \geqslant 1} \tfrac{1}{j} \, \mathrm{e}^{-\gamma j - \alpha_j} \, ; \tag{4.1}$$

see [4, 16]. Second, the following estimates were obtained in [6]: There exist κ and $0 < c < C < \infty$ such that for any $n \geqslant 1$,

$$c\, n^{\kappa} \leqslant h_n \leqslant C\, n^{\kappa}. \tag{4.2}$$

The exponent turns out to be $\kappa = \mathrm{e}^{-\alpha} - 1$ when $\alpha_j \to \alpha$, and $\kappa = -1 - \gamma$ when $\alpha_j = \gamma \log j$; see [6]. The results of the present article actually extend to other cycle weights, as long as these bounds hold true.

4.1. **Macroscopic occupation of the zero mode.** We use a strategy that is inspired by Buffet and Pulé in their study of the ideal Bose gas [8]. It consists in looking at the Laplace (or Fourier) transform of the distribution of $\frac{n_0}{N}$. Let $Y(N)$ be the normalization of Eq. (2.4). We now put the explicit dependence on N because it is going to vary. Notice that $Y(N)$ also depends on Λ, but the domain is fixed throughout.

We have

$$\mathrm{Prob}(n_0 = j) = \frac{h_j}{Y(N)} \sum_{\substack{\boldsymbol{n} \in \mathcal{N}_{\Lambda,N} \\ n_0 = j}} \prod_{k \neq 0} \mathrm{e}^{-\varepsilon(k) n_k}\, h_{n_k} = h_j \frac{\check{Y}(N-j)}{Y(N)}, \tag{4.3}$$

with

$$\check{Y}(N) = \sum_{\substack{\boldsymbol{n} \in \mathcal{N}_{\Lambda,N} \\ n_0 = 0}} \prod_{k \in \Lambda^*} \mathrm{e}^{-\varepsilon(k) n_k}\, h_{n_k}. \tag{4.4}$$

Notice the relation

$$Y(N) = \sum_{j=0}^{N} h_j \check{Y}(N-j). \tag{4.5}$$

We introduce the following probability measure on $[0, \infty)$:

$$\mu_\Lambda = \frac{1}{\check{Z}} \sum_{N \geqslant 0} \check{Y}(N) \delta_{N/|\Lambda|}. \tag{4.6}$$

The motivation for μ_Λ is that the distribution of the occupation of the zero mode can be expressed as

$$\begin{aligned} \mathrm{Prob}(\tfrac{n_0}{N} \geqslant a) &= \sum_{j=\lceil aN \rceil}^{N} h_j \frac{\check{Y}(N-j)}{Y(N)} = \sum_{j=0}^{\lfloor (1-a)N \rfloor} h_{N-j} \frac{\check{Y}(j)}{Y(N)} \\ &= \frac{\int_0^{(1-a)\rho} h(|\Lambda|(\rho - s))\, \mathrm{d}\mu_\Lambda(s)}{\int_0^{\rho} h(|\Lambda|(\rho - s))\, \mathrm{d}\mu_\Lambda(s)}. \end{aligned} \tag{4.7}$$

We can write a useful expression for the normalization $\check{Z}$. Let $\mathcal{N}_\Lambda = \cup_{N \geqslant 0} \mathcal{N}_{\Lambda,N}$ be the set of unconstrained occupation numbers on Λ^*. Then

$$\begin{aligned} \check{Z} = \sum_{N \geqslant 0} \check{Y}(N) &= \sum_{\substack{\boldsymbol{n} \in \mathcal{N}_\Lambda \\ n_0 = 0}} \prod_{k \in \Lambda^*} \mathrm{e}^{-\varepsilon(k) n_k}\, h_{n_k} \\ &= \prod_{k \neq 0} \Bigl(\sum_{n \geqslant 0} \mathrm{e}^{-\varepsilon(k) n}\, h_n \Bigr) = \exp\Bigl(\sum_{j \geqslant 1} \tfrac{1}{j}\, \mathrm{e}^{-\alpha_j} \sum_{k \neq 0} \mathrm{e}^{-j\varepsilon(k)} \Bigr). \end{aligned} \tag{4.8}$$

The last identity follows from (4.1). Finally, we introduce a Riemann approximation to the critical density (2.9), namely

$$\rho_{\rm c}^{(\Lambda)} = \sum_{j \geqslant 1} \mathrm{e}^{-\alpha_j} \frac{1}{|\Lambda|} \sum_{k \neq 0} \mathrm{e}^{-j\varepsilon(k)} . \tag{4.9}$$

We now have all the elements that allow to state and to prove the key properties leading to the macroscopic occupation of the zero Fourier mode.

Proposition 4.1.

(a) $\mu_\Lambda \to \delta_{\rho_{\rm c}}$ *weakly as* $|\Lambda| \to \infty$.

(b) *If* $|\lambda(\Lambda)| \leqslant |\Lambda|^{\frac{1-\eta/d}{2}}$, *then*

$$E_{\mu_\Lambda}\big(\mathrm{e}^{\lambda(\Lambda)(s-\rho_{\rm c}^{(\Lambda)})} \big) \to 1.$$

The parameter η in the claim (b) is the one that appears in the condition for $\varepsilon(k)$, see the paragraph after Eq. (2.6). The relevant aspect of the claim (b) is that the expectation is bounded uniformly in the domain even though $\lambda(\Lambda)$ diverges. A consequence is the following concentration property for $|\Lambda|$ large enough, which will be used later:

$$\begin{aligned} E_{\mu_\Lambda}(1_{|s-\rho_{\rm c}|>\epsilon}) &\leqslant \mathrm{e}^{-\frac12 \epsilon |\Lambda|^{\frac{1-\eta/d}{2}}} E_{\mu_\Lambda}\big(\mathrm{e}^{|\Lambda|^{\frac{1-\eta/d}{2}}(s-\rho_{\rm c}^{(\Lambda)})} + \mathrm{e}^{-|\Lambda|^{\frac{1-\eta/d}{2}}(s-\rho_{\rm c}^{(\Lambda)})} \big) \\ &\leqslant 3\,\mathrm{e}^{-\frac12 \epsilon |\Lambda|^{\frac{1-\eta/d}{2}}} . \end{aligned} \tag{4.10}$$

Proof of Proposition 4.1. For (a) we show that the characteristic function of μ_Λ converges to that of the Dirac measure $\delta_{\rho_{\rm c}}$. First, by the same calculations as in (4.8) for $\check Z$, we have

$$\begin{aligned} E_{\mu_\Lambda}(\mathrm{e}^{-\mathrm{i}\lambda s}) &= \int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda s}\, \mathrm{d}\mu_\Lambda(s) = \frac{1}{\check Z} \sum_{N \geqslant 0} \check Y(N)\, \mathrm{e}^{-\mathrm{i}\lambda N/|\Lambda|} \\ &= \frac{1}{\check Z} \sum_{\substack{\boldsymbol{n} \in \mathcal{N}_\Lambda \\ n_0 = 0}} \prod_{k \neq 0} \mathrm{e}^{-(\varepsilon(k) + \mathrm{i}\lambda/|\Lambda|) n_k} h_{n_k} \\ &= \frac{1}{\check Z} \exp\Big(\sum_{j \geqslant 1} \tfrac{1}{j} \mathrm{e}^{-\alpha_j} \sum_{k \neq 0} \mathrm{e}^{-j(\varepsilon(k) + \mathrm{i}\lambda/|\Lambda|)} \Big) \\ &= \exp\Big(-\mathrm{i}\lambda \sum_{j \geqslant 1} \mathrm{e}^{-\alpha_j} \frac{1}{|\Lambda|} \sum_{k \neq 0} \mathrm{e}^{-j\varepsilon(k)} \frac{|\Lambda|}{\mathrm{i}\lambda j} \big(1 - \mathrm{e}^{-\mathrm{i}\lambda j/|\Lambda|} \big) \Big). \end{aligned} \tag{4.11}$$

For all $\lambda \neq 0$ we have

$$\Big| \frac{|\Lambda|}{\mathrm{i}\lambda j} \big(1 - \mathrm{e}^{-\mathrm{i}\lambda j/|\Lambda|} \big) \Big| \leqslant 1, \qquad \lim_{|\Lambda| \to \infty} \frac{|\Lambda|}{\mathrm{i}\lambda j} \big(1 - \mathrm{e}^{-\mathrm{i}\lambda j/|\Lambda|} \big) = 1. \tag{4.12}$$

One can use dominated convergence to show that (4.11) converges to $\mathrm{e}^{-\mathrm{i}\lambda\rho_{\rm c}}$ as $|\Lambda| \to \infty$. Indeed, one can use the lower bound $\varepsilon(k) \geqslant a\|k\|^\eta$ for small k; the Riemann sum is then essentially bounded by the integral, yielding a term of the form $\frac{\rm const}{j^{d/\eta}}$ which is summable for $\eta < d$. For large k one can use $\mathrm{e}^{-j\varepsilon(k)} \leqslant \mathrm{e}^{-cj}\, \mathrm{e}^{-c\varepsilon(k)}$ with a constant $c > 0$ that is uniform in j, k.

For (b) we can repeat the calculations above so as to get

$$E_{\mu_\Lambda}\big(\,\mathrm{e}^{\lambda(\Lambda)(s-\rho_{\mathrm{c}}^{(\Lambda)})}\,\big) = \exp\Big(\sum_{j\,\geqslant\,1} \mathrm{e}^{-\alpha_j}\,\frac{1}{|\Lambda|}\sum_{k\neq0}\mathrm{e}^{-j\varepsilon(k)}\,\frac{|\Lambda|}{j}\big(\,\mathrm{e}^{j\lambda(\Lambda)/|\Lambda|}\,-1-j\tfrac{\lambda(\Lambda)}{|\Lambda|}\big)\Big). \tag{4.13}$$

We now use $\frac{1}{|x|}|\,\mathrm{e}^{x}\,-1-x| \leqslant |x|\,\mathrm{e}^{|x|}$ (which is easy to check using Taylor series) with $x = j\lambda(\Lambda)/|\Lambda|$. Observe also that

$$\mathrm{e}^{-\frac12 j\varepsilon(k)}\,j\frac{\lambda^2(\Lambda)}{|\Lambda|}\,\mathrm{e}^{j|\lambda(\Lambda)|/|\Lambda|}\,\leqslant 1 \tag{4.14}$$

for all j, all Λ large enough, and all $k\neq0$; this holds because $\varepsilon(k)\geqslant a\|k\|^\eta$ and because of the upper bound on $\lambda(\Lambda)$. We can then check that the term in the exponential in (4.13) is less than

$$\sum_{j\,\geqslant\,1}\mathrm{e}^{-\alpha_j}\,\frac{1}{|\Lambda|}\sum_{k\neq0}\mathrm{e}^{-\frac12 j\varepsilon(k)}\,.$$

Again summing separately over small and large k, as in the proof of (a), we can use dominated convergence and take the limit $|\Lambda|\to\infty$ under the sum over j. The term in the exponential in (4.13) converges then to zero. $\square$

Proposition 4.2. *Suppose that $\rho>\rho_{\mathrm{c}}$. Then, in the thermodynamic limit $|\Lambda|, N\to\infty$,*

$$\mathrm{Prob}(\tfrac{n_0}{N}>a)\to\begin{cases}1 & \textit{if } a<1-\frac{\rho_{\mathrm{c}}}{\rho},\\ 0 & \textit{if } a>1-\frac{\rho_{\mathrm{c}}}{\rho}.\end{cases}$$

Proof. We use the expression (4.6) which involves the measure μ_Λ. It can be written as

$$\mathrm{Prob}(\tfrac{n_0}{N}>a) = \frac{J_1(a)}{J_1(a)+J_2(a)} = \frac{1}{1+J_2(a)/J_1(a)}, \tag{4.15}$$

with

$$\begin{aligned} J_1(a) &= \frac{1}{|\Lambda|^\kappa}\int_0^{(1-a)\rho} h(|\Lambda|(\rho-s))\mathrm{d}\mu_\Lambda(s),\\ J_2(a) &= \frac{1}{|\Lambda|^\kappa}\int_{(1-a)\rho}^{\rho} h(|\Lambda|(\rho-s))\mathrm{d}\mu_\Lambda(s). \end{aligned} \tag{4.16}$$

Above, κ is the constant of (4.2). Using the latter estimates, we get

$$c\int_0^{(1-a)\rho}(\rho-s)^\kappa\mathrm{d}\mu_\Lambda(s)\leqslant J_1(a)\leqslant C\int_0^{(1-a)\rho}(\rho-s)^\kappa\mathrm{d}\mu_\Lambda(s), \tag{4.17}$$

and by Proposition 4.1 (a) we find that $\lim_{|\Lambda|\to\infty}J_1(a)=0$ if $(1-a)\rho<\rho_{\mathrm{c}}$, while $\liminf_{|\Lambda|\to\infty}J_1(a)\geqslant c(\rho-\rho_{\mathrm{c}})$ if $(1-a)\rho>\rho_{\mathrm{c}}$.

For $J_2(a)$, we write

$$J_2(a) = J_{2,0}(a)+J_\epsilon(a), \tag{4.18}$$

with

$$J_\epsilon(a) = \frac{1}{|\Lambda|^\kappa}\int_{\rho-\epsilon}^{\rho} h(|\Lambda|(\rho-s))\mathrm{d}\mu_\Lambda(s) \tag{4.19}$$

and $0<\epsilon<\min(\frac{\rho-\rho_{\mathrm{c}}}{2},a\rho)$. Then in the same way as above, $\lim_{|\Lambda|\to\infty}J_{2,0}(a)=0$ if $(1-a)\rho>\rho_{\mathrm{c}}$, and $\liminf_{|\Lambda|\to\infty}J_{2,0}(a)\geqslant c(\rho-\rho_{\mathrm{c}})$ if $(1-a)\rho<\rho_{\mathrm{c}}$. We now want to show that $\lim_{|\Lambda|\to\infty}J_\epsilon(a)=0$. In the case $\kappa>0$, this is done in the same way as with J_1 and $J_{2,0}$ above. But for $\kappa<0$, the function $s\mapsto(s-\rho)^\kappa$ is not bounded when $s \to \rho$, and thus Proposition 4.1 (a) is not strong enough. However, we can check that

$$J_\epsilon(a) \leqslant C|\Lambda|^{|\kappa|} E_{\mu_\Lambda}(1_{s>\rho_c+\epsilon}), \tag{4.20}$$

which goes to zero as $|\Lambda| \to \infty$ by (4.10). Inserting these findings into (4.15) proves the claim. □

4.2. **No macroscopic occupation below the critical density.** For $\rho < \rho_c$, the support of the Dirac measure to which the measures μ_Λ converge lies outside of the interval $[0, \rho]$, and the above argument fails. In its place, we use the formula

$$\mathrm{Prob}(n_0 \geqslant j) = \frac{1}{Y(N)} \sum_{\substack{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N} \\ n_0 \geqslant j}} \prod_{k\in\Lambda^*} \mathrm{e}^{-n_k \varepsilon(k)}\, h_{n_k} = \frac{Y(N-j,j)}{Y(N)}, \tag{4.21}$$

where

$$Y(m,j) = \sum_{\boldsymbol{n}\in\mathcal{N}_{\Lambda,m}} \frac{h_{n_0+j}}{h_{n_0}} \prod_{k\in\Lambda^*} \mathrm{e}^{-n_k\varepsilon(k)}\, h_{n_k}. \tag{4.22}$$

We apply summation by parts $E(f(X)) = f(0) + \sum_{j=1}^N \big[f(j) - f(j-1)\big] P(x \geqslant j)$ to the function $\mathrm{e}^{-\lambda n_0/N}$, and we find

$$\begin{aligned} E(\,\mathrm{e}^{-\lambda n_0/N}\,) &= 1 + \frac{(1-\,\mathrm{e}^{\lambda/N}\,)}{Y(N)} \sum_{j=0}^N \mathrm{e}^{-\lambda j/N}\, Y(N-j,j) \\ &= 1 + \frac{\mathrm{e}^{-\rho}\,(1-\,\mathrm{e}^{\lambda/N}\,)}{Y(N)} \sum_{j=0}^N \mathrm{e}^{\lambda j/N}\, Y(j, N-j). \end{aligned} \tag{4.23}$$

Proposition 4.3. *Suppose that $\rho \leqslant \rho_c$. In the thermodynamic limit $|\Lambda|$, $N \to \infty$,*

$$\mathrm{Prob}(\tfrac{n_0}{N} > \delta) \to 0.$$

for all $\delta > 0$.

Proof. By (4.23),

$$|E(\,\mathrm{e}^{\frac{\lambda n_0}{N}}\,) - 1| \leqslant \frac{\mathrm{const}}{N} \left(\sum_{j=0}^{(1-\epsilon)N} \frac{Y(j,N-j)}{Y(N)} + \sum_{j=(1-\epsilon)N}^{N} \frac{Y(j,N-j)}{Y(N)} \right). \tag{4.24}$$

From (4.21) it is obvious that $\frac{Y(j,N-j)}{Y(N)} \leqslant 1$, as it is a probability. Thus the second term above, along with the prefactor $1/N$, is bounded by const$\rho\epsilon$. For the first term, we use the inequality

$$\sup\{h_{r+j}/h_r : 0 \leqslant r \leqslant N\} \leqslant \frac{C}{c}(1+j)^{|\kappa|} \tag{4.25}$$

which follows from (4.2). Then, $Y(N,j) \leqslant Y(N)\frac{C}{c}(1+j)^{|\kappa|}$, and

$$\sum_{j=0}^{\lfloor(1-\epsilon)N\rfloor} \frac{Y(j,N-j)}{Y(N)} \leqslant \frac{C}{c} \sum_{j=0}^{\lfloor(1-\epsilon)N\rfloor} \frac{Y(j)}{Y(N)} (N-j+1)^{|\kappa|} \tag{4.26}$$

Now

$$\frac{Y(j)}{Y(N)} = \mathrm{e}^{-|\Lambda|(q_\Lambda(j/|\Lambda|) - q_\Lambda(\rho))}\,, \tag{4.27}$$

where $q_\Lambda(\rho) = -\frac{1}{|\Lambda|}\log Y(|\Lambda|\rho)$ is the finite volume free energy associated with the partition function Y. It was shown in [4], under conditions on the coefficients α_j that are more general than the present ones, that q_Λ converges uniformly on compact intervals to a convex function q, and that $\rho \mapsto q(\rho)$ is strictly decreasing for $\rho < \rho_{\rm c}$. Thus for each $\epsilon > 0$ there is $b_\epsilon > 0$ such that $q_\Lambda(j/|\Lambda|) - q_\Lambda(\rho) > b_\epsilon$ for all $|\Lambda|$ large enough, and all $j \leqslant (1-\epsilon)N$. So

$$\sum_{j=0}^{\lfloor(1-\epsilon)N\rfloor} \frac{Y(j,N-j)}{Y(N)} \leqslant \tfrac{C}{c}\,{\rm e}^{-b_\epsilon|\Lambda|} \sum_{j=0}^{\lfloor(1-\epsilon)N\rfloor} (N-j+1)^{|\kappa|} \leqslant {\rm e}^{-b_\epsilon N/\rho}\, N^{|\kappa|+1}, \tag{4.28}$$

which converges to zero as $N \to \infty$. Since ϵ was arbitrary, we have shown $E(\,{\rm e}^{\lambda n_0/N}\,) \to 1$ for all $\lambda > 0$, which implies the claim. ☐

4.3. **Occupation of nonzero modes.** We now turn to the modes $k \neq 0$. The basic estimate is

Lemma 4.4. *Let c, C be the constants from* (4.2). *Then for $0 < \sigma < 1$, $k \in \Lambda^*$ and $j \geqslant 0$ we have*

$$\mathrm{Prob}(n_k \geqslant j) \leqslant \frac{C^2}{c^2}\sigma^{-2|\kappa|}\,{\rm e}^{-j(1-\sigma)\varepsilon(k)}\,.$$

Proof. We begin by observing that due to (4.2),

$$\sup_{n,j:\, n\,\geqslant\,\sigma j} \Bigl\{\frac{h_{n+(1-\sigma)j}}{h_n}\Bigr\} \leqslant \frac{C}{c}\sup_{n\,\geqslant\,\sigma j}\Bigl(1+\tfrac{(1-\sigma)j}{n}\Bigr)^{\kappa} \leqslant \frac{C}{c}\Bigl(1+\tfrac{1-\sigma}{\sigma}\Bigr)^{|\kappa|} = \frac{C}{c}\sigma^{-|\kappa|}. \tag{4.29}$$

and

$$\sup_{n,j:\, n\,\geqslant\, j} \Bigl\{\frac{h_{n-(1-\sigma)j}}{h_n}\Bigr\} \leqslant \frac{C}{c}\sup_{n\,\geqslant\, j}\Bigl(1-\frac{(1-\sigma)j}{n}\Bigr)^{\kappa} \leqslant \frac{C}{c}\sigma^{-|\kappa|}. \tag{4.30}$$

Using the shorthand $C_0 = \frac{C}{c}\sigma^{-|\kappa|}$, we get

$$\begin{aligned}
\mathrm{Prob}(n_k \geqslant j) &= \frac{1}{Y(N)} \sum_{\substack{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N}\\ n_k\,\geqslant\, j}} \prod_{k'\in\Lambda^*} {\rm e}^{-n_{k'}\varepsilon(k')}\, h_{n_{k'}} \\
&= \frac{1}{Y(N)} \sum_{\substack{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N-(1-\sigma)j}\\ n_k\,\geqslant\,\sigma j}} {\rm e}^{-(1-\sigma)j\varepsilon(k)}\,\frac{h_{n_k+(1-\sigma)j}}{h_{n_k}} \prod_{k'\in\Lambda^*} {\rm e}^{-n_{k'}\varepsilon(k')}\, h_{n_{k'}} \\
&\leqslant \frac{C_0}{Y(N)}\,{\rm e}^{-(1-\sigma)j\varepsilon(k)} \sum_{\substack{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N-(1-\sigma)j}\\ n_k\,\geqslant\,\sigma j}} \prod_{k'\in\Lambda^*} {\rm e}^{-n_{k'}\varepsilon(k')}\, h_{n_{k'}}\,.
\end{aligned} \tag{4.31}$$

We get an upper bound by replacing the constraint $n_k \geqslant \sigma j$ by $n_0 \geqslant \sigma j$. Then

$$\begin{aligned}
\operatorname{Prob}(n_k \geqslant j) &\leqslant \frac{C_0}{Y(N)}\, \mathrm{e}^{-(1-\sigma)j\varepsilon(k)} \sum_{\substack{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N-(1-\sigma)j}\\ n_0 \geqslant \sigma j}} \prod_{k'\in\Lambda^*} \mathrm{e}^{-n_{k'}\varepsilon(k')}\, h_{n_{k'}} \\
&= \frac{C_0}{Y(N)}\, \mathrm{e}^{-(1-\sigma)j\varepsilon(k)} \sum_{\substack{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N}\\ n_0 \geqslant j}} \frac{h_{n_0-(1-\sigma)j}}{h_{n_0}} \prod_{k'\in\Lambda^*} \mathrm{e}^{-n_{k'}\varepsilon(k')}\, h_{n_{k'}} \\
&\leqslant \frac{C_0^2}{Y(N)}\, \mathrm{e}^{-(1-\sigma)j\varepsilon(k)} \sum_{\substack{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N}\\ n_0 \geqslant j}} \prod_{k'\in\Lambda^*} \mathrm{e}^{-n_{k'}\varepsilon(k')}\, h_{n_{k'}} \\
&\leqslant C_0^2\, \mathrm{e}^{-(1-\sigma)j\varepsilon(k)}\,.
\end{aligned} \tag{4.32}$$

□

We now define three sets of occupation numbers, each of which will be shown to have measure close to one. Let $\tilde\nu = \max(0, 1-\frac{\rho_{\rm c}}{\rho})$; we will prove in the next section that $\tilde\nu = \nu$, but we do not know this yet. The sets are

$$\begin{aligned}
A_\epsilon &= \Big\{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N} : \big|\tfrac{n_0}{N} - \tilde\nu\big| < \epsilon\Big\} \\
B_{\epsilon,\delta} &= \Big\{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N} : \sum_{0<|k|<\delta} n_k < \epsilon N\Big\} \\
C_{\epsilon,\delta,M} &= \Big\{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N} : \sum_{\substack{k\in\Lambda^*, |k| \geqslant \delta\\ n_k \geqslant M}} n_k < \epsilon N\Big\}.
\end{aligned} \tag{4.33}$$

Proposition 4.5. *For any $\rho > 0$, we have in the thermodynamic limit $N, |\Lambda| \to \infty$:*

(a) *For any $\epsilon > 0$, $\operatorname{Prob}(A_\epsilon) \to 1$.*
(b) *For any $\epsilon > 0$, there exists $\delta > 0$ such that $\liminf \operatorname{Prob}(B_{\epsilon,\delta}) > 1-\epsilon$.*
(c) *For any $\epsilon, \delta > 0$ there exists $M > 0$ such that $\liminf \operatorname{Prob}(C_{\epsilon,\delta,M}) > 1-\epsilon$.*

Proof. The claim (a) immediately follows from Propositions 4.2 and 4.3. For (b), we use Lemma 4.4 with $\sigma = 1/2$ to get

$$E(n_k) = \sum_{i \geqslant 1} \operatorname{Prob}(n_k \geqslant i) \leqslant C_0 \sum_{i \geqslant 1} \mathrm{e}^{-\varepsilon(k)i/2} = \frac{1}{\mathrm{e}^{\varepsilon(k)/2} - 1}. \tag{4.34}$$

For every $\delta > 0$ we get, by Markov's inequality,

$$\operatorname{Prob}(B^{\rm c}_{\epsilon,\delta}) \leqslant \frac{C_0}{\epsilon N} \sum_{0<|k|<\delta} \frac{1}{\mathrm{e}^{\varepsilon(k)/2} - 1} \overset{N\to\infty}{\longrightarrow} \frac{C_0}{\epsilon\rho} \int_{|k|<\delta} \frac{\mathrm{d}k}{\mathrm{e}^{\varepsilon(k)/2} - 1}. \tag{4.35}$$

By the assumption $\varepsilon(k) \geqslant a\|k\|^\eta$ with $\eta < d$, the integral is finite, and thus δ can be chosen so small that $\liminf \operatorname{Prob}(B^{\rm c}_{\epsilon,\delta}) < \epsilon$.

For (c), we define $F(\boldsymbol{n}) = \sum_{k\in\Lambda^*, |k| \geqslant \delta} n_k 1_{n_k \geqslant M}$, and we note that $\operatorname{Prob}(C^{\rm c}_{\varepsilon,\delta,M}) = \operatorname{Prob}(F \geqslant \varepsilon N)$. Now

$$\begin{aligned}
E(F/N) &= \frac{1}{N} \sum_{k\in\Lambda^*, |k| \geqslant \delta} E(n_k 1_{n_k \geqslant M}) \\
&= \frac{1}{N} \sum_{k\in\Lambda^*, |k| \geqslant \delta} \Big(M \operatorname{Prob}(n_k \geqslant M) + \sum_{j>M} \operatorname{Prob}(n_k \geqslant j) \Big),
\end{aligned} \tag{4.36}$$

where the last equality is summation by parts. By Lemma 4.4,

$$\sum_{j>M} \operatorname{Prob}(n_k \geqslant j) \leqslant C_0 \sum_{j=M+1}^{\infty} \mathrm{e}^{-j\varepsilon(k)/2} = C_0 \,\mathrm{e}^{-M\varepsilon(k)/2} \frac{1}{\mathrm{e}^{\varepsilon(k)/2} - 1}. \tag{4.37}$$

Define $c(\delta) = \inf_{|k| \geqslant \delta} \varepsilon(k)$. Note that $c(\delta) > 0$ for all $\delta > 0$. Then,

$$E(F/N) \leqslant C_0 M \,\mathrm{e}^{-Mc(\delta)/4} \frac{1}{N} \sum_{k\in\Lambda^*, |k| \geqslant \delta} \mathrm{e}^{-M\varepsilon(k)/4} \Big(1 + \frac{1}{M(\mathrm{e}^{\varepsilon(k)/2} - 1)}\Big). \tag{4.38}$$

The sum above, along with the factor $1/N$, converges to a Riemann integral which is finite thanks to our conditions on $\varepsilon(k)$. Therefore $\limsup E(F/N) \leqslant C_1 M \,\mathrm{e}^{-Mc(\delta)/4}$, and Markov's inequality implies that $\limsup \operatorname{Prob}(C^{\mathrm{c}}_{\epsilon,\delta,M}) \leqslant \frac{M}{\epsilon} \,\mathrm{e}^{-Mc(\delta)/4}$. Choosing M large enough for given ϵ, δ proves the claim. □

5. Cycle lengths of spatial permutations

We now prove the claims of Theorems 2.1 and 2.2, starting with the fraction ν of points in infinite cycles. We denote by $\operatorname{Prob}_n(\pi)$ the probability of a permutation $\pi \in \mathcal{S}_n$ in the nonspatial model with cycle weights. That is,

$$\operatorname{Prob}_n(\pi) = \frac{1}{h_n n!} \prod_{j \geqslant 1} \mathrm{e}^{-\alpha_j r_j(\pi)} \tag{5.1}$$

with h_n the normalization defined in (3.8). We also write E_n for the corresponding expectation. We keep the notation Prob, E for probability and expectation with respect to the spatial model.

Recall that we defined $\tilde\nu = \max(0, 1 - \frac{\rho_{\mathrm{c}}}{\rho})$.

Proposition 5.1. *Under the assumptions of Theorems 2.1 or 2.2, we have $\nu = \tilde\nu$.*

Proof. We use the Fourier modes decomposition of Section 3. Recall that $\boldsymbol{\pi} = (\pi_k)$, $r_{jk} = r_j(\pi_k)$, and $r_j = \sum_k r_{jk}$. We have

$$\begin{aligned} E\Big(\frac{1}{N} \sum_{i:\ell^{(i)}>K} \ell^{(i)}\Big) &= E\Big(\frac{1}{N} \sum_{j>K} j r_j\Big) \\ &= E\Big(\frac{1}{N} \sum_{j>K} j r_{j0}\Big) + E\Big(\frac{1}{N} \sum_{0<|k|<\delta} \sum_{j>K} j r_{jk}\Big) + E\Big(\frac{1}{N} \sum_{|k| \geqslant \delta} \sum_{j>K} j r_{jk}\Big). \end{aligned} \tag{5.2}$$

The first term of the right-hand side is equal to

$$E\Big(\frac{1}{N} \sum_{j>K} j r_{j0}\Big) = \sum_{n \geqslant 0} \frac{n}{N} \operatorname{Prob}(n_0 = n) E_{n_0}\Big(\frac{1}{n} \sum_{j>K} j r_j\Big). \tag{5.3}$$

It follows from Proposition 4.5 (a) that $\frac{n_0}{N} \to \tilde\nu$ as $|\Lambda|, N \to \infty$. In addition, we have

$$E_n\Big(\frac{1}{n} \sum_{j>K} j r_j\Big) = \operatorname{Prob}(\ell_1 > K), \tag{5.4}$$

where ℓ_1 is the length of the cycle that contains the index 1. It was shown in [16, 6] that the latter converges to 1 as $n \to \infty$. We have thus proved that, for any finite K,

$$\lim_{|\Lambda|, N\to\infty} E\Big(\frac{1}{N} \sum_{j>K} j r_{j0}\Big) = \tilde\nu. \tag{5.5}$$

The second term in the right-hand side of (5.2) is less than $E(\frac{1}{N}\sum_{0<|k|<\delta} n_k)$ and this is as small as we want by choosing δ small, see Proposition 4.5 (b). The last term is less than

$$E\Big(\frac{1}{N}\sum_{|k|\,\geqslant\,\delta} n_k 1_{n_k>K}\Big).$$

For any $\delta>0$, this can be made small by choosing K large, see Proposition 4.5 (c). This shows that both ν_K and $\bar\nu_K$ converge to $\tilde\nu$ as $K\to\infty$. □

The next step is to relate the distribution of long cycles of the spatial model with that of nonspatial random permutations. Let

$$A=[a_1,b_1]\times\cdots\times[a_m,b_m]\subset(0,1)^m. \tag{5.6}$$

Proposition 5.2. *If $\nu>0$, we have for any $m\geqslant 1$,*

$$\lim_{|\Lambda|,N\to\infty}\mathrm{Prob}\Big(\Big(\frac{\ell^{(1)}}{\nu N},\dots,\frac{\ell^{(m)}}{\nu N}\Big)\in A\Big)=\lim_{n\to\infty}\mathrm{Prob}_n\Big(\Big(\frac{\ell^{(1)}}{n},\dots,\frac{\ell^{(m)}}{n}\Big)\in A\Big).$$

Proof. We clearly have

$$\mathrm{Prob}\Big(\sup_{k\neq0}\frac{\ell_k^{(1)}}{N}>\epsilon\Big)\leqslant\mathrm{Prob}\Big(\sup_{k\neq0}\frac{n_k}{N}>\epsilon\Big). \tag{5.7}$$

It follows from Proposition 4.5 (b) and (c) that the right-hand side vanishes in the limit $|\Lambda|,N\to\infty$. The zero Fourier mode is consequently the only one that matters, i.e.

$$\begin{aligned}\lim_{|\Lambda|,N\to\infty}\mathrm{Prob}\Big(\Big(\frac{\ell^{(1)}}{\nu N},\dots,\frac{\ell^{(m)}}{\nu N}\Big)\in A\Big)&=\lim_{|\Lambda|,N\to\infty}\mathrm{Prob}\Big(\Big(\frac{\ell_0^{(1)}}{\nu N},\dots,\frac{\ell_0^{(m)}}{\nu N}\Big)\in A\Big)\\&=\lim_{|\Lambda|,N\to\infty}\mathrm{Prob}\Big(\Big(\frac{\ell_0^{(1)}}{n_0},\dots,\frac{\ell_0^{(m)}}{n_0}\Big)\in A\Big).\end{aligned} \tag{5.8}$$

The last identity follows from Proposition 4.5 (a). Since $n_0\to\infty$ as $|\Lambda|,N\to\infty$, the last term converges to the asymptotic joint probability of the m largest cycles in nonspatial random permutations with cycle weights. □

Finally, we prove that the distribution of cycle lengths of nonspatial weighted random permutations is asymptotically equal to Poisson-Dirichlet.

Proposition 5.3. *Assume that $\alpha_j\to\alpha$ as in Theorem 2.1. Then*

$$\Big(\frac{\ell^{(1)}}{n},\dots,\frac{\ell^{(m)}}{n}\Big)\Rightarrow\mathrm{PD}(\,\mathrm{e}^{-\alpha}\,).$$

Proof. Let us order the cycles of a permutation π according to some rule, such as their smallest element. That is, the first cycle is the one that contains the index 1; the second cycle is the one that contains the smallest element that is not already in the first cycle; and so on... Let $\ell_1,\ell_2,\dots$ be the cycle lengths with respect to this order. We prove that

$$\Big(\frac{\ell_1}{n},\frac{\ell_2}{n-\ell_1},\dots,\frac{\ell_m}{n-\ell_1-\dots-\ell_{m-1}}\Big)$$

converges (in distribution) to i.i.d. beta random variables with parameters $(1,\,\mathrm{e}^{-\alpha}\,)$. It then immediately follows that $(\frac{\ell_1}{n},\dots,\frac{\ell_m}{n})$ converges to $\mathrm{GEM}(\,\mathrm{e}^{-\alpha}\,)$, and that

$(\frac{\ell^{(1)}}{n}, \dots, \frac{\ell^{(m)}}{n})$ converges to PD($\mathrm{e}^{-\alpha}$). We proceed by induction on m. The case $m = 1$ is just the law for $\frac{\ell_1}{n}$, whose convergence to the beta random variable was proved in [16, 6]. For $m > 1$, let

$$A = [a_1, b_1] \times \cdots \times [a_{m-1}, b_{m-1}] \subset (0,1)^{m-1}. \tag{5.9}$$

Then

$$\begin{aligned} \mathrm{Prob}_n\Big(\big(\tfrac{\ell_1}{n}, \dots, \tfrac{\ell_m}{n-\ell_1-\dots-\ell_{m-1}} \big) \in A \times [a_m, b_m] \Big) \\ = \mathrm{Prob}_n\Big(\big(\tfrac{\ell_1}{n}, \dots, \tfrac{\ell_{m-1}}{n-\ell_1-\dots-\ell_{m-2}} \big) \in A \Big) \\ \cdot \mathrm{Prob}_n\Big(\tfrac{\ell_m}{n-\ell_1-\dots-\ell_{m-1}} \in [a_m, b_m] \,\Big|\, \big(\tfrac{\ell_1}{n}, \dots, \tfrac{\ell_{m-1}}{n-\ell_1-\dots-\ell_{m-2}} \big) \in A \Big) \end{aligned} \tag{5.10}$$

It is not hard to check that

$$\begin{aligned} \mathrm{Prob}_n\big(\tfrac{\ell_m}{n-\ell_1-\dots-\ell_{m-1}} \in [a_m, b_m] \,\big|\, \ell_1 = c_1, \dots, \ell_{m-1} = c_{m-1} \big) \\ = \mathrm{Prob}_{n-c_1-\dots-c_{m-1}}\big(\tfrac{\ell_1}{n-c_1-\dots-c_{m-1}} \in [a_m, b_m] \big). \end{aligned} \tag{5.11}$$

As $n \to \infty$ in (5.10), we necessarily have $n - \ell_1 - \cdots - c_{m-1} \to \infty$, so the last term of the right-hand side converges to the beta measure of $[a_m, b_m]$. The first term of the right-hand side converges to a product of beta measures of the set $\times_{i=1}^{m-1} [a_i, b_i]$ by the induction hypothesis. $\square$

Theorem 2.1 clearly follows from Propositions 5.1, 5.2, and 5.3. Theorem 2.2 follows from Propositions 5.1 and 5.2, and from the fact that $\frac{\ell_1}{n} \Rightarrow 1$ for random permutations with cycle weights of the form $\mathrm{e}^{-\alpha_j} = j^{-\gamma}$ with $\gamma > 0$, see [6]. Notice that Proposition 5.2 is trivial here for $m \geqslant 2$, as both sides of the identity converge to zero.

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